Derivations of Qasi-Geostrophic Equations from the Bussinesq Equations

Embid & Majda (1998) showed rigorously that QG dynamic result in the limits

\[ \frac{\Omega}{L} \rightarrow 0 \quad \frac{\Omega}{2L^2} \rightarrow 0 \]

2 other "Formal" derivations.

Alternative derivation by restricting nonlinear interactions to include only vortical = PV modes

Recall the full nonlinear equations:

\[
\frac{\partial}{\partial t} \left[ \alpha_{mk}(k,t) \hat{e} \right] = - \left( \Phi^m_k(k) \right)^H \wedge \hat{NL}(k) \hat{e}
\]
\[ \hat{N}(k) = \sum_{k=p+q} \sum_{m_p, m_q} \frac{1}{2} \left\{ \right. \]

\[ \text{amp}(p, t) \left( \tilde{\phi}^{m_p}(p) \cdot i q \right) \text{amp}(q, t) \tilde{\phi}^{m_q}(q) \]

\[ + \text{amp}(q, t) \left( \tilde{\phi}^{m_q}(q) \cdot i p \right) \text{amp}(p, t) \tilde{\phi}^{m_p}(p) \left\{ \right. \]

Consider interactions among \( \tilde{\phi}^0 \) modes only:

\[ \frac{\partial}{\partial t} a_0(k, t) = - \left( \tilde{\phi}^0(k) \right)^H \cdot \hat{N}(k) \]

\[ = - \sum_{k=p+q} \frac{1}{2} \left\{ a_0(p, t) \left( \tilde{\phi}^0(p) \cdot i q \right) a_0(q, t) \tilde{\phi}^0(q) \cdot \tilde{\phi}^0(k) \right. \]

\[ + a_0(q, t) \left( \tilde{\phi}^0(q) \cdot i p \right) a_0(p, t) \tilde{\phi}^0(p) \cdot \tilde{\phi}^0(k) \left\{ \right. \]

Since \( \tilde{\phi}^0(k) \) is real
Recall \( \phi^0(k) = \frac{1}{|\sigma_k|} \begin{bmatrix} Nk_y & -Nk_x & 0 \end{bmatrix} k \)

\[ |\sigma_k| = |\sigma^+_k| \quad \sigma^+_k = \pm \left( N^2k_h^2 + f^2k_z^2 \right)^{1/2} \]

Compute the products:

\[ \Phi^0(p) \cdot i\frac{\mathbf{q}}{\rho} = \frac{iN}{10q/\rho} \mathbf{\hat{z}} \cdot (\mathbf{q} \times \mathbf{p}) \]

\[ \Phi^0(q) \cdot i\mathbf{f} = \frac{iN}{10q/\rho} (-\mathbf{\hat{z}}) \cdot (\mathbf{q} \times \mathbf{p}) \]

\[ \Phi^0(q) \cdot \Phi^0(k) = \frac{1}{10q/\rho} \sum_{j=0}^{\infty} \frac{N^2k_yq_y}{10q/\rho} \]

\[ + \frac{N^2k_xq_x + f^2k_zq_z}{10q/\rho} \]

\[ \Phi^0(p) \cdot \Phi^0(k) = \ldots \]
Plug in $\Rightarrow$
\[
\frac{\partial}{\partial t} a_0(k,t) = \sum_{k=p+q} C_{kpq} a_0(p,t) a_0(q,t)
\]
\[
C_{kpq} = -\frac{1}{2} \frac{iN}{k p q l \sigma_k l \sigma_p l \sigma_q l}
\]
\[
\begin{aligned}
\{ & \hat{z} \cdot (q \times p) \\
& \left( N^2 q^2 + F^2 q^2 \right) - \left( N^2 p^2 + F^2 p^2 \right) \}
\end{aligned}
\]

Let $\hat{\Psi}_0(k,t) = \frac{-N a_0(k,t)}{i l \sigma_k l k}$

Use $\hat{\psi}_0(q \times p) = (\hat{z} \times q) \cdot p = -(\hat{z} \times p) \cdot q$
\[
\frac{\partial}{\partial t} (\sigma_k^2 k^2 \Phi_0 (k,t)) = \sum \sum \frac{i}{2} e^{-i k \cdot x} \\
\{ \sum_{p} \xi_{p} \Phi_{0} (p,t) e^{i p \cdot x} - i \int (-N_{p}^{2} k - F_{p}^{2} k) \Phi_{0} (p,t) e^{i p \cdot x} \\
+ \sum_{q} \xi_{q} \Phi_{0} (q,t) e^{i q \cdot x} - i \int (-N_{q}^{2} k - F_{q}^{2} k) \Phi_{0} (q,t) e^{i q \cdot x} \} \}
\]

* Use \( \sigma_k^2 k^2 = -(-N_{k_{h}}^{2} k - F_{k}^{2} k) \)

* Multiply both sides by \( e^{-i k \cdot x} \)

* Sum over \( \hat{k} \)

* Inverse transform

* Identify

\[
\sum \sum \frac{i}{2} e^{-i k \cdot x} \xi_{p} \Phi_{0} (p,t) e^{i p \cdot x} = \sum \sum \int \xi_{q} \Phi_{0} (q,t) e^{i q \cdot x} e^{i p \cdot x} (x,t) = U_{\#} (x,t)
\]
\[-\frac{\partial}{\partial t} \left( (N^2 \nabla_H^2 + F^2 \partial_z^2) \Psi(x,t) \right) \]
\[= (u_H(x,t) \cdot \nabla_H) \left[ N^2 \nabla_H^2 + F^2 \partial_z^2 \right] \Psi(x,t) \]

or

\[\left( \frac{\partial}{\partial t} + u_H \cdot \nabla_H \right) \Psi = 0 \]

\[\left( \frac{1}{N} \right) \Psi = \nabla_H^2 \Psi + \frac{F^2}{N^2} \frac{\partial^2 \Psi}{\partial z^2} \]

\[u = \frac{-\partial \Psi}{\partial y}, \quad v = \frac{\partial \Psi}{\partial x}, \quad \theta = \frac{-F \frac{\partial \Psi}{\partial z}}{N} \]

\[\Psi = -N \omega \cdot \hat{z} + F \hat{z} \cdot \nabla \theta \]

is the linear part of the

\[PV = (\omega_a \cdot \nabla \theta) \left( \frac{\Psi}{b_0} \right)^{1/2} \]

is the linear part of the
More traditional derivation

\[
\frac{Du}{Dt} + F \hat{z} \times u = -\nabla p - N \Theta \hat{z} + \nu \nabla^2 u
\]

\[\nabla \cdot u = 0, \quad \frac{D\theta}{Dt} - N(u \cdot \hat{z}) = \kappa \nabla^2 \theta\]

\[p = \rho_0 - b \theta + \rho', \quad \rho' = \left(\frac{b \rho_0}{g}\right)^{1/2}, \quad N = \left(\frac{gb}{\rho_0}\right)^{1/2}\]

Assume \( F, N \) large \( \Rightarrow \) \( R_0, F_r \) small
inviscid for simplicity

Then \( F \hat{z} \times u = -\nabla p - N \Theta \hat{z} \)

or

\[
F \hat{z} \times u_\# = -\nabla_\# p
\]

\[
\frac{dp}{dz} = -N \Theta
\]

Some vector algebra

\[
\hat{z} \times (F \hat{z} \times u_\#) = -\nabla_\# p
\]

\[
F \left[ \hat{z} (\hat{z} \cdot u_\#) - u_\# (\hat{z} \cdot \hat{z}) \right] = -\hat{z} \times \nabla_\# p
\]
\[ \Rightarrow \mathbf{u}_H = \frac{1}{\rho} \mathbf{v} \times \nabla \cdot \mathbf{P} \]

\[ = \frac{1}{\rho} \nabla \times \mathbf{F} \]

So \( \mathbf{F} \) is a stream function with

\[ u = -\frac{1}{\rho} \frac{\partial P}{\partial y}, \quad v = \frac{1}{\rho} \frac{\partial P}{\partial x} \]

and

\[ \nabla \cdot \mathbf{u}_H = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{\partial^2 P}{\partial x \partial y} + \frac{\partial^2 P}{\partial y \partial x} = 0 \]
Find corrections to $u_1^0$.

E.g. in the $\hat{x}$-direction

$$\frac{\partial u_1^0}{\partial t} + u_1^0 \frac{\partial u_1^0}{\partial x} + v_1^0 \frac{\partial u_1^0}{\partial y} = F(v_1^0 + v') = -\frac{\partial}{\partial x}(p^0 + p') + \text{heat}$$

So corrections to $u_1^0$:

(i) $Fv = F(v_1^0 + v')$

$$= \frac{\partial p}{\partial x} + \frac{\partial u_1^0}{\partial t} + u_1^0 \frac{\partial u_1^0}{\partial x} + v_1^0 \frac{\partial u_1^0}{\partial y}$$

(ii) $Fv = F(u_1^0 + u')$

$$= -\frac{\partial p}{\partial y} - \frac{\partial v_1^0}{\partial t} + u_1^0 \frac{\partial v_1^0}{\partial x} + v_1^0 \frac{\partial v_1^0}{\partial y}$$

Now $\rho = \rho^0 + \rho'$

Take $\frac{\partial}{\partial y} (i) + \frac{\partial}{\partial x} (ii) \Rightarrow$
\[ F \left( \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} \right) = - \left( \frac{\partial \psi'}{\partial t} + u' \frac{\partial \psi'}{\partial x} + v' \frac{\partial \psi'}{\partial y} \right) \]

\[ - \frac{\partial u^0}{\partial x} \frac{\partial \psi'}{\partial x} - \frac{\partial v^0}{\partial y} \frac{\partial \psi'}{\partial x} + \frac{\partial u^0}{\partial y} \frac{\partial \psi'}{\partial x} + \frac{\partial v^0}{\partial y} \frac{\partial \psi'}{\partial y} \]

but continuity \[ \nabla \cdot \left( \mathbf{v}^0 + u' + w' \right) = 0 \]

\[ \Rightarrow \frac{\partial w'}{\partial z} = - \frac{\partial u'}{\partial x} - \frac{\partial v'}{\partial y} \]

\[ \Rightarrow \frac{\partial}{\partial t} s^0 = - \frac{\partial w'}{\partial z} \]

(all other terms cancel out)

where \[ s^0 = \frac{\partial v^0}{\partial x} - \frac{\partial u^0}{\partial y} \] is the lowest order vertical vorticity

\[ \Rightarrow \text{background vorticity is advected and stretched} \]

\[ s^0 = \frac{\partial v^0}{\partial x} - \frac{\partial u^0}{\partial y} = \frac{1}{\gamma} \frac{\partial^2 \rho^0}{\partial C^2} \]
Find corrections to $w^o = 0$:

the $\theta$-equation gives

$$NW' = \frac{\partial \theta^o}{\partial t} + u^o \frac{\partial \theta^o}{\partial x} + v^o \frac{\partial \theta^o}{\partial y}$$

with $\theta^o = -\frac{1}{N} \frac{\partial P^o}{\partial z}$ (vertical momentum)

So we have

$$\frac{D_t^o \theta^o}{Dz} = -\frac{\partial w^o}{\partial z}$$

$$\frac{D_t^o \theta^o}{Dz} = NW'$$

Eliminating $w'$ gives $\theta_0$ as a solvability constraint

$$\left(\frac{\partial}{\partial t} + u^o \frac{\partial}{\partial x}\right) \left( \frac{V_{11}^2}{N^2} + \frac{F^2}{N^2} \frac{\partial^2}{\partial z^2} \right) \psi = 0$$

$$\psi = \frac{P^o}{F}$$

$u^o = \frac{\partial \psi}{\partial y}$, $v^o = \frac{\partial \psi}{\partial x}$, $\theta = -F \frac{\partial \psi}{N \partial z}$
Conservation Laws

\[ \left( \frac{\partial}{\partial t} + u_h^0 \cdot \nabla_h \right) u_h^0 + F^h x (u_h^0 + u') = -\nabla_h (p^0 + p') \]

splits into

\[ \begin{aligned}
F^h x u_h^0 &= -\nabla_h p^0 \\

\end{aligned} \]

\[ \begin{aligned}
\left( \frac{\partial}{\partial t} + u_h^0 \cdot \nabla_h \right) u_h^0 + F^h x u' &= -\nabla_h p' \\

\end{aligned} \]

We also have \( \nabla_h \cdot u_h^0 = 0 \), \( \frac{\partial w'}{\partial z} = -\frac{\partial u'}{\partial x} - \frac{\partial v'}{\partial y} \)

\[ \begin{aligned}
\frac{\partial p^0}{\partial z} &= -N \theta^0 \\
\theta^0 &= \frac{\partial \theta^0}{\partial t} + u_h^0 \cdot \nabla_h \theta^0 = N w' \\

\end{aligned} \]

\[ \begin{aligned}
u^1 - \left\{ F^h x u_h^0 = -\nabla_h p^0 \right\} \text{ diagnostic} \\
u_h^0 - \left\{ \left( \frac{\partial}{\partial t} + u_h^0 \cdot \nabla_h \right) u_h^0 + F^h x u' = -\nabla_h p' \right\} \text{ evolution equations} \\
\theta^0 - \left\{ \frac{\partial \theta^0}{\partial t} + u_h^0 \cdot \nabla_h \theta^0 = N w' \right\} \text{ diagnostic} \\
w' - \left\{ \frac{\partial \theta^0}{\partial z} = -N \theta^0 \right\} \text{ diagnostic} \]
Add them up using $u_0^* \cdot (\hat{a} \times u')$

$$= -u_1^* \cdot (\hat{a} \times u_0^*)$$

Let $k = u_0^* \cdot u_0^* \frac{\hat{a}}{2}, \quad p = \frac{\hat{a}^2}{2}, \quad E = k + p$

$$\Rightarrow \frac{\partial E}{\partial t} + \nabla_0 \cdot (E u_0^*) + \nabla_0 \cdot (p^0 u_1^*) + \nabla_0 \cdot (p^1 u_0^*) = 0$$

Integrate over the domain assuming periodic boundary conditions and using divergence thm

$$\Rightarrow \frac{d}{dt} \int_V E dV = 0$$

2nd quadratic invariant $\mathcal{L} = \frac{\hat{a}^2}{2}$ potential energy

$$\left(\frac{\partial}{\partial t} + u_0^* \cdot \nabla_0\right) q = 0 \Rightarrow$$

$$\frac{\partial}{\partial t} \mathcal{L} + \nabla_0 \cdot (\mathcal{L} u_0^*) = 0$$
\[ \int_V \frac{d \mathbf{n}}{dt} \, dV + \int_A \mathbf{n} \cdot \hat{n} \, dA = 0 \]

\[ \Rightarrow \frac{1}{\mathcal{V}} \int_V \mathbf{n} \, dV = 0 \text{ in a periodic domain} \]

Homework: Make the analogy between conservation of energy/enstrophy in 2D non-rotating flow and conservation of energy/potential enstrophy in QG flow to argue for an inverse cascade of energy in QG dynamics.
We showed that QG comes from

\[ \frac{\partial}{\partial t} a_k^o = \sum_{k+p+q=0} C_{kpq} (a_p^o)^* (a_q^o)^* \]

\[ C_{kpq} = \frac{iN (p \times q)^{\wedge} (\sigma_q^2 q^2 - \sigma_p^2 p^2)}{\sigma_k \sigma_p \sigma_q} \]

with \( \sigma_k = (N^2 k_n^2 + F^2 k_z^2)^{1/2} \)

Show that \( k + p + q = 0 \implies \)

\[ C_{kpq} + C_{pqk} + C_{qkp} = 0 \]

\[ \sigma_k^2 k^2 C_{kpq} + \sigma_p^2 p^2 C_{pqk} + \sigma_q^2 q^2 C_{qkp} = 0 \]

and therefore triad interactions have 2 quadratic invariants
(1) The total energy (kinetic + potential)
\[ |a_k|^2 + |a_p|^2 + |a_q|^2 \]

(2) The quadratic part of the potential enstrophy
\[ \xi_k^2 |a_k|^2 + \xi_p^2 |a_p|^2 + \xi_q^2 |a_q|^2 \]