NCAR/IMAGe Summer Workshop Lecture 3

Derivations of Quasi-Geostrophic Equations from the Boussinesq Equations

Embid & Majda (1998) showed rigorously that QG dynamic result in the limit

$$Fr = \frac{U}{LN} \longrightarrow 0 \qquad Ro = \frac{U}{2L\Omega} = \frac{U}{fL} \longrightarrow 0$$

2 other "Formal" derivations.

Alternative derivation by restricting nonlinear interactions to include only vortical = PV modes

Recall the full nonlinear equations:

$$\frac{\partial}{\partial t}\left[a_{m_K}(\underline{k},t)\,e^{i\sigma_{m_K}(\underline{k})t}\right]$$

$$= -\left(\underline{\phi}^{m_K}(\underline{k})\right)^H \cdot \hat{NL}(\underline{k})\,e^{i\sigma_{m_K}(\underline{k})t}$$

$$\hat{NL}(\underline{k}) = \sum_{(\underline{k}=\underline{p}+\underline{q})} \sum_{m_p} \sum_{m_q} \frac{1}{2} \Big\{$$

$$a_{m_p}(\underline{p},t)\left(\underline{\phi}^{m_p}(\underline{p})\cdot i\underline{q}\right) a_{m_q}(\underline{q},t)\,\underline{\phi}^{m_q}(\underline{q})$$

$$+\, a_{m_q}(\underline{q},t)\left(\underline{\phi}^{m_q}(\underline{q})\cdot i\underline{p}\right) a_{m_p}(\underline{p},t)\,\underline{\phi}^{m_p}(\underline{p}) \Big\}$$

Consider interactions among $\underline{\phi}^{0}$ modes only:

$$\frac{\partial}{\partial t} a_0(\underline{k},t) = -\left(\underline{\phi}^{0}(\underline{k})\right)^{H} \cdot \hat{NL}(\underline{k})$$

$$= -\sum_{\underline{k}=\underline{p}+\underline{q}} \frac{1}{2} \Big\{ a_0(\underline{p},t)\left(\underline{\phi}^{0}(\underline{p})\cdot i\underline{q}\right) a_0(\underline{q},t)\,\underline{\phi}^{0}(\underline{q})\cdot\underline{\phi}^{0}(\underline{k})$$

$$+\, a_0(\underline{q},t)\left(\underline{\phi}^{0}(\underline{q})\cdot i\underline{p}\right) a_0(\underline{p},t)\,\underline{\phi}^{0}(\underline{p})\cdot\underline{\phi}^{0}(\underline{k}) \Big\}$$

since $\underline{\phi}^{0}(\underline{k})$ is real

Recall $\underline{\Phi}^{0}(\underline{k}) = \dfrac{1}{|\sigma_k| k} \left[Nk_y \quad -Nk_x \quad 0 \quad Fk_z \right]^T$

$|\sigma_k| = |\sigma_k^{\pm}| \qquad \sigma_k^{\pm} = \dfrac{\pm \left(N^2 k_h^2 + F^2 k_z^2 \right)^{1/2}}{k}$

Compute the products:

$$\underline{\Phi}^{0}(\underline{p}) \cdot i\underline{q} = \frac{iN}{|\sigma_p| p} \hat{z} \cdot (\underline{q} \times \underline{p})$$

$$\underline{\Phi}^{0}(\underline{q}) \cdot i\underline{p} = \frac{iN}{|\sigma_q| q} (-\hat{z}) (\underline{q} \times \underline{p})$$

$$\underline{\Phi}^{0}(\underline{q}) \cdot \underline{\Phi}^{0}(\underline{k}) = \frac{1}{|\sigma_q| q |\sigma_p| p} \left\{ N^2 k_y q_y + N^2 k_x q_x + F^2 k_z q_z \right\}$$

$$\underline{\Phi}^{0}(\underline{p}) \cdot \underline{\Phi}^{0}(\underline{k}) = \cdots$$

Plug in $\Longrightarrow$

$$\frac{\partial}{\partial t} a_0(\underline{k},t) = \sum_{\underline{k}=\underline{p}+\underline{q}} C_{kpq}\, a_0(\underline{p},t)\, a_0(\underline{q},t)$$

$$C_{kpq} = -\frac{1}{2} \frac{iN}{kpq\,|\sigma_k||\sigma_p||\sigma_q|}\, \hat{z}\cdot(\underline{q}\times\underline{p}) \left\{ \left(N^2 q_h^2 + f^2 q_z^2\right) - \left(N^2 p_h^2 + f^2 p_z^2\right) \right\}$$

Let $\hat{\psi}^0(\underline{k},t) = \dfrac{-N a_0(\underline{k},t)}{i\,|\sigma_k|\,k}$

use $\hat{z}\cdot(\underline{q}\times\underline{p}) = (\hat{z}\times\underline{q})\cdot\underline{p} = -(\hat{z}\times\underline{p})\cdot\underline{q}$

$$\frac{\partial}{\partial t}\left(\sigma_k^2 k^2 \hat{\psi}_0(\underline{k},t)\right) = \sum_{\underline{p}} \sum_{\underline{q}} \frac{1}{2} e^{-i\underline{k}\cdot\underline{x}}$$

$$\Big\{ \hat{z} \times i\underline{p}\, \hat{\psi}_0(\underline{p},t) e^{i\underline{p}\cdot\underline{x}} \cdot i\underline{q}\left(-N^2 q_h^2 - f^2 q_z^2\right) \hat{\psi}^0(\underline{q},t) e^{i\underline{q}\cdot\underline{x}}$$

$$+ \hat{z} \times i\underline{q}\, \hat{\psi}_0(\underline{q},t) e^{i\underline{q}\cdot\underline{x}} \cdot i\underline{p}\left(-N^2 p_h^2 - f^2 p_z^2\right) \hat{\psi}^0(\underline{p},t) e^{i\underline{p}\cdot\underline{x}} \Big\}$$

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* use $\sigma_k^2 k^2 = -\left(-N^2 k_h^2 - f^2 k_z^2\right)$
* multiply both sides by $e^{i\underline{k}\cdot\underline{x}}$
* sum over $\underline{k}$
* inverse transform
* identify

$$\sum_{\underline{p}} \hat{z} \times i\underline{p}\, \hat{\psi}_0(\underline{p},t) e^{i\underline{p}\cdot\underline{x}} = \hat{z} \times \nabla \psi(\underline{x},t) = \underline{u}_H(\underline{x},t)$$

$\Longrightarrow$

$$\boxed{\begin{aligned} &-\frac{\partial}{\partial t}\left[\left(N^2\nabla_H^2 + f^2\partial_z^2\right)\psi(\underline{x},t)\right] \\ &= \left(\underline{u}_H(\underline{x},t)\cdot\underline{\nabla}_H\right)\left[N^2\nabla_H^2 + f^2\partial_z^2\right]\psi(\underline{x},t)\end{aligned}}$$

or

$$\left(\frac{\partial}{\partial t} + \underline{u}_H\cdot\underline{\nabla}_H\right) q = 0$$

$$\left(\frac{1}{N}\right) q = \nabla_H^2\psi + \frac{f^2}{N^2}\frac{\partial^2\psi}{\partial z^2}$$

$$u = \frac{-\partial\psi}{\partial y}, \quad v = \frac{\partial\psi}{\partial x}, \quad \theta = \frac{-f}{N}\frac{\partial\psi}{\partial z}$$

$$q = -N\underline{\omega}\cdot\hat{z} + f\,\hat{z}\cdot\underline{\nabla}\theta$$

is the linear part of the

$$PV = (\underline{\omega}_a\cdot\underline{\nabla}\rho)\left(\frac{g}{b\rho_0}\right)^{1/2}$$

$\boxed{\text{More traditional derivation}}$

$$\frac{D\underline{u}}{Dt} + F\hat{z}\times\underline{u} = -\underline{\nabla}P - N\theta\hat{z} + \nu\nabla^2\underline{u}$$

$$\underline{\nabla}\cdot\underline{u} = 0, \quad \frac{D\theta}{Dt} - N(\underline{u}\cdot\hat{z}) = \kappa\nabla^2\theta$$

$$\rho = \rho_0 - bz + \rho', \quad \rho' = \left(\frac{b\rho_0}{g}\right)^{1/2}\theta, \quad N = \left(\frac{gb}{\rho_0}\right)^{1/2}$$

Assume F, N large $\Rightarrow$ Ro, Fr small
inviscid for simplicity

Then $\quad F\hat{z}\times\underline{u} = -\underline{\nabla}P - N\theta\hat{z}$

$$\text{or} \quad \left.\begin{aligned} F\hat{z}\times\underline{u}_H &= -\underline{\nabla}_H P \\ \frac{\partial P}{\partial z} &= -N\theta \end{aligned}\right\}$$

Some vector algebra

$$\hat{z}\times\left(F\hat{z}\times\underline{u}_H = -\underline{\nabla}_H P\right)$$

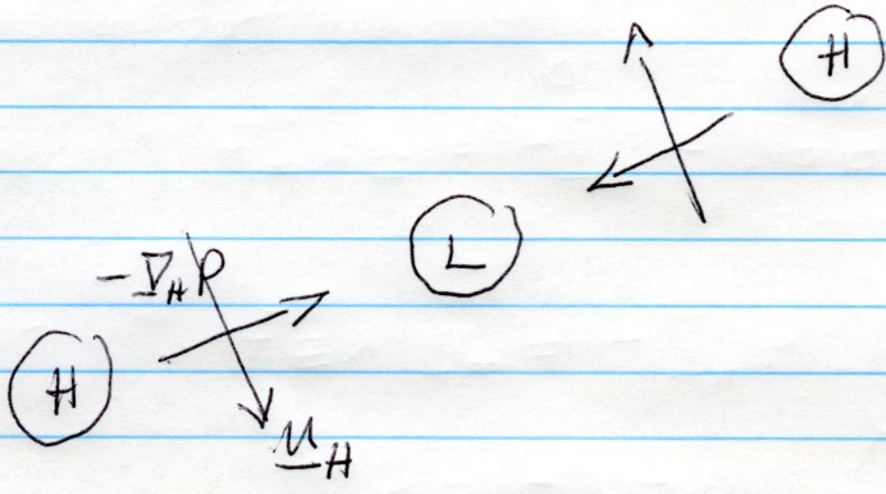
$$F\left[\hat{z}(\hat{z}\cdot\underline{u}_H) - \underline{u}_H(\hat{z}\cdot\hat{z})\right] = -\hat{z}\times\underline{\nabla}_H P$$

$$\Rightarrow \quad \underline{u}_H = \frac{1}{f}\,\hat{z} \times \underline{\nabla}_H\, p \qquad \left[= \frac{-1}{f}\,\underline{\nabla} \times p\hat{z} \right]$$

So $\frac{p}{f}$ is a streamfunction with

$$u = \frac{-1}{f}\frac{\partial p}{\partial y}, \quad v = \frac{1}{f}\frac{\partial p}{\partial x} \quad \text{and}$$

$$\underline{\nabla}_H \cdot \underline{u}_H = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{\partial^2 p}{\partial x\,\partial y} + \frac{\partial^2 p}{\partial y\,\partial x} = 0$$



$\boxed{\text{Find corrections to } \underline{u}_H^0 :}$

e.g. in the $\hat{x}$-direction

$$\frac{\partial u^0}{\partial t} + u^0 \frac{\partial u^0}{\partial x} + v^0 \frac{\partial u^0}{\partial y} - f(v^0 + v') = -\frac{\partial}{\partial x}(p^0 + p') + \text{h.o.t.}$$

So corrections to $\underline{u}_H^0$:

(i) $fv = f(v^0 + v')$

$$= \frac{\partial p}{\partial x} + \frac{\partial u^0}{\partial t} + \frac{u^0 \partial u^0}{\partial x} + \frac{v^0 \partial u^0}{\partial y}$$

(ii) $fu = f(u^0 + u')$

$$= -\frac{\partial p}{\partial y} - \frac{\partial v^0}{\partial t} + u^0 \frac{\partial v^0}{\partial x} + \frac{v^0 \partial v^0}{\partial y}$$

Now $p = p^0 + p'$

Take $\frac{\partial}{\partial y}$ (i) $+ \frac{\partial}{\partial x}$ (ii) $\Rightarrow$

$$F\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right) = -\left(\frac{\partial \zeta^0}{\partial t} + u^0\frac{\partial \zeta^0}{\partial x} + v^0\frac{\partial \zeta^0}{\partial y}\right)$$

$$-\frac{\partial u^0}{\partial x}\frac{\partial v^0}{\partial x} - \frac{\partial v^0}{\partial x}\frac{\partial v^0}{\partial y} + \frac{\partial u^0}{\partial y}\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}\frac{\partial u^0}{\partial y}$$

but continuity $\underline{\nabla}\cdot(\underline{u}_H^0 + \underline{u}_H^1 + w') = 0$

$$\Rightarrow \frac{\partial w'}{\partial z} = -\frac{\partial u'}{\partial x} - \frac{\partial v'}{\partial y}$$

$$\Rightarrow \boxed{\frac{D_H^0}{Dt}\zeta^0 = F\frac{\partial w'}{\partial z}}$$ (all other terms cancel out)

where $\zeta^0 = \frac{\partial v^0}{\partial x} - \frac{\partial u^0}{\partial y}$ is the lowest order vertical vorticity

$\Rightarrow$ background vorticity is advected and stretched

$$\zeta^0 = \frac{\partial v^0}{\partial x} - \frac{\partial u^0}{\partial y} = \frac{1}{F}\nabla_H^2 p^0$$

$\boxed{\text{Find corrections to } w^\circ = 0 \text{ :}}$

the θ - equation gives

$$N w' = \frac{\partial \theta^\circ}{\partial t} + u^\circ \frac{\partial \theta^\circ}{\partial x} + v^\circ \frac{\partial \theta^\circ}{\partial y}$$

with $\theta^\circ = -\frac{1}{N} \frac{\partial p^\circ}{\partial z}$ (vertical momentum)

So we have

$$\boxed{\begin{aligned} &\frac{D_H^\circ}{Dt} \zeta^\circ = F \frac{\partial w'}{\partial z} \,, \qquad \zeta^\circ = \frac{1}{F} \nabla_H^2 p^\circ \\ &\frac{D_H^\circ}{Dt} \theta^\circ = N w' \,, \qquad \theta^\circ = -\frac{1}{N} \frac{\partial p^\circ}{\partial z} \end{aligned}}$$

Eliminating w' gives QG as a solvability constraint

$$\left(\frac{\partial}{\partial t} + \underline{u}_H^\circ \cdot \underline{\nabla}_H \right) \left(\nabla_H^2 + \frac{F^2}{N^2} \frac{\partial^2}{\partial z^2} \right) \psi = 0$$

$$\psi = \frac{p^\circ}{F} \,, \quad u^\circ = \frac{-\partial \psi}{\partial y} \,, \quad v^\circ = \frac{\partial \psi}{\partial x} \,, \quad \theta = \frac{-F}{N} \frac{\partial \psi}{\partial z}$$

Conservation Laws

$$\left(\frac{\partial}{\partial t} + \underline{u}_H^0 \cdot \underline{\nabla}_H\right)\underline{u}_H^0 + f\hat{z} \times (\underline{u}_H^0 + \underline{u}^1) = -\underline{\nabla}_H(p^0 + p^1)$$

splits into

$$\left.\begin{aligned} & f\hat{z} \times \underline{u}_H^0 = -\underline{\nabla}_H p^0 \\ & \left(\frac{\partial}{\partial t} + \underline{u}_H^0 \cdot \underline{\nabla}_H\right)\underline{u}_H^0 + f\hat{z} \times \underline{u}^1 = -\underline{\nabla}_H p^1 \end{aligned}\right\}$$

We also have $\underline{\nabla}_H \cdot \underline{u}_H = 0$, $\dfrac{\partial w^1}{\partial z} = -\dfrac{\partial u^1}{\partial x} - \dfrac{\partial v^1}{\partial y}$

$$\frac{\partial p^0}{\partial z} = -N\theta^0, \quad \frac{\partial \theta^0}{\partial t} + \underline{u}_H^0 \cdot \underline{\nabla}_H \theta^0 = N w^1$$

$$\underline{u}^1 \cdot \left\{ f\hat{z} \times \underline{u}_H^0 = -\underline{\nabla}_H p^0 \right\} \qquad \text{diagnostic}$$

$$* \ \underline{u}_H^0 \cdot \left\{ \left(\frac{\partial}{\partial t} + \underline{u}_H^0 \cdot \underline{\nabla}_H\right)\underline{u}_H^0 + f\hat{z} \times \underline{u}^1 = -\underline{\nabla}_H p^1 \right\}$$

$$* \ \theta^0 \left\{ \frac{\partial \theta^0}{\partial t} + \underline{u}_H^0 \cdot \underline{\nabla}_H \theta^0 = N w^1 \right\} \qquad \text{evolution equations}$$

$$w^1 \left\{ \frac{\partial p^0}{\partial z} = -N\theta^0 \right\} \qquad \text{diagnotic}$$

Add them up using $\underline{u}_H^0 \cdot (\hat{z} \times \underline{u}')$

$$= -\underline{u}' \cdot (\hat{z} \times \underline{u}_H^0)$$

Let $K = \dfrac{\underline{u}_H^0 \cdot \underline{u}_H^0}{2}$, $P = \dfrac{\theta^2}{2}$, $E = K + P$

$$\Rightarrow \frac{\partial E}{\partial t} + \underline{\nabla}_H \cdot (E\,\underline{u}_H^0) + \underline{\nabla} \cdot (p^0 \underline{u}') + \underline{\nabla}_H \cdot (p' \underline{u}_H^0) = 0$$

integrate over the domain assuming periodic boundary conditions and using divergence thm

$$\Rightarrow \frac{d}{dt}\int_V E\,dV = 0$$

2^{nd} quadratic invariant $\Omega = q^2/2$ potential enstrophy

$$\left(\frac{\partial}{\partial t} + \underline{u}_H^0 \cdot \underline{\nabla}_H\right) q = 0 \Rightarrow$$

$$\frac{\partial}{\partial t}\Omega + \underline{\nabla}_H \cdot (\Omega\, \underline{u}_H^0) = 0$$

$$\int_V \frac{\partial \Omega}{\partial t}\, dV + \int_A \Omega\, \underline{u}^{\circ}_{H} \cdot \hat{n}\, dA = 0$$

$$\Rightarrow \frac{d}{dt}\int_V \Omega\, dV = 0 \quad \text{in a periodic domain}$$

$\boxed{\text{Homework}}$ Make the analogy between conservation of energy/enstrophy in 2D non-rotating flow and conservation of energy/potential enstrophy in QG flow to argue for an inverse cascade of energy in QG dynamics.

$\boxed{\text{Homework}}$

We showed that QG comes from

$$\frac{\partial}{\partial t} a_k^0 = \sum_{\underline{k}+\underline{p}+\underline{q}=0} C_{kpq} (a_p^0)^* (a_q^0)^*$$

$$C_{kpq} = \frac{iN(\underline{p} \times \underline{q} \cdot \hat{z})}{\sigma_k k \, \sigma_p p \, \sigma_q q} \left(\sigma_q^2 q^2 - \sigma_p^2 p^2\right)$$

with $\sigma_k k = (N^2 k_h^2 + f^2 k_z^2)^{1/2}$

Show that $\underline{k} + \underline{p} + \underline{q} = 0 \implies$

$$C_{kpq} + C_{pqk} + C_{qkp} = 0$$

$$\sigma_k^2 k^2 C_{kpq} + \sigma_p^2 p^2 C_{pqk} + \sigma_q^2 q^2 C_{qkp} = 0$$

and therefore triad interactions have 2 quadratic invariants

① the total energy (kinetic + potential)

$$|a_k^0|^2 + |a_p^0|^2 + |a_q^0|^2$$

② the quadratic part of the potential enstrophy

$$\sigma_k^2 k^2 |a_k^0|^2 + \sigma_p^2 p^2 |a_p^0|^2 + \sigma_q^2 q^2 |a_q^0|^2$$