NCAR/IMAGe Summer Workshop Lecture 2

Linear and Nonlinear Structure of Boussinesq

Boussinesq in a rotating frame: $\underline{\Omega} = \Omega \hat{z}$

$$\frac{D\underline{u}}{Dt} + \hat{z}2\Omega \times \underline{u} + N\theta\hat{z} = -\underline{\nabla}P + \nu\nabla^2\underline{u}$$

$$\underline{\nabla}\cdot\underline{u} = 0, \quad \frac{D\theta}{Dt} - N(\underline{u}\cdot\hat{z}) = \kappa\nabla^2\theta$$

$$\rho' = \left(\frac{b\rho_0}{g}\right)^{1/2}\theta, \quad N = \left(\frac{gb}{\rho_0}\right)^{1/2}, \quad P = p'/\rho_0$$

$$\rho = \rho_0 + \tilde{\rho}(\underline{x},t), \quad |\tilde{\rho}| \ll \rho_0$$

$$\tilde{\rho} = -bz + \rho'(\underline{x},t)$$

$$p = p_{hydro}(z) + p'(\underline{x},t)$$

θ has dimensions of velocity

[not potential temperature!]

Conservation Laws in the inviscid limit

Energy $\frac{\underline{u}\cdot\underline{u}}{2} + \frac{\theta^2}{2}$ is conserved globally

$$\underline{u}\cdot\left[\frac{\partial \underline{u}}{\partial t} + \underline{\omega}\times\underline{u} + f\hat{z}\times\underline{u} = -\underline{\nabla} p^* - N\theta\hat{z}\right]$$

$$\theta\left[\frac{\partial \theta}{\partial t} + (\underline{u}\cdot\underline{\nabla})\theta - N(\underline{u}\cdot\hat{z}) = 0\right]$$

where $(\underline{u}\cdot\underline{\nabla})\underline{u} = \omega\times\underline{u} + \underline{\nabla}\left(\frac{\underline{u}\cdot\underline{u}}{2}\right)$

Add $\Rightarrow$ $\frac{\partial E}{\partial t} + \underline{u}\cdot\underline{\nabla} p^* + \underline{u}\cdot\underline{\nabla}\left(\theta^2/2\right) = 0$

$$\frac{\partial E}{\partial t} = -\underline{u}\cdot\underline{\nabla}\left[p^* + \frac{\theta^2}{2}\right] = -\underline{\nabla}\cdot\left[\phi\,\underline{u}\right]$$

$$E = \frac{\underline{u}\cdot\underline{u}}{2} + \frac{\theta^2}{2}, \quad \phi = p^* + \theta^2/2$$

$$\int_V \frac{\partial}{\partial t}\left(\frac{\underline{u}\cdot\underline{u}}{2} + \frac{\theta^2}{2}\right) dV = \int_V -\underline{\nabla}\cdot\left[\phi\,\underline{u}\right]$$

$$= -\int_A \phi\,\underline{u}\cdot\hat{n}\,dA$$

For periodic boundary conditions $\Longrightarrow$

$$\boxed{\frac{d}{dt}\int_V E\, dV = 0}$$

Conservation of Potential Vorticity PV
Following fluid particles

$$\frac{D}{Dt}(\underline{\omega}_a \cdot \underline{\nabla}\rho) = 0, \quad \underline{\omega}_a = 2\Omega\hat{z} + \underline{\omega},$$

$$\underline{\omega} = \underline{\nabla} \times \underline{u}, \quad \rho = \rho_0 - bz + \rho', \quad f = 2\Omega$$

$$(\underline{\omega}_a \cdot \underline{\nabla}\rho)\left(\frac{g}{b\rho_0}\right)^{1/2} = \underline{\omega}_a \cdot \left[-N\hat{z} + \underline{\nabla}\theta\right]$$

$$= \underbrace{-fN}_{\text{constant}} \underbrace{- N\underline{\omega}\cdot\hat{z} + f\hat{z}\cdot\underline{\nabla}\theta}_{\text{linear part}} + \underbrace{\underline{\omega}\cdot\underline{\nabla}\theta}_{\text{quadratic part}}$$

$\underline{\nabla} \times$ [inviscid momentum] $\Longrightarrow$

$$\frac{D}{Dt}\underline{\omega} - f(\hat{z}\cdot\underline{\nabla})\underline{u} - (\underline{\omega}\cdot\underline{\nabla})\underline{u}$$

$$+ N\,\underline{\nabla} \times \theta\hat{z} = \underline{0}$$

or $$\boxed{\frac{D\underline{\omega}_a}{Dt} - (\underline{\omega}_a \cdot \underline{\nabla})\underline{u} + N\underline{\nabla} \times \theta\hat{z} = \underline{0}}$$

$\underline{\nabla}$ [inviscid θ-equation] $\Longrightarrow$

$$\frac{D}{Dt}\underline{\nabla}\theta + \underline{\nabla}\underline{u} \cdot \underline{\nabla}\theta - N\underline{\nabla}w = 0, \quad w = \underline{u}\cdot\hat{z}$$

or

$$\boxed{\frac{D}{Dt}\left[-N\hat{z} + \underline{\nabla}\theta\right] + \underline{\nabla}\underline{u} \cdot \underline{\nabla}\theta - N\underline{\nabla}w = 0}$$

$$\left[-N\hat{z} + \underline{\nabla}\theta\right] \cdot \left\{\frac{D\underline{\omega}_a}{Dt} = \cdots\right\}$$

$$\underline{\omega}_a \cdot \left\{\frac{D}{Dt}\left[-N\hat{z} + \underline{\nabla}\theta\right] = \cdots\right\}$$

Add $\Longrightarrow$

$$\boxed{\begin{aligned}\frac{D}{Dt}\left[\underline{\omega}_a \cdot \left[-N\hat{z} + \underline{\nabla}\theta\right]\right] &= 0 \\ = \frac{D}{Dt}\left(\underline{\omega}_a \cdot \underline{\nabla}\rho\right)\end{aligned}}$$

Note: $\frac{D}{Dt}(\underline{\omega}_a \cdot \underline{\nabla}\rho) = 0$ is the QG

equation when we keep only the linear part of the PV

$$q = -N\underline{\omega}\cdot\hat{z} + F\hat{z}\cdot\underline{\nabla}\theta$$ with

$$\frac{D_H}{Dt} = \frac{\partial}{\partial t} + \underline{u}_H\cdot\underline{\nabla}_H \quad , \quad \underline{u}_H = u\hat{x} + v\hat{y}$$

$$u = -\frac{\partial\psi}{\partial y} \, , \quad v = \frac{\partial\psi}{\partial x} \, , \quad \theta = -\frac{F}{N}\frac{\partial\psi}{\partial z}$$

$$\frac{D_H}{Dt}q = \left(\frac{\partial}{\partial t} + \underline{u}_H\cdot\underline{\nabla}_H\right)\left(-N\,\underline{\nabla}_H^2 - \frac{F^2}{N}\frac{\partial^2}{\partial z^2}\right)\psi = 0$$

$$\Rightarrow \boxed{\left(\frac{\partial}{\partial t} + \underline{u}_H\cdot\underline{\nabla}_H\right)\left(\nabla_H^2 + \frac{F^2}{N^2}\frac{\partial^2}{\partial z^2}\right)\psi = 0}$$

Now consider the $\boxed{\text{linear, inviscid limit}}$

$$\frac{\partial \underline{u}}{\partial t} + f\hat{z} \times \underline{u} + N\theta\hat{z} + \underline{\nabla}P = 0$$

$$\underline{\nabla} \cdot \underline{u} = 0, \quad \frac{\partial \theta}{\partial t} - N(\underline{u} \cdot \hat{z}) = 0$$

Assuming periodic boundary conditions:

$$\begin{bmatrix} \underline{u}(\underline{x}, t; \underline{k}) \\ \theta(\underline{x}, t; \underline{k}) \\ p(\underline{x}, t; \underline{k}) \end{bmatrix} = \begin{bmatrix} \hat{\underline{u}}(\underline{k}) \\ \hat{\theta}(\underline{k}) \\ \hat{p}(\underline{k}) \end{bmatrix} \exp\left[i(\underline{k} \cdot \underline{x} - \sigma(\underline{k})t)\right]$$

Plug in $\Rightarrow$

$$-i\sigma\hat{\underline{u}} + f\hat{z} \times \hat{\underline{u}} + N\hat{\theta}\hat{z} + i\underline{k}\hat{p} = 0$$

$$\underline{k} \cdot \hat{\underline{u}} = 0, \quad -i\sigma\hat{\theta} - N\hat{\underline{u}} \cdot \hat{z} = 0$$

with $\underline{x} = [\hat{\underline{u}} \ \hat{\theta} \ i\hat{p}]^T$, $\underline{\underline{A}}\,\underline{x} = \lambda \underline{\underline{B}}\,\underline{x}$

$\underline{\underline{A}}^H = -\underline{\underline{A}}$, $\underline{\underline{B}}^H = \underline{\underline{B}}$, $\det(\underline{\underline{B}}) = 0$

$\lambda = i\sigma$, last row of $\underline{\underline{B}}$ is zero

"Easy" to show that

* λ is pure imaginary $\Rightarrow$ σ real
* Defining $\hat{\underline{V}} = [\hat{\underline{u}} \;\; \theta]^T$:

For $\lambda_1, \hat{\underline{V}}_1$ $\quad \lambda_2, \hat{\underline{V}}_2$

$$\hat{\underline{V}}_1^H \hat{\underline{V}}_2 = \hat{\underline{V}}_2^H \underline{V}_1 = 0$$

Lets find $\hat{\underline{V}}$'s [3D vector space because of the constraint of incompressibility]

① $-i\sigma \hat{\underline{u}} + f\hat{z} \times \hat{\underline{u}} + N\theta\hat{z} + i\underline{k}\hat{p} = 0$

② $\underline{k} \cdot \hat{\underline{u}} = 0$ ③ $-i\sigma\hat{\theta} - N\hat{w} = 0$

$\boxed{\text{Step 1:}}$ $\boxed{\text{Take } \underline{k} \times ①}$

to eliminate $\hat{p}$ $\Rightarrow$

$$\boxed{\underline{k} \times \hat{\underline{u}} = \frac{i}{\sigma}\left[f k_z \hat{\underline{u}} - N\hat{\theta}(\underline{k} \times \hat{z})\right]}$$

Step 2: Take $\underline{k} \times (\underline{k} \times ①)$

$$i\sigma k^2 \underline{\hat{u}} - f k_z (\underline{k} \times \underline{\hat{u}}) + N\hat{\theta}(k_z \underline{k} - \hat{z} k^2) = 0$$

Step 3: Use $\underline{k} \times \underline{\hat{u}} = \cdots$ from

Step 2 in Step 3 result $\Rightarrow$

$$** \quad (\sigma^2 k^2 - f^2 k_z^2)\underline{\hat{u}} + f k_z N \hat{\theta}(\underline{k} \times \hat{z})$$

$$- i\sigma N \hat{\theta}(k_z \underline{k} - k^2 \hat{z}) = 0$$

we still have

② $\underline{k} \cdot \underline{\hat{u}} = 0$ ③ $-i\sigma \hat{\theta} = N\hat{w}$

$\boxed{\text{Wave Modes: } \sigma \neq 0 \quad \hat{w} \neq 0}$

Take $\hat{z} \cdot$ (**) ; use $-i\sigma\hat{\theta} = N\hat{w}$

$$\left(\sigma^2 k^2 - f^2 k_z^2\right)\hat{w} + N^2\left(k_z^2 - k^2\right)\hat{w} = 0$$

$$\boxed{\sigma^{\pm}(\underline{k}) = \pm\frac{\left(f^2 k_z^2 + N^2 k_h^2\right)^{1/2}}{k}}$$

Find 2 waves modes

$$\underline{\hat{\phi}}^{\pm} = \begin{bmatrix} \hat{u} \\ \hat{\theta} \end{bmatrix}^{\pm} \quad \text{For } \sigma^{\pm}$$

From (**)

$$\begin{bmatrix} \hat{u} \\ \hat{v} \\ \hat{w} \end{bmatrix} + \frac{i\hat{w} f k_z}{\sigma k_h^2}\begin{bmatrix} -k_y \\ k_x \\ 0 \end{bmatrix} - \frac{\hat{w}}{k_h^2}\begin{bmatrix} k_x k_z \\ k_y k_z \\ -k_h^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

For orthonormal vectors pick

$$\hat{w} = \frac{-k_h}{\sqrt{2}\,k} \; ; \quad \text{Then} \quad \hat{\theta} = \frac{-i N k_h}{\sqrt{2}\,\sigma\, k}$$

$$\underline{\hat{\phi}}^{\pm} = \frac{1}{\sqrt{2}\,\sigma^{\pm} k_h k} \begin{bmatrix} \sigma^{\pm} k_x k_z + i F k_y k_z \\ \sigma^{\pm} k_y k_z - i F k_x k_z \\ -k_h^2 \sigma^{\pm} \\ -i N k_h^2 \end{bmatrix}$$

$k_h \neq 0$

singular if $k_h = 0$ so choose

$$\underline{\hat{\phi}}^{\pm}(0, 0, k_z) = \begin{bmatrix} \frac{1}{2}(1 + i \operatorname{sgn}(k_z)) \\ \frac{1}{2}(1 - i \operatorname{sgn}(k_z)) \\ 0 \\ 0 \end{bmatrix}$$

VSHF

$\hat{w} = 0$, $\underline{\hat{\omega}}_z = 0$, no potential vorticity

$\sigma^{\pm} = \pm F$

inertial oscillations

see Embid & Majda (1996, 1998)
Smith & Waleffe (2002)

Embid & Majda (1996, 1998)
Smith & Waleffe (2002)

showed VSHF important for

$$Fr = \frac{U}{NL} \to 0, \quad Ro = \frac{U}{FL} = O(1)$$

$$\boxed{\begin{array}{l}\text{Vortical Modes = PV Modes} \\ \text{with } \sigma = 0 \implies \hat{w} = 0\end{array}}$$

** becomes $f^2 k_z^2 \hat{\underline{u}} = f k_z N \hat{\theta} \, (\underline{k} \times \hat{z})$

or

$$k_z f \begin{bmatrix} \hat{u} \\ \hat{v} \\ \hat{w} \end{bmatrix} = N\hat{\theta} \begin{bmatrix} k_y \\ -k_x \\ 0 \end{bmatrix}$$

and one can see that

$$\hat{u} = k_y \hat{\eta}, \quad \hat{v} = -k_x \hat{\eta}, \quad \hat{\theta} = \frac{k_z f}{N} \hat{\eta}$$

where $\hat{\eta}$ is a stream function

For $\hat{\eta} = \dfrac{N}{k|\sigma^{\pm}|} \Longrightarrow$

$$\underline{\hat{\phi}}^{0} = \begin{bmatrix} \hat{u} \\ \hat{v} \\ \hat{w} \\ \hat{\theta} \end{bmatrix}^{0} = \frac{1}{k|\sigma^{\pm}|} \begin{bmatrix} k_y N \\ -k_x N \\ 0 \\ k_z F \end{bmatrix}$$

For $k_h = 0$: $\underline{\hat{\phi}}^{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

True Statements

$$\underline{\hat{\phi}}^{0H} \underline{\hat{\phi}}^{0} = \underline{\hat{\phi}}^{\pm H} \underline{\hat{\phi}}^{\pm} = \underline{1}$$

$$\underline{k} \cdot \underline{\hat{\phi}}^{0} = \underline{k} \cdot \underline{\hat{\phi}}^{\pm} = 0$$

$$\underline{\hat{\phi}}^{0H} \underline{\hat{\phi}}^{\pm} = \underline{\hat{\phi}}^{\pm H} \underline{\hat{\phi}}^{0} = 0$$

$\boxed{\text{Linearized PV conservation}}$

$$\frac{\partial}{\partial t}\left(F\hat{z}\cdot\underline{\nabla}\theta - N\hat{z}\cdot\underline{\omega}\right) = 0$$

in wave space $\Rightarrow$

$$\sigma\left(iFk_z\hat{\theta} - N\hat{\omega}_z\right) = 0$$

requires either $\sigma = 0$ PV modes

or $iFk_z\hat{\theta} = N\hat{\omega}_z$ which

is satisfied by the wave modes

$\Rightarrow$ the vortical = PV modes have all the linear PV

Homework

* Fill in the missing steps on pages (2-3) → (2-4) to show PV conservation

* Fill in the missing steps on pages (2-7) → (2-12) to find the linear eigenmodes

* Consider linearized PV conservation

$$\frac{\partial}{\partial t}\left(f\hat{z}\cdot\underline{\nabla}\theta - N\hat{z}\cdot\underline{\omega}\right) = 0$$

$$\hat{u}(\underline{k}) = \hat{b}(\underline{k})\,\hat{\phi}_i^0(\underline{k})\,\delta_{i1} = \hat{b}(\underline{k})\,k_y\,\hat{\eta}(\underline{k})$$

$$\hat{v}(\underline{k}) = \hat{b}(\underline{k})\,\hat{\phi}_i^0(\underline{k})\,\delta_{i2} = \hat{b}(\underline{k})\,(-k_x)\,\hat{\eta}(\underline{k})$$

$$\hat{w}(\underline{k}) = \hat{b}(\underline{k})\,\hat{\phi}_i^0(\underline{k})\,\delta_{i4} = \hat{b}(\underline{k})\left(\frac{k_z f}{N}\right)\hat{\eta}(\underline{k})$$

Show that the choice

$\hat{\eta}(\underline{k}) = -i$ leads to

$$-\frac{i\sigma}{N}\left[F^2 k_z^2 + N^2 k_h^2\right]\hat{b}(\underline{k}) = 0$$

$\Longrightarrow$ $\hat{b}(\underline{k}) = \hat{\psi}(\underline{k})$ gives

linearized QG equation:

$$\frac{\partial}{\partial t}\left[\nabla_H^2 + \frac{F^2}{N^2}\frac{\partial^2}{\partial z^2}\right]\psi = 0$$

$$u = -\frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \psi}{\partial x}, \quad \theta = -\frac{F}{N}\frac{\partial \psi}{\partial z}$$

NCAR/IMAGe Lecture 2 continued

Return to the full nonlinear problem

Let
$$\begin{bmatrix} \underline{u}(\underline{x},t) \\ \theta(\underline{x},t) \end{bmatrix} = \sum_{\underline{k}} \begin{bmatrix} \hat{\underline{u}}(\underline{k},t) \\ \hat{\theta}(\underline{k},t) \end{bmatrix} e^{i\underline{k}\cdot\underline{x}}$$

$$\begin{bmatrix} \hat{u}(\underline{k},t) \\ \hat{\theta}(\underline{k},t) \end{bmatrix} = \sum_{s_k} a_{s_k}(\underline{k},t)\, \underline{\phi}^{s_k}(\underline{k}) \qquad s_k = 0, +1, -1$$

$$= \sum_{s_k} b_{s_k}(\underline{k},t)\, \underline{\phi}^{s_k}(\underline{k})\, e^{-i\sigma_{s_k}(\underline{k})t}$$

$$\sigma_0 = 0 \quad , \quad \sigma_\pm(\underline{k}) = \pm \frac{(N^2 k_h^2 + f^2 k_z^2)^{1/2}}{k}$$

Also $p(\underline{x},t) = \sum_{\underline{k}} \hat{p}(\underline{k},t)\, e^{i\underline{k}\cdot\underline{x}}$

Plug into the nonlinear Boussinesq Equations
[inviscid case for simplicity]

momentum equation $\Longrightarrow$

$$\frac{\partial}{\partial t}\sum_{\underline{k}} \underline{\hat{u}}(\underline{k},t)e^{i\underline{k}\cdot\underline{x}} + f\hat{z}\times\sum_{\underline{k}}\underline{\hat{u}}(\underline{k},t)e^{i\underline{k}\cdot\underline{x}}$$

$$+\sum_{\underline{p}}\underline{\hat{u}}(\underline{p},t)e^{i\underline{p}\cdot\underline{x}}\cdot\sum_{\underline{q}} i\underline{q}\,\underline{\hat{u}}(\underline{q},t)e^{i\underline{q}\cdot\underline{x}}$$

$$+N\hat{z}\sum_{\underline{k}}\hat{\theta}(\underline{k},t)e^{i\underline{k}\cdot\underline{x}} = -\sum_{\underline{k}} i\underline{k}\,\hat{p}(\underline{k},t)e^{i\underline{k}\cdot\underline{x}}$$

continuity $\sum_{\underline{k}} i\underline{k}\cdot\underline{\hat{u}}(\underline{k},t)\,e^{i\underline{k}\cdot\underline{x}} = 0$

θ - equation $\Longrightarrow$

$$\frac{\partial}{\partial t}\sum_{\underline{k}}\hat{\theta}(\underline{k},t)e^{i\underline{k}\cdot\underline{x}} - N\hat{z}\cdot\sum_{\underline{k}}\underline{\hat{u}}(\underline{k},t)e^{i\underline{k}\cdot\underline{x}}$$

$$+\sum_{\underline{p}}\underline{\hat{u}}(\underline{p},t)e^{i\underline{p}\cdot\underline{x}}\cdot\sum_{\underline{q}} i\underline{q}\,\hat{\theta}(\underline{q},t)e^{i\underline{q}\cdot\underline{x}} = 0$$

Now project onto $e^{i\underline{k}\cdot\underline{x}}$: multiply by $e^{-i\underline{k}\cdot\underline{x}}$ and integrate over $d\underline{x}$

use orthogonality

$$\frac{1}{L^3}\int_0^L\int_0^L\int_0^L e^{i\underline{l}\cdot\underline{x}}\, e^{i\underline{k}\cdot\underline{x}}\, d\underline{x} = \begin{cases} 0 & \underline{l} \neq -\underline{k} \\ 1 & \underline{l} = -\underline{k} \end{cases}$$

$k_n = \frac{2n\pi}{L}$ $\quad n = 0, \pm 1, \pm 2 \cdots$ in a periodic domain of size L^3

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* $$\frac{\partial}{\partial t}\hat{\underline{u}}(\underline{k},t) + f\hat{z}\times\hat{\underline{u}}(\underline{k},t)$$

$$+ \sum_{\underline{p}+\underline{q}=\underline{k}} \frac{1}{2}\left\{ \hat{\underline{u}}(\underline{p},t)\cdot i\underline{q}\,\hat{\underline{u}}(\underline{q},t) + \hat{\underline{u}}(\underline{q},t)\cdot i\underline{p}\,\hat{\underline{u}}(\underline{p},t)\right\}$$

$$+ N\hat{z}\,\hat{\theta}(\underline{k},t) = -i\underline{k}\,\hat{p}(\underline{k},t)$$

* $$\underline{k}\cdot\hat{\underline{u}}(\underline{k},t) = 0$$

* $$\frac{\partial\hat{\theta}(\underline{k},t)}{\partial t} - N\hat{z}\cdot\hat{u}(\underline{k},t)$$

$$+ \sum_{\underline{p}+\underline{q}=\underline{k}} \frac{1}{2}\left\{ \hat{\underline{u}}(\underline{p},t)\cdot i\underline{q}\,\hat{\theta}(\underline{q},t) + \hat{u}(\underline{q},t)\cdot i\underline{p}\,\hat{\theta}(\underline{p},t)\right\} = 0$$

$$\frac{\partial}{\partial t}\begin{bmatrix}\hat{\underline{u}} \\ \hat{\theta}\end{bmatrix} + L\begin{bmatrix}\hat{\underline{u}} \\ \hat{\theta}\end{bmatrix} + \hat{NL}(\underline{k}) = 0$$

Now let $\begin{bmatrix}\hat{\underline{u}}(\underline{k},t) \\ \hat{\theta}(\underline{k},t)\end{bmatrix} = \sum_{s_k} a_{s_k}(\underline{k},t)\,\underline{\phi}^{s_k}(\underline{k})$

multiply by $\left(\underline{\phi}^{m_k}(\underline{k})\right)^H$ and use

$$\left(\underline{\phi}^{m_k}(\underline{k})\right)^H \underline{\phi}^{s_k}(\underline{k}) = \begin{cases} 1 & s_k = m_k \\ 0 & s_k \neq m_k \end{cases}$$

[orthogonality]

$\left(\underline{\phi}^{m_k}(\underline{k})\right)^H \cdot \underline{k} = 0$ [eigenmodes are divergence free]

Also $a_{s_k}(\underline{k},t) = b_{s_k}(\underline{k},t)\,e^{-i\sigma_{s_k}(\underline{k})t}$

$\Longrightarrow$

$$\frac{\partial}{\partial t} b_{m_k}(\underline{k},t) = -\left(\underline{\phi}^{m_k}(\underline{k})\right)^{H} \cdot \hat{NL}(\underline{k})\, e^{i\sigma_{m_k}(\underline{k})t}$$

$$\hat{NL}(\underline{k}) = \sum_{\underline{k}=\underline{p}+\underline{q}} \sum_{m_p} \sum_{m_q} \frac{1}{2} e^{i[\sigma_{m_k}(\underline{k}) - \sigma_{m_p}(\underline{p}) - \sigma_{m_q}(\underline{q})]t}$$

$$\Big\{ b_{m_p}(\underline{p},t)\left(\underline{\phi}^{m_p}(\underline{p}) \cdot i\underline{q}\right) b_{m_q}(\underline{q},t)\, \underline{\phi}^{m_q}(\underline{q})$$

$$+ b_{m_q}(\underline{q},t)\left(\underline{\phi}^{m_q}(\underline{q}) \cdot i\underline{p}\right) b_{m_p}(\underline{p},t)\, \underline{\phi}^{m_p}(\underline{p}) \Big\}$$

where $\underline{\phi}^{m_p}(\underline{p}) \cdot i\underline{q} = \underline{\phi}^{m_p}(\underline{p}) \cdot i \begin{bmatrix} q_x \\ q_y \\ q_z \\ 0 \end{bmatrix}$

Finally one can rewrite using reality conditions $\Longrightarrow$

$$\boxed{\frac{\partial}{\partial t} b_{m_k} = \sum_{\underline{k}+\underline{p}+\underline{q}=0} \sum_{m_p} \sum_{m_q} C_{\underline{k}\underline{p}\underline{q}}^{m_k m_p m_q} \, b_{m_p}^{*} \, b_{m_q}^{*} \, e^{i[\sigma_{m_k} + \sigma_{m_p} + \sigma_{m_q}]t}}$$

reality: $\hat{\underline{u}}(\underline{k}) = \hat{\underline{u}}^{*}(-\underline{k})$

Homework Show that there are no 3-wave exact resonances with $\underline{k}+\underline{p}+\underline{q}=0$ and

$$\sigma^{\pm}(\underline{k}) + \sigma^{\pm}(\underline{p}) + \sigma^{\pm}(\underline{q}) = 0$$

in the range $\frac{1}{2} < N/f < 2$.

$\boxed{\text{Homework}}$

Find $C_{\underline{k}pq}^{s_k s_p s_q}$ explicitly for pure rotation with

$$\underset{\underline{k} \neq (0,0,k_z)}{\underline{\phi}^s(\underline{k})} = \frac{\underline{k} \times (\underline{k} \times \hat{z})}{k\,|\underline{k} \times \hat{z}|} + is\,\frac{\underline{k} \times \hat{z}}{|\underline{k} \times \hat{z}|}$$

$$\underline{\phi}^s(0,0,k_z) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1+i\,\text{sgn}(k_z) & 1-i\,\text{sgn}(k_z) & 0 \end{bmatrix}^T$$

$s = +1, -1$ (no PV mode)

$$\sigma_s(\underline{k}) = sf\,k_z/k$$

$$\left(\underline{\phi}^m(\underline{k})\right)^H \underline{\phi}^s(\underline{k}) = \begin{cases} 2 & m=s \\ 0 & m\neq s \end{cases}$$

$$\underline{k} \cdot \underline{\phi}^m(\underline{k}) = 0$$