From Boussinesq to QG and "In Between"

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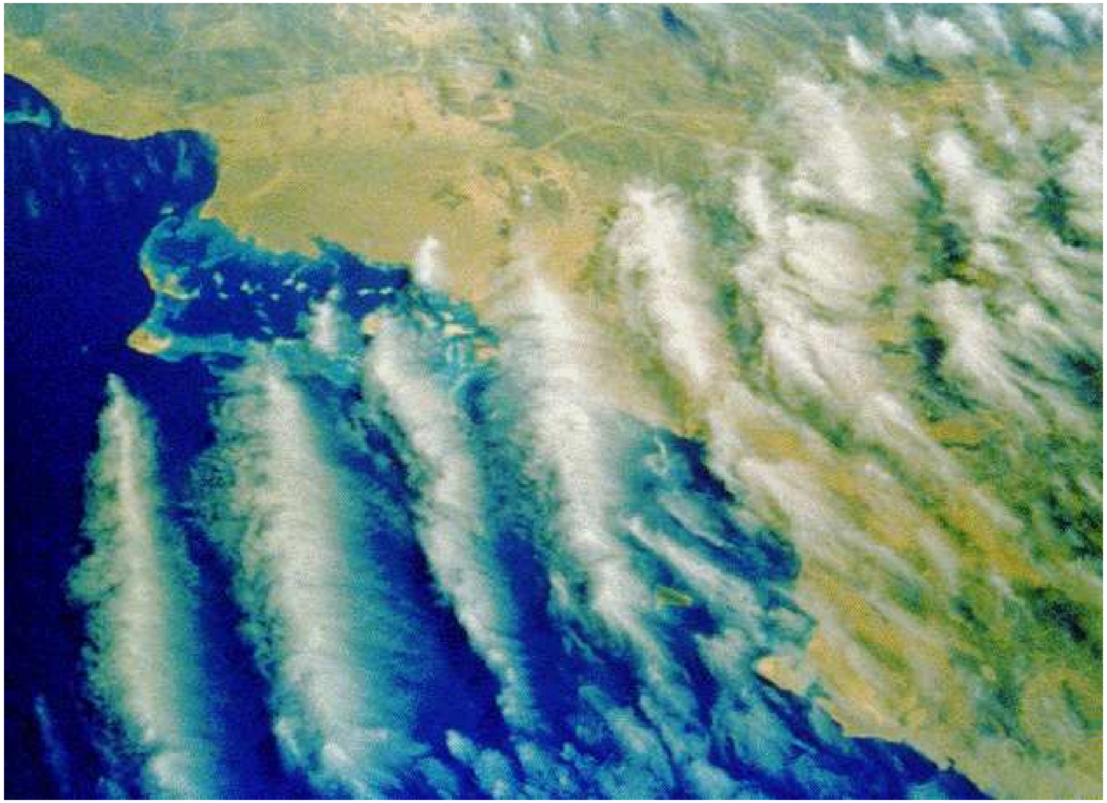


Roll clouds in the jet stream over Saudi Arabia/Red Sea

Vortical-Wave Mode Interactions in Atmosphere-Ocean Flows

- The rotating Boussinesq equations have a vortical mode and a wave mode in the linear limit
- The QG reduced model includes only nonlinear interactions among vortical modes
- When are wave modes important and how can we systematically include them in new reduced models?

Issue 1: a large range of space and time scales:

The QG model without wave-vortical interactions works well to describe large-scale flows on short time scales, e.g. short-term weather prediction.

However, wave mode interactions contribute in many physical situations, e.g. flow over topography,

and influence large-scale coherent flows on long times

e.g. they can *generate* jets, vortices and layers on long time scales

Issue 2: direction of energy transfer:

In QG, energy is transferred predominantly from small to large scales

In strongly stratified RBE with weak rotation, energy is transferred predominantly from large to small scales,

with leakage of energy from small to large VSHF modes

How does the change in direction of energy transfer happen as the rotation rate changes?

Outline

I. Review Analytical Properties of the Governing Equations

- Solution as a superposition of linear eigenmodes
- Reduced models

II. Example Numerical Results for Boussinesq

- large-scale forcing, small-scale forcing

III. Derivation of PDE reduced models to understand wave-vortical mode interactions

The rotating Boussinesq equations

Conservation laws for vertically stratified flow rotating about the vertical $\hat{\mathbf{z}}$-axis:

$$momentum: \quad \frac{D\mathbf{u}}{Dt} + f\hat{\mathbf{z}} \times \mathbf{u} = -\nabla\phi - N\theta\hat{\mathbf{z}} + \nu\nabla^2\mathbf{u}$$

$$mass: \quad \nabla \cdot \mathbf{u} = 0$$

$$energy: \quad \frac{D\theta}{Dt} - Nw = \kappa\nabla^2\theta, \quad \theta = \frac{g}{N\rho_o}\rho'$$

$$f = 2\Omega, \qquad Ro = \frac{U}{fL}$$

$$\rho = \rho_o - bz + \rho', \quad \rho' \ll \rho_o, |bz|, \quad N^2 = \frac{gb}{\rho_o}, \quad Fr = \frac{U}{NH}$$

Rossby and Froude numbers in geophysical flows

Pedlosky (1986) estimates:

- $Ro \approx 0.14$ for typical synoptic-scale winds at mid-latitudes
  $U \approx 10 \;\; \mathrm{m} \;\; \mathrm{s}^{-1}$, $L \approx 1000 \;\; \mathrm{km}$
- $Ro \approx 0.07$ in the western Atlantic
  $U \approx 5 \;\; \mathrm{cm} \;\; \mathrm{s}^{-1}$, $L \approx 100 \;\; \mathrm{km}$

Typical values are $N/f \approx 100$ in the stratosphere

and $N/f \approx 10$ in the oceans.

Flows with $N/f \approx L/H ==> Fr \approx 0.1$ (Burger number unity).

Solutions in the unforced, linear, inviscid limit

$$[\mathbf{u}, \theta]^T(\mathbf{x}, t; \mathbf{k}) = \phi(\mathbf{k}) \exp\left[i\Big(\mathbf{k}\cdot\mathbf{x} - \sigma(\mathbf{k})t\Big)\right] + \text{ c.c.}$$

with eigenmodes $\phi(\mathbf{k})$ and eigenvalues $\sigma(\mathbf{k})$.

- Wave modes $\phi_+(\mathbf{k})$ and $\phi_-(\mathbf{k})$ with

$$\sigma_\pm(\mathbf{k}) = \pm\frac{(N^2k_h^2 + f^2k_z^2)^{1/2}}{k}$$

- A non-wave (vortical or geostophic) mode $\phi_0(\mathbf{k})$ with

$$\sigma_0(\mathbf{k}) = 0$$

.

Slow wave modes (as important as slow vortical modes!)

- Rotation-dominated flows

$$\sigma_{\pm}(\mathbf{k}) \approx \pm \frac{f k_z}{k}$$

slow when $k_z = 0$, e.g. vortical columns.

- Stratification-dominated flows

$$\sigma_{\pm}(\mathbf{k}) \approx \pm \frac{N k_h}{k}$$

slow when $k_h = 0$, e.g. horizontal shear layers (VSHF)

Eigenmode representation for nonlinear flows

Since $\boldsymbol{\phi}_s(\mathbf{k})$, $s = \pm, 0$ form an orthogonal basis

$$[\mathbf{u}, \theta]^T(\mathbf{x}, t) = \sum_{\mathbf{k}} \sum_{s} b_s(t; \mathbf{k}) \boldsymbol{\phi}_s(\mathbf{k}) \exp\left[i \left(\mathbf{k} \cdot \mathbf{x} - \sigma_s(\mathbf{k}) t \right) \right]$$

and the equations become

$$\frac{\partial}{\partial t} b_{s_{\mathbf{k}}} = \sum_{\triangle} \sum_{s_{\mathbf{p}}, s_{\mathbf{q}}} C_{\mathbf{kpq}}^{s_{\mathbf{k}} s_{\mathbf{p}} s_{\mathbf{q}}} \; b^*_{s_{\mathbf{p}}} \, b^*_{s_{\mathbf{q}}} \exp\left[i \left(\sigma_{s_{\mathbf{k}}} + \sigma_{s_{\mathbf{p}}} + \sigma_{s_{\mathbf{q}}} \right) t \right]$$

27 interaction types, including 3-wave interactions

Exact and near resonances dominate: $|\sigma_{s_{\mathbf{k}}} + \sigma_{s_{\mathbf{p}}} + \sigma_{s_{\mathbf{q}}}| \ll 1.$

Reduced Models

Reduced models resulting from restriction of the sum

$$\frac{\partial}{\partial t} b_{s_{\mathbf{k}}} = \sum_{\triangle} \sum_{s_{\mathbf{p}}, s_{\mathbf{q}}} C^{s_{\mathbf{k}} s_{\mathbf{p}} s_{\mathbf{q}}}_{\mathbf{kpq}} \; b^*_{s_{\mathbf{p}}} \, b^*_{s_{\mathbf{q}}} \exp\left[i \left(\sigma_{s_{\mathbf{k}}} + \sigma_{s_{\mathbf{p}}} + \sigma_{s_{\mathbf{q}}} \right) t \right]$$

automatically conserve energy because each triad $(\mathbf{k}, \mathbf{p}, \mathbf{q})$ satisfies the detailed balance:

$$C^{s_{\mathbf{k}} s_{\mathbf{p}} s_{\mathbf{q}}}_{\mathbf{kpq}} \; + \; C^{s_{\mathbf{p}} s_{\mathbf{q}} s_{\mathbf{k}}}_{\mathbf{pqk}} \; + \; C^{s_{\mathbf{q}} s_{\mathbf{k}} s_{\mathbf{p}}}_{\mathbf{qkp}} = 0$$

Reduced Models for 3D Boussinesq

Keeping only slow vortical mode interactions $==>$ the symmetric 3D quasi-geostrophic equation.

In Fourier space

$$\frac{d}{dt} b_0(t;\mathbf{k}) = \sum_{\triangle} C^{000}_{\mathbf{kpq}} \; b_0^*(\mathbf{p}) \; b_0^*(\mathbf{q})$$

An inverse transform gives 3DQG

$$\left(\frac{\partial}{\partial t} + \mathbf{u}_H \cdot \boldsymbol{\nabla}\right) q = 0, \qquad q = \left(\nabla_H^2 + \frac{f^2}{N^2}\frac{\partial^2}{\partial z^2}\right)\psi(\mathbf{x},t)$$

$$\nabla_H^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad \mathbf{u}_H = \hat{\mathbf{z}} \times \boldsymbol{\nabla}\psi, \quad \theta = -\frac{f}{N}\frac{\partial \psi}{\partial z}$$

3D Pure Rotation

keeping only slow wave modes with $k_z = 0 ==>$ symmetric 2D flow for (u,v); w is a passive scalar.

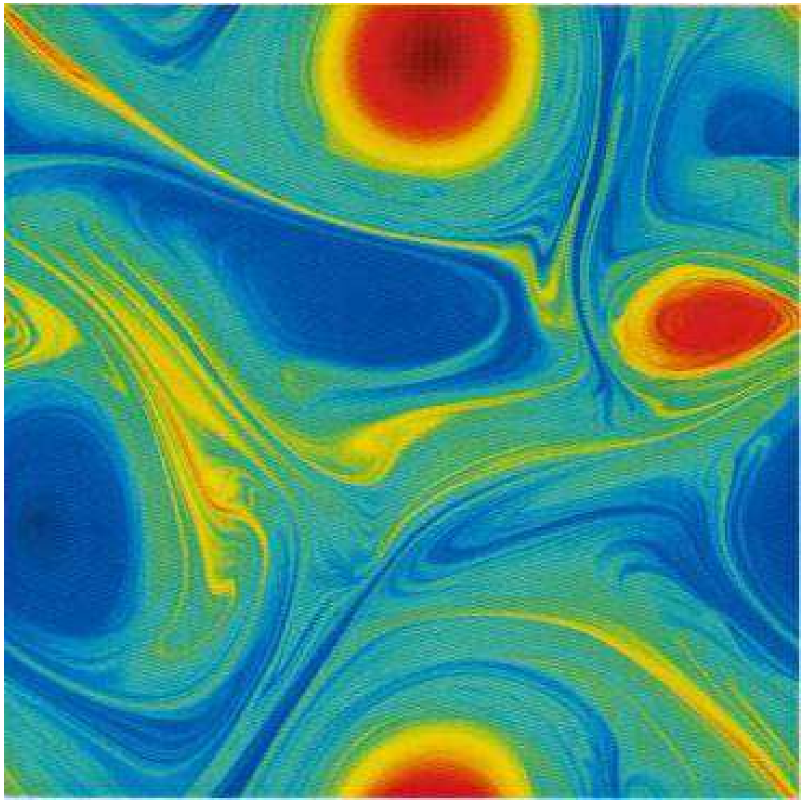
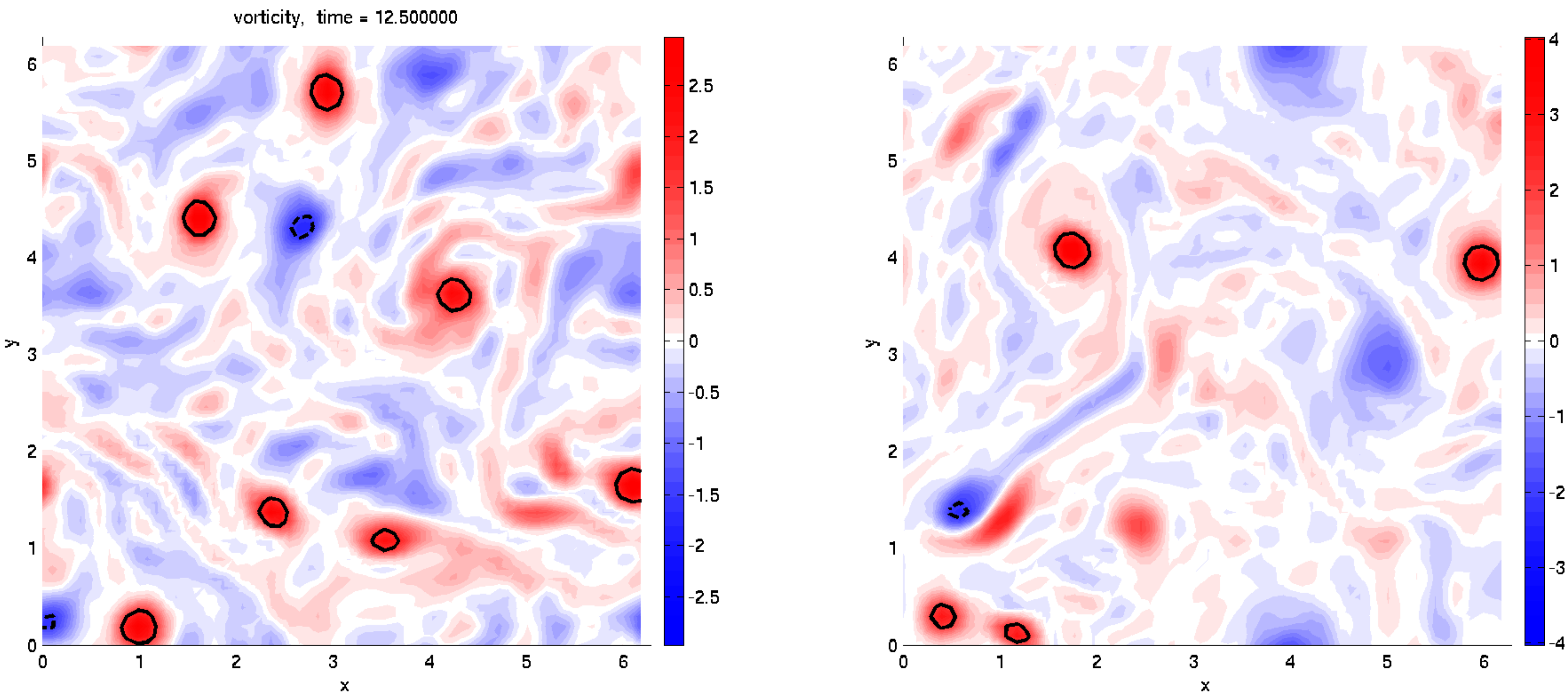


Fig. shows (symmetric) 2D decay with large-scale drag.

Embid & Majda 1996, 1998, Babin et al. 2002 all interactions with $|\sigma_{s_{\mathbf{k}}} + \sigma_{s_{\mathbf{p}}} + \sigma_{s_{\mathbf{q}}}| < Ro ==>$ cyclone dominance (but is not a PDE !).



Full

Near resonances (12%)

Smith & Lee 2005

Part II: Numerical Simulations of RBE:

I. QG-like range with $1/2 < N/f < 2$ and $Fr \approx Ro$ small

a. small-scale forcing

b. large-scale forcing

II. Strongly stratified flow with $N \gg f$, $Fr \ll 1$

a. small-scale forcing; generation of VSHF ($k_h = 0$)

b. large-scale forcing; vortical mode continues to be significant at small scales

III. Strongly rotating flows with $f \gg N$, $Ro \ll 1$

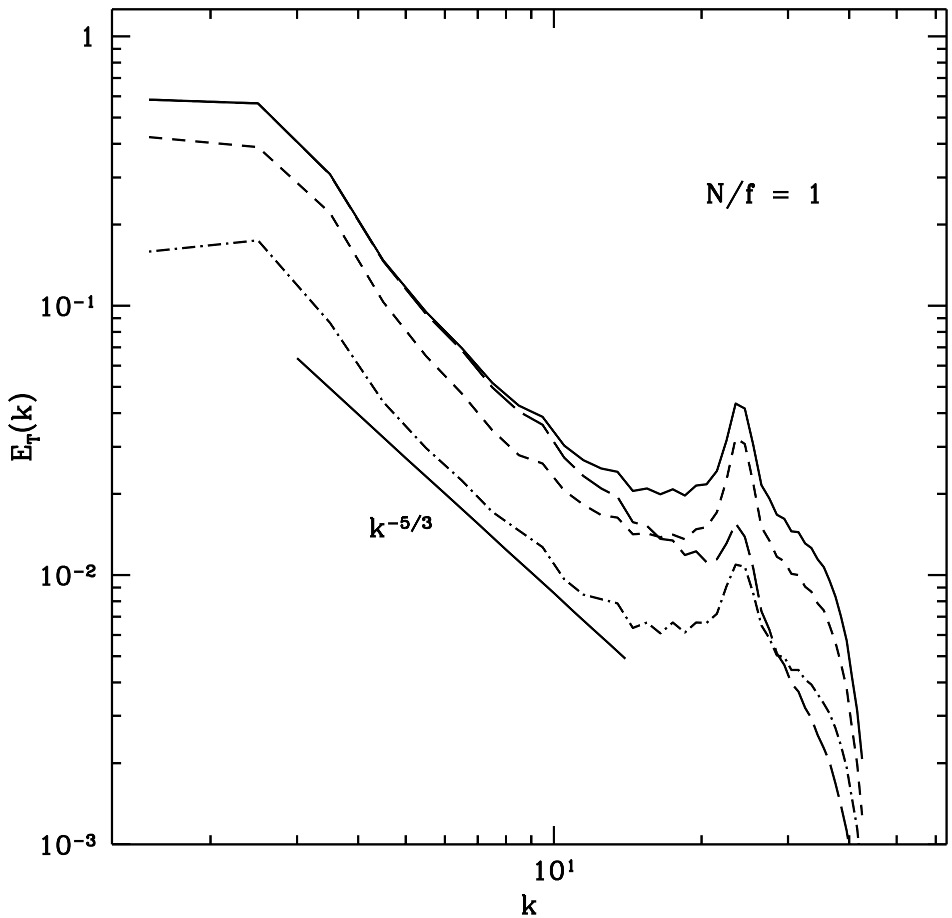
a. small-scale forcing; generation of vortical columns ($k_z = 0$)

b. large-scale forcing; small-scales are wave dominated

I. QG-like range with $1/2 < N/f < 2$ and $Fr \approx Ro$ small

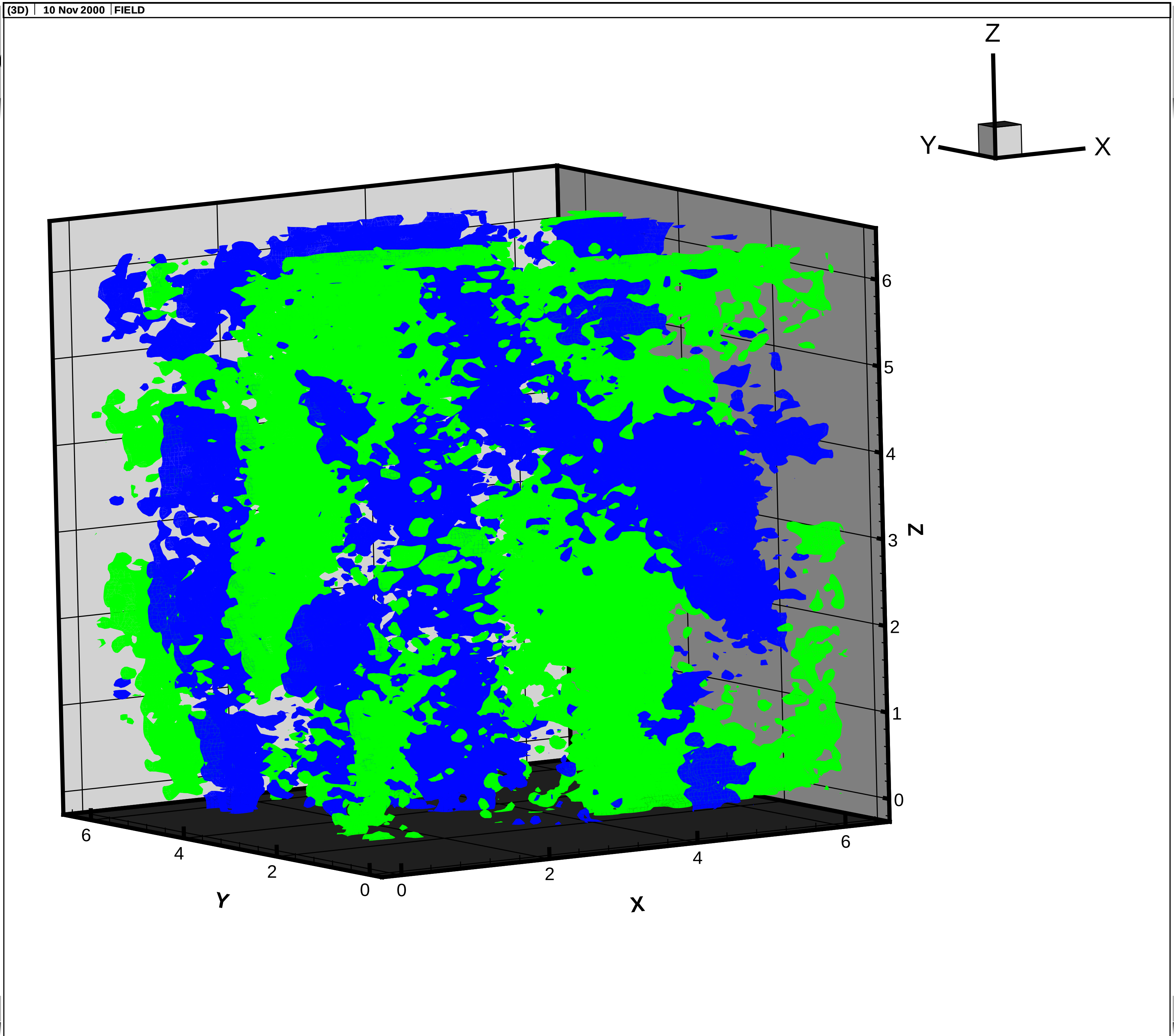
- QG is the limiting dynamics for $Fr \approx Ro \rightarrow 0$ (Embid & Majda 1998)
- No 3-wave exact resonances
- Expect dominance of vortical modes at large scales with $E(k) \propto k^{-5/3}, k < k_f$
- Expect vortical modes to have small-scale spectrum $E(k) \propto k^{-3}, k > k_f$

N/f=1, Fr = 0.2; Full flow is close to QG

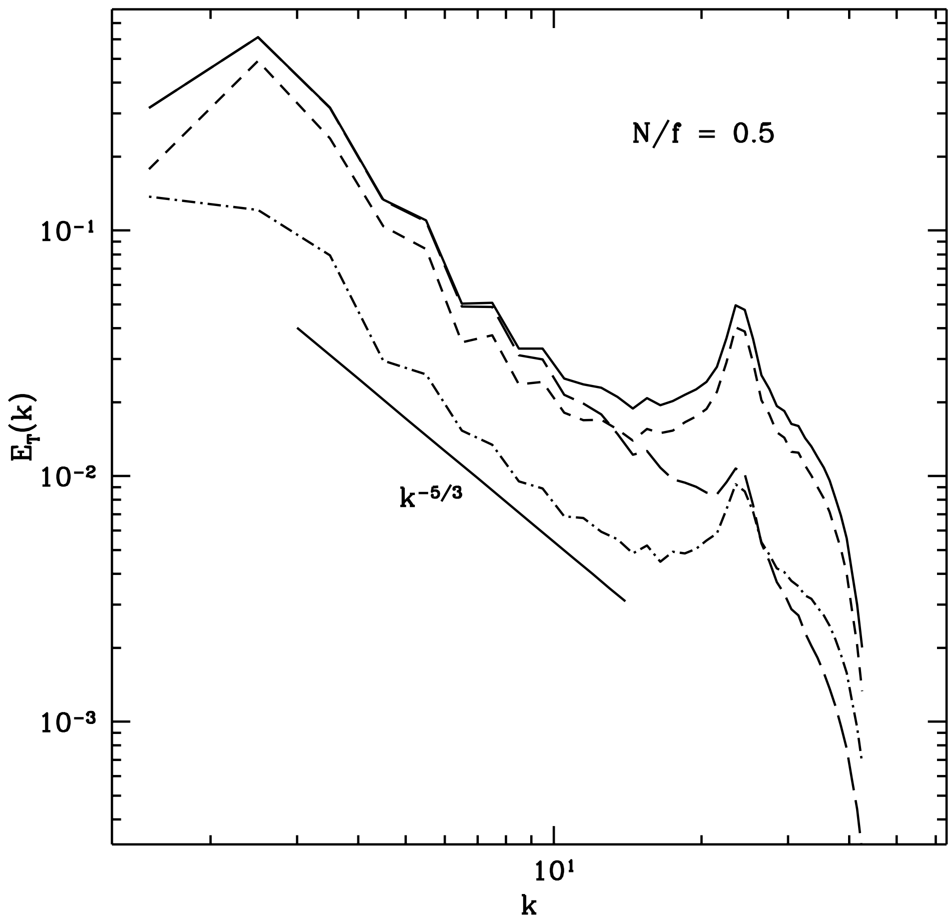


Solid: $E_T(k)$; Long dash: $E_{PV}(k)$; Dash: $K(k)$; Dot-dash: $P(k)$

C

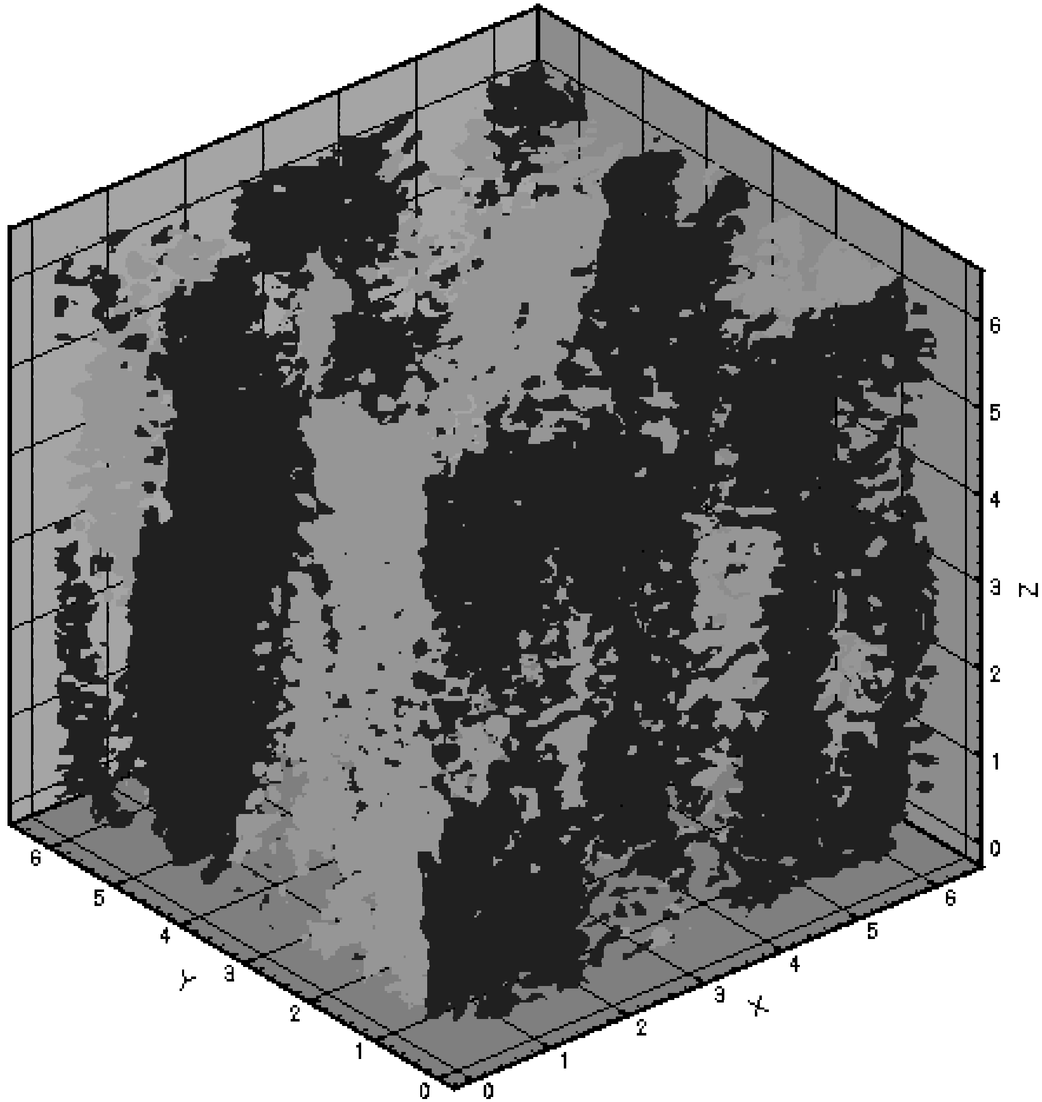


N/f=1/2, Fr = 0.2; Full flow remains close to QG

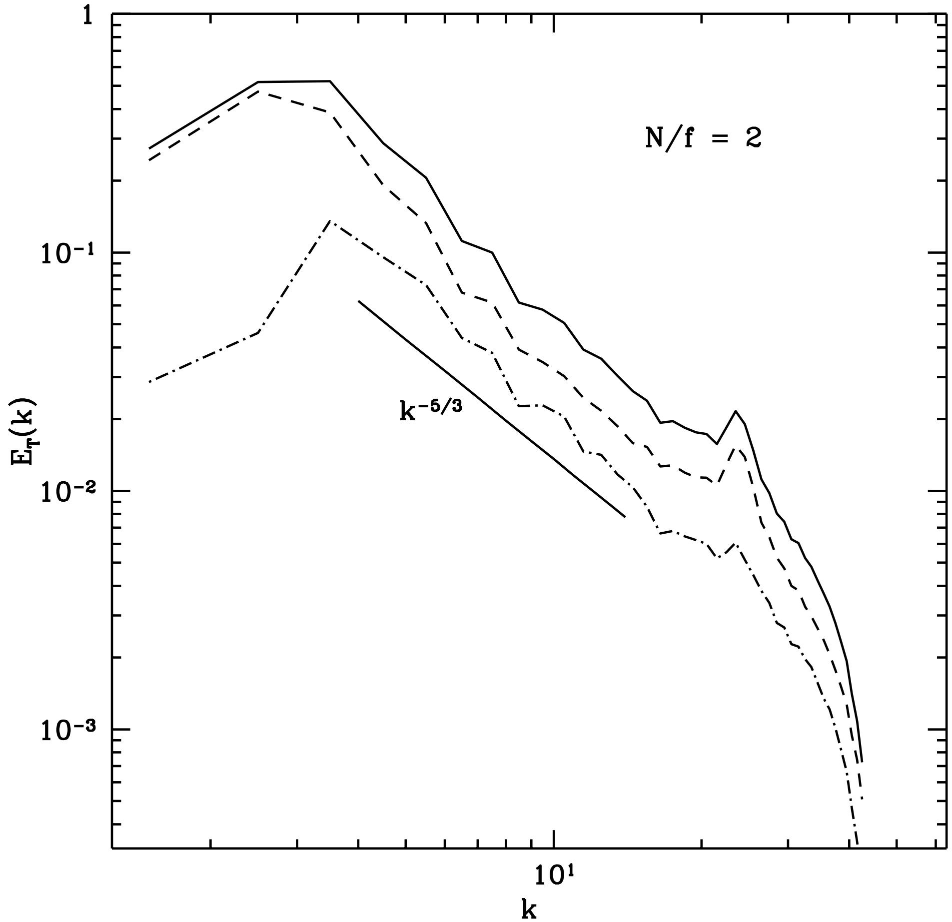


Solid: $E_T(k)$; Long dash: $E_{PV}(k)$; Dash: $K(k)$; Dot-dash: $P(k)$

Contours of zonal velocity for $N/f = 1/2$, $Fr = 0.2$

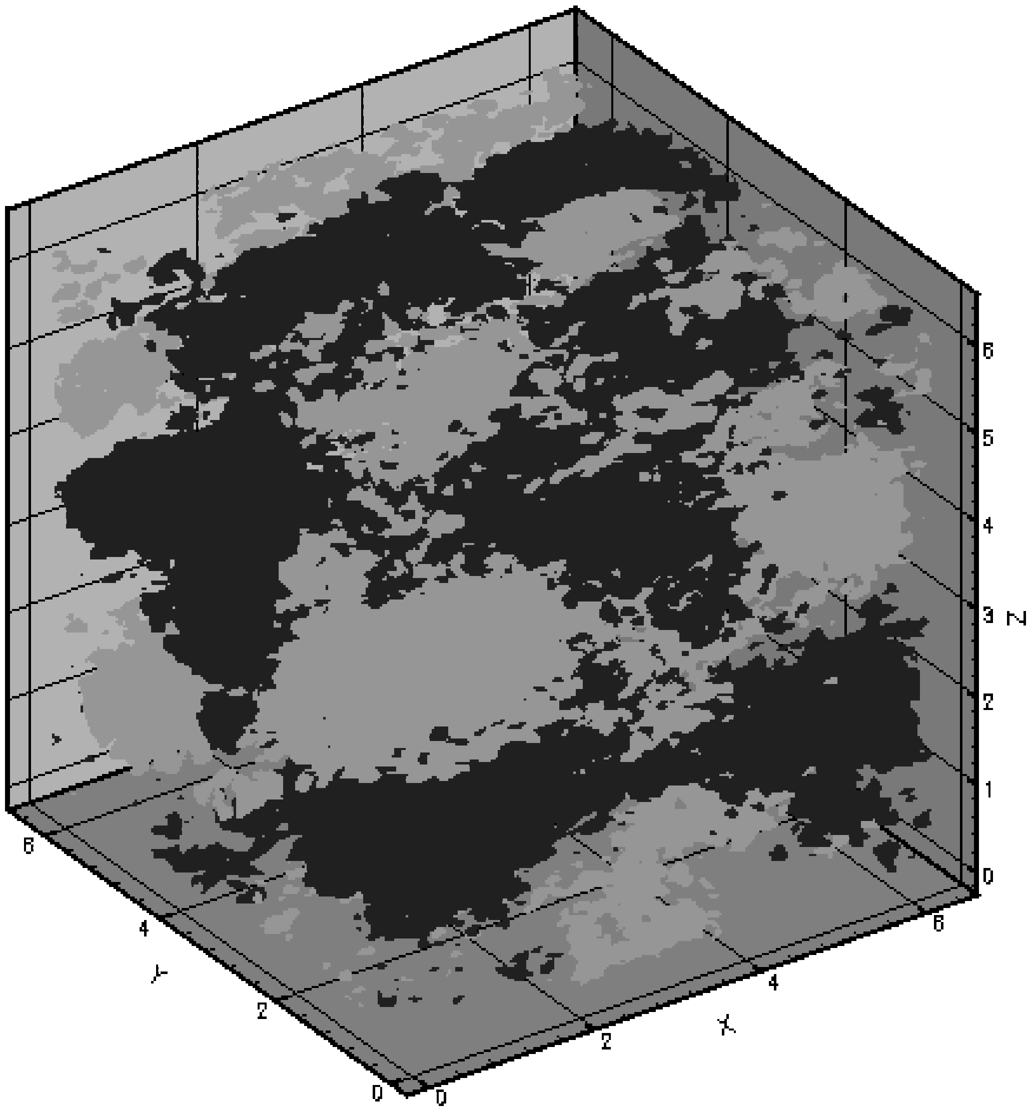


N/f=2, Fr = 0.2; Full flow remains close to QG

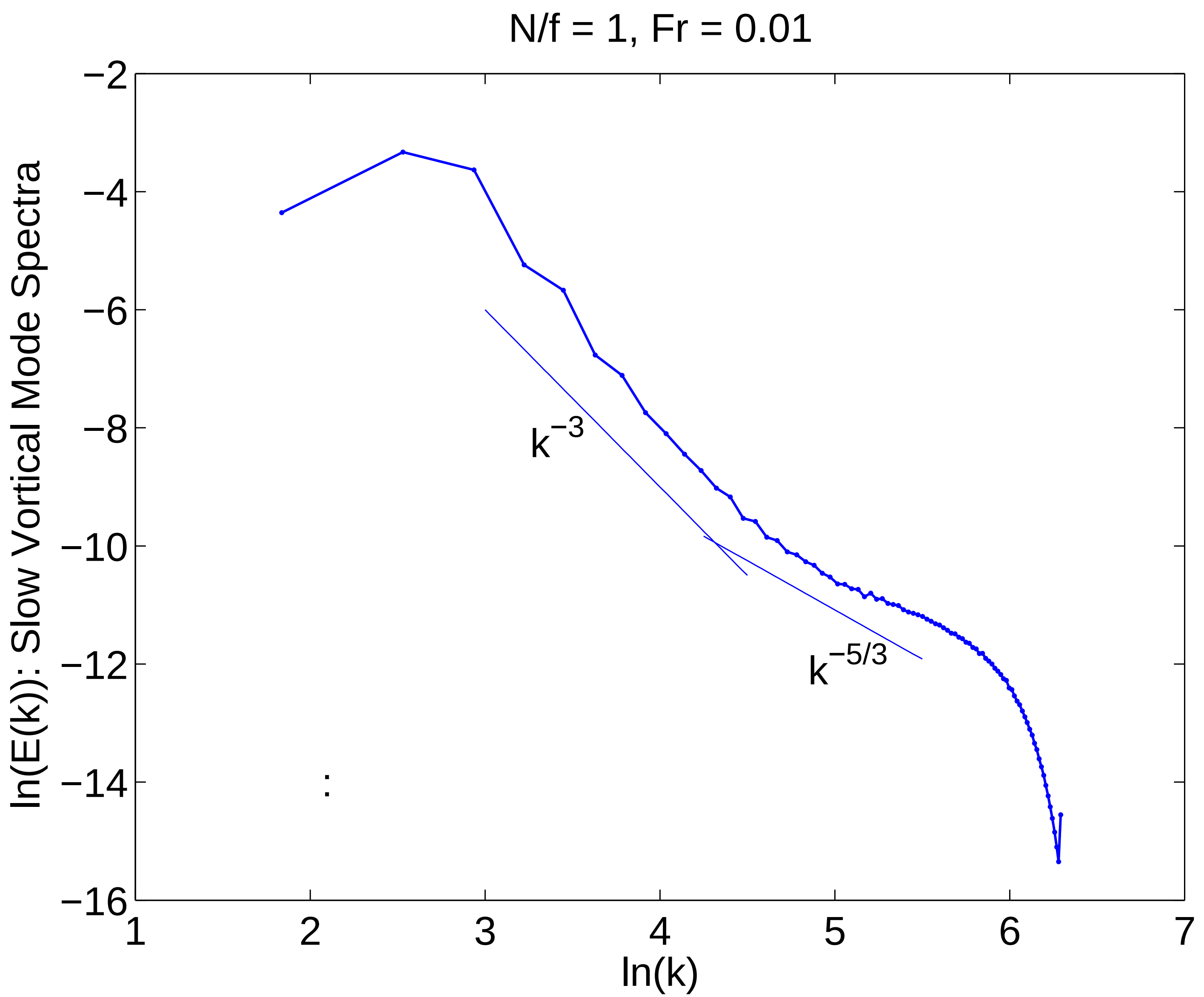


Solid: $E_T(k)$; Long dash: $E_{PV}(k)$; Dash: $K(k)$; Dot-dash: $P(k)$

Contours of zonal velocity for $N/f = 2$, $Fr = 0.2$



Small-scale vortical mode spectrum for $N/f = 1$, $Fr = 0.01$



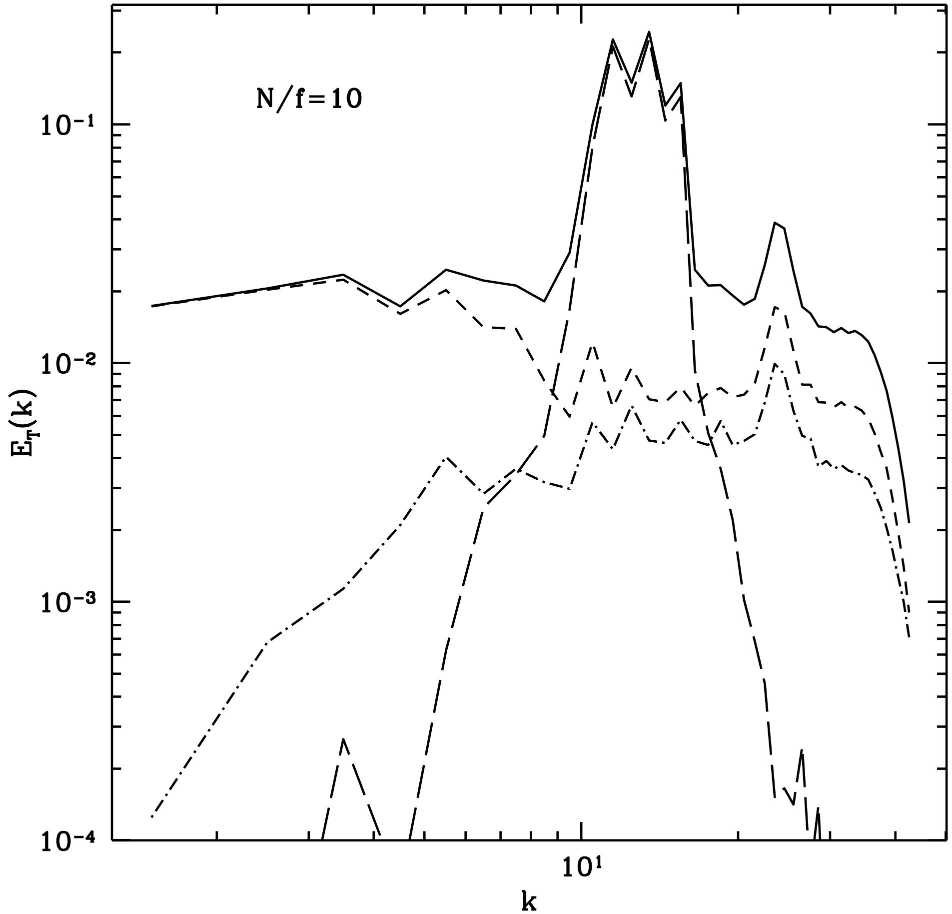
I. Strongly stratified flow with $N/f \gg 2$ and Fr small

- The limiting dynamics for $Fr \to 0$, $Ro = O(1)$ is a system with QG decoupled from VSHF
  (Embid & Majda 1998)

- For intermediate scale forcing:
  there is a slow leakage to large-scale VSHF at $k < k_f$
  energy transfer is predominantly to small scales $k > k_f$.

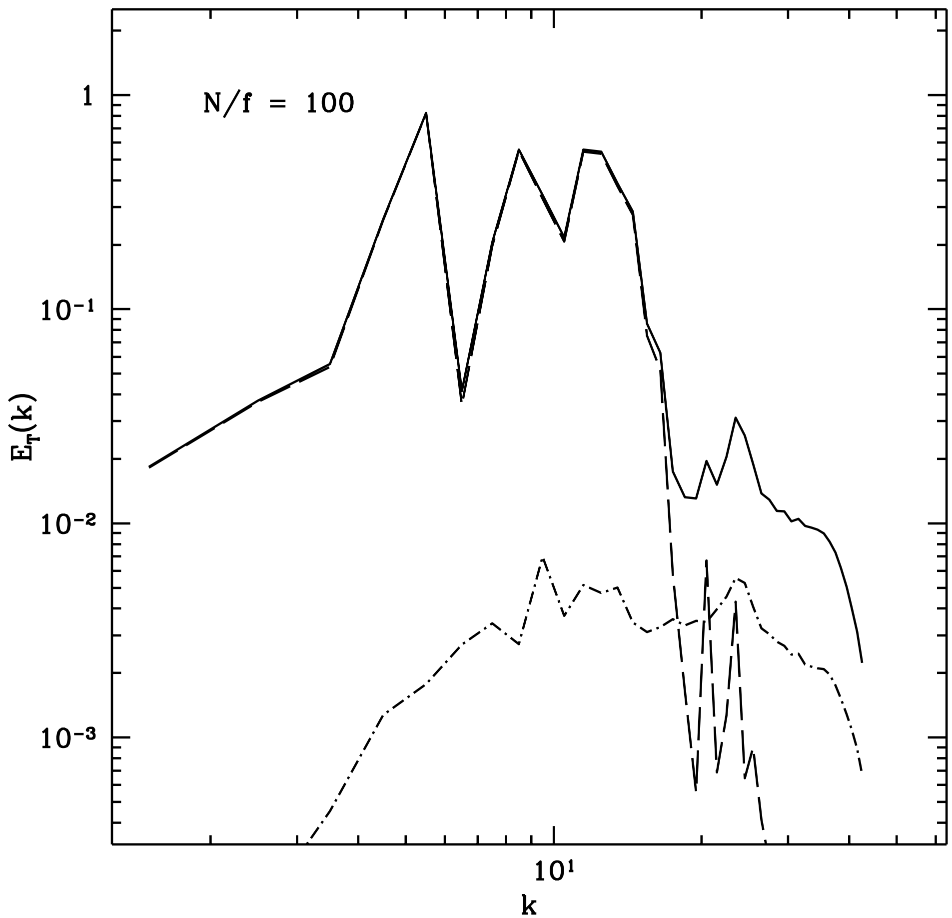
- The scaling of energy spectra for $k > k_f$ may not be universal and appears to depend on type of forcing, aspect ratio, etc.
  Lindborg et al. (2007), Waite & Bartello (2005)

N/f=10, Fr = 0.2; VSHF compete with PV modes



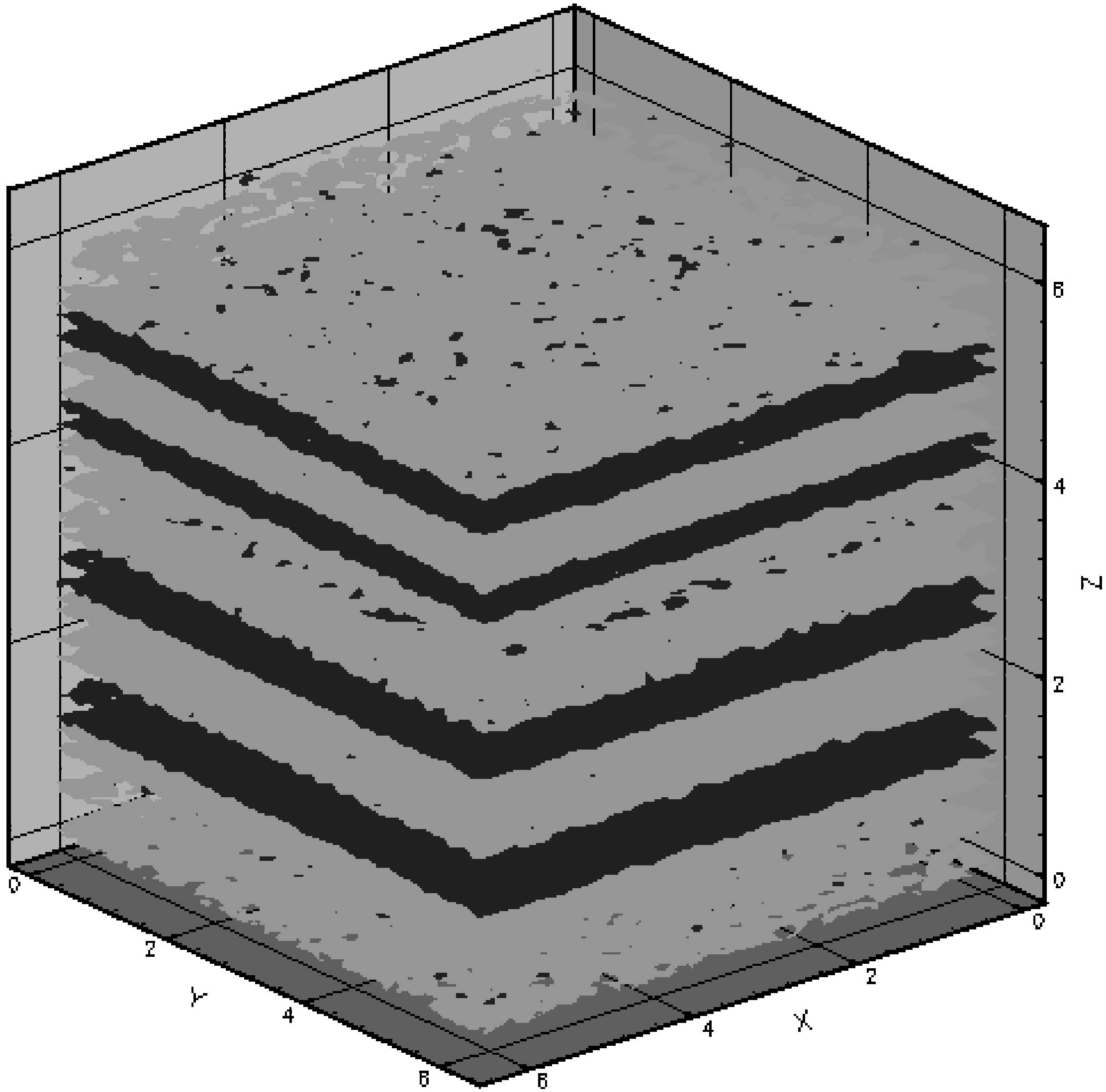
Solid: $E_T(k)$; Long dash: VSHF; Dash: PV; Dot-dash: $P(k)$

N/f=100, Fr = 0.2; VSHF dominate at large scales

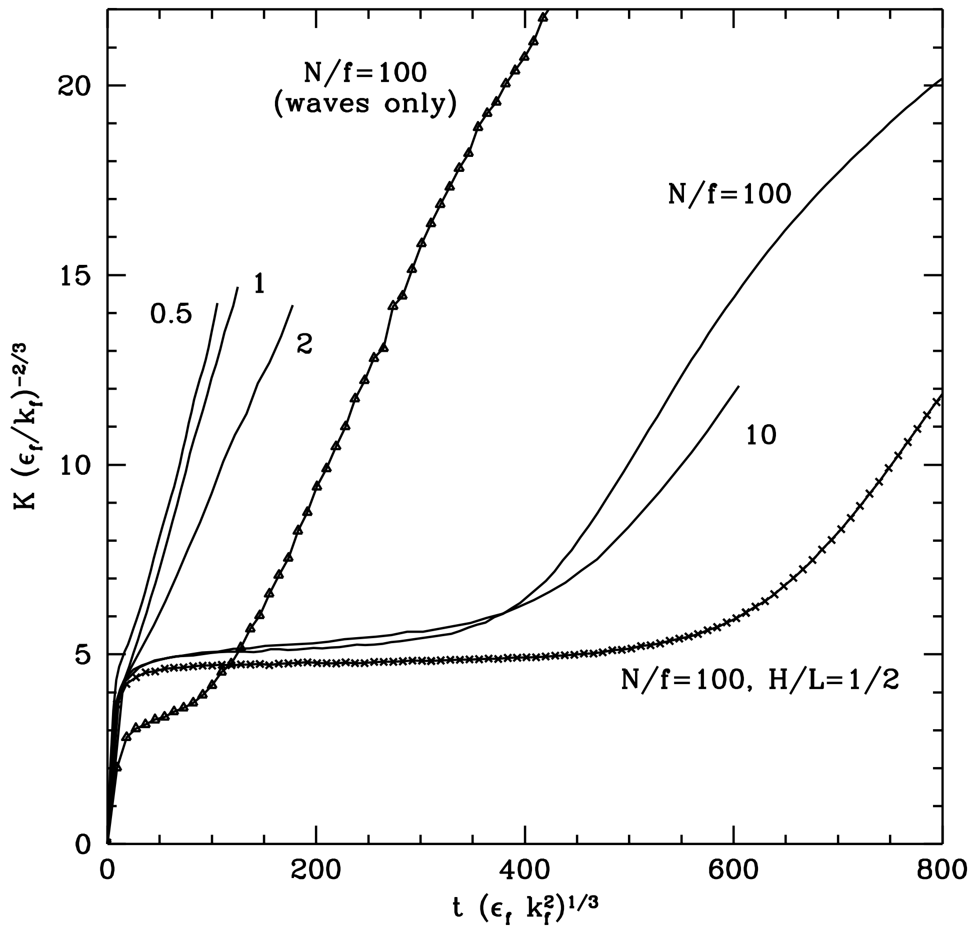


Solid: $E_T(k)$; Long dash: VSHF; Dot-dash: $P(k)$

Contours of zonal velocity for $N/f = 100$, $Fr = 0.2$



Kinetic energy vs. time, Fr = 0.2 and varying N/f values



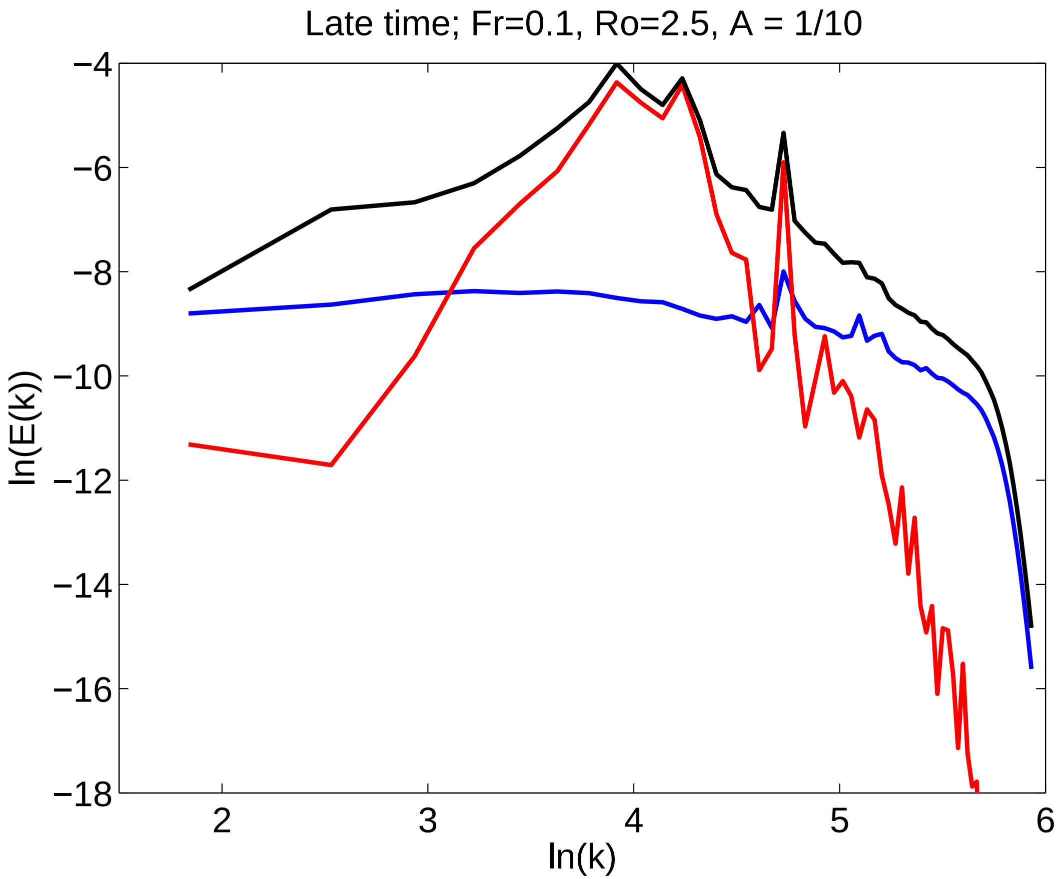
Aspect Ratio Effects

Question: Does small aspect ratio make the flow more QG-like?

Answer: YES and NO

For small aspect ratio, the growth of the VSHF is delayed but not arrested (resolutions issues not entirely settled)

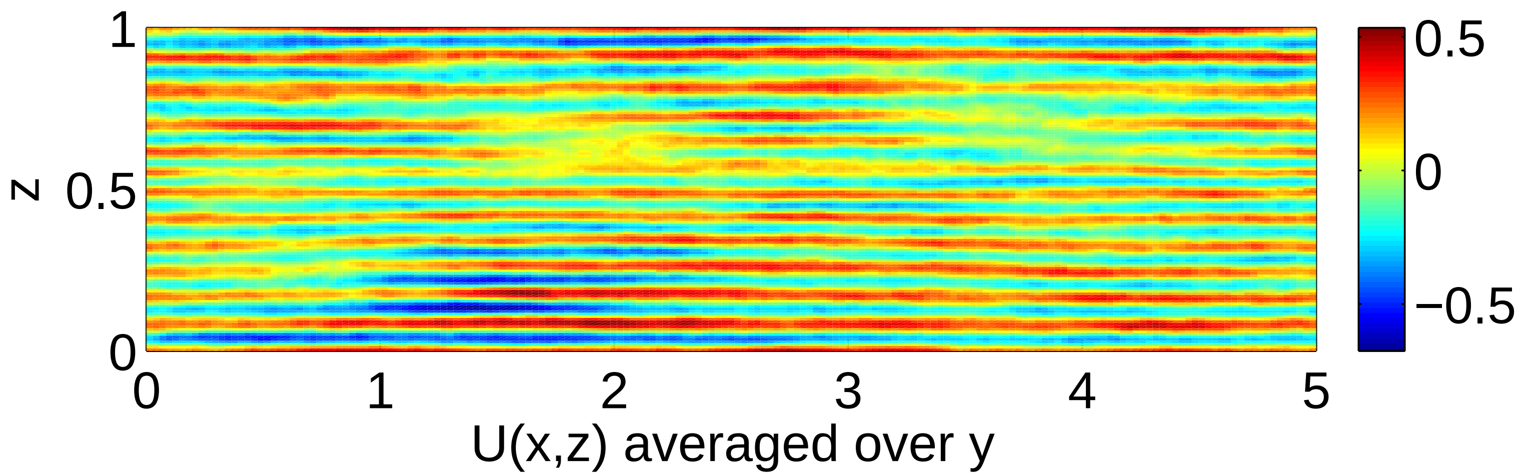
Aspect Ratio 1/10

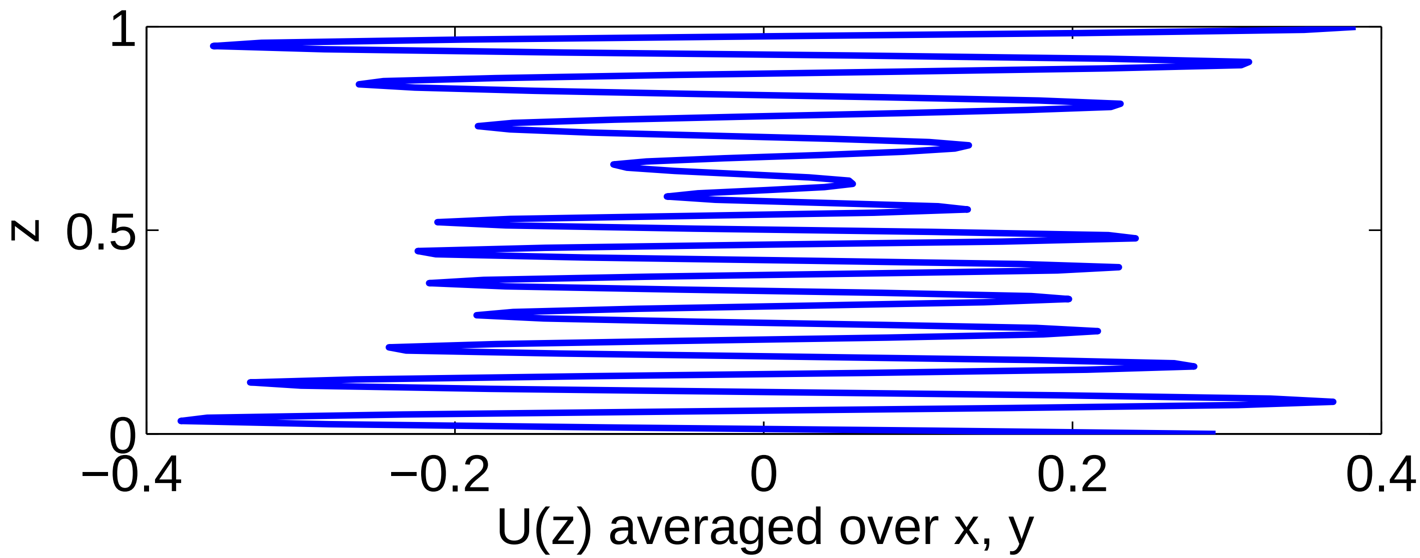


Black: Total; Blue: PV; Red: VSHF

Generation of VSHF; Bu = fL/(NH)=1; H/L=1/5

Ro = U/(fL) = 0.1; Fr = U/(NH) = 0.1



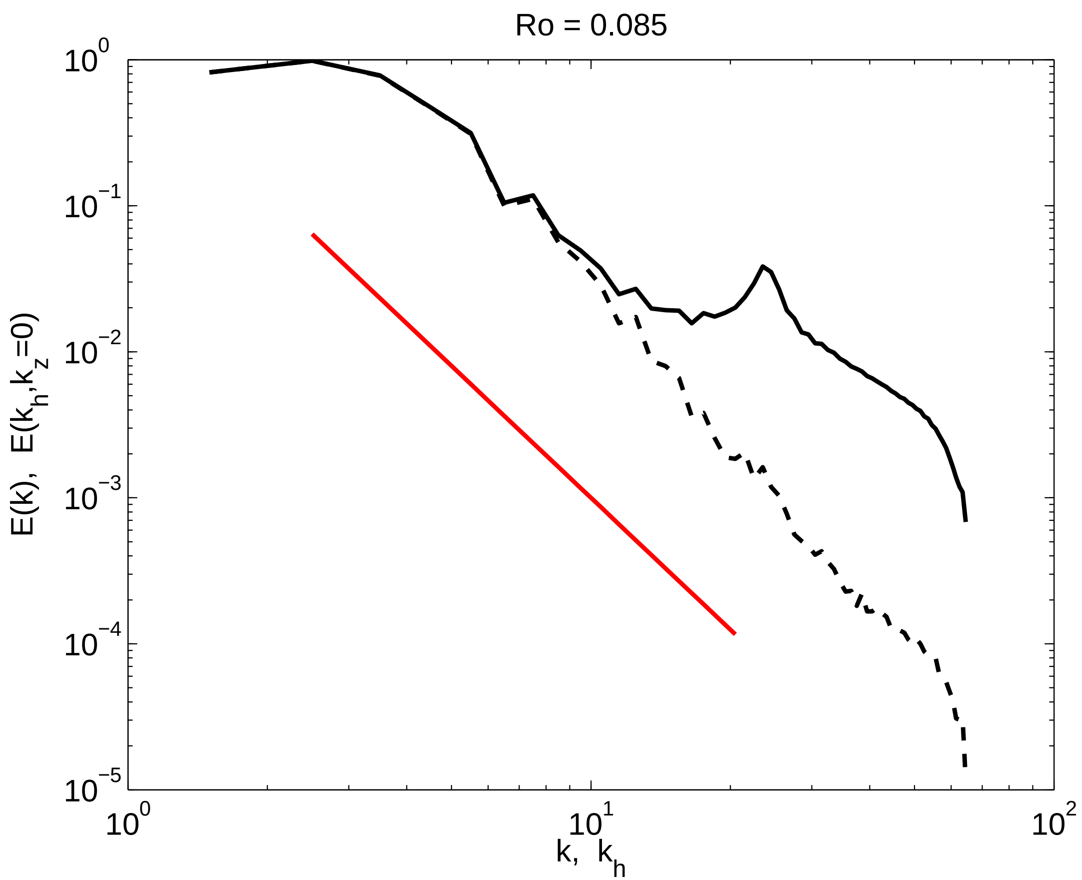


Rotation dominated flow with $f \gg N$, Ro small

Speculation: The PV modes quickly become irrelevant for $f/N \gg 2$ since the PV mode vanishes for purely rotating flow

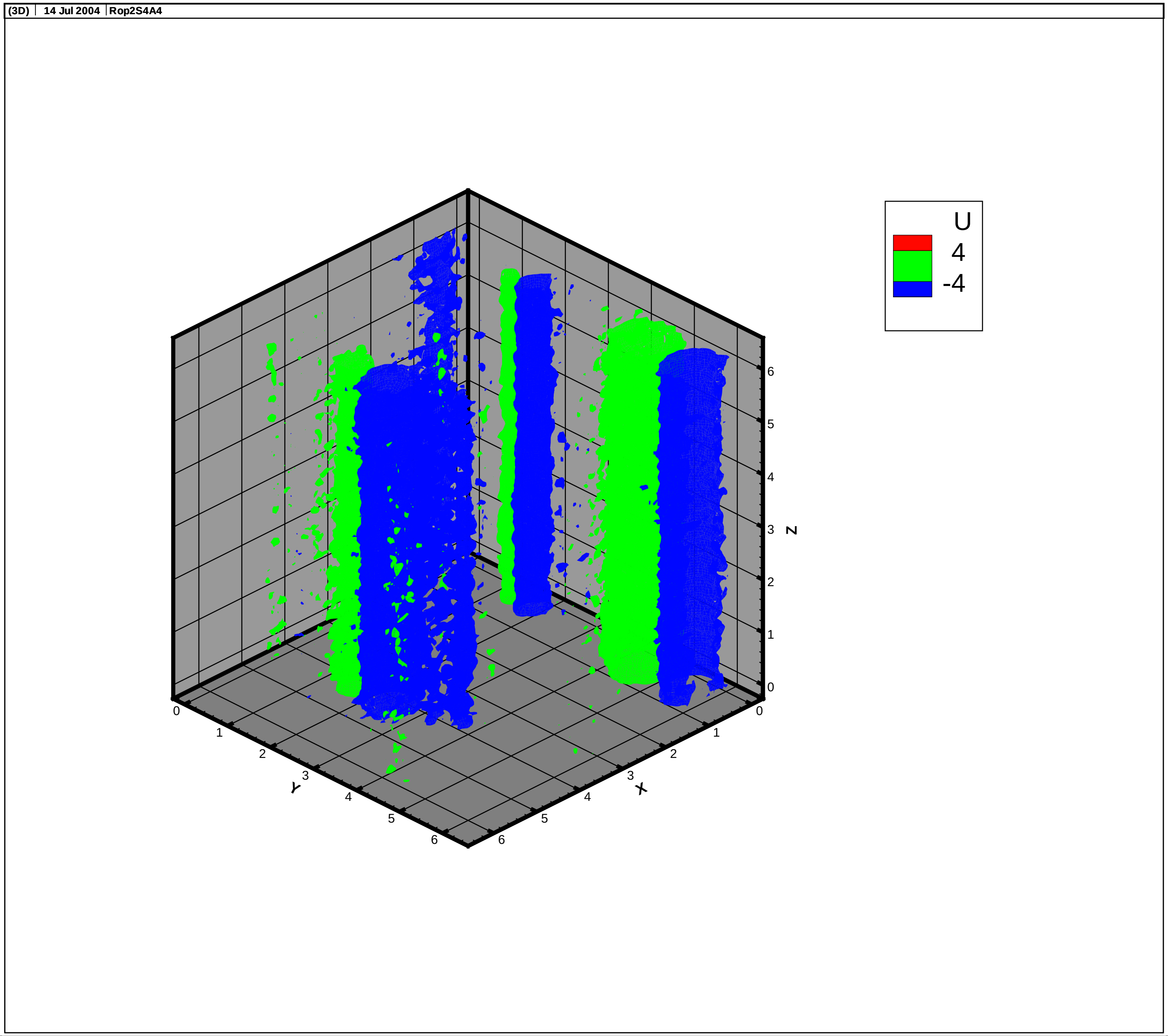
Then cyclonic vortical columns will dominate (as in purely rotating flow) as long as Ro small enough (smaller than about 0.2)

A 128^3 simulation of purely rotating flow with $Ro = 0.085$

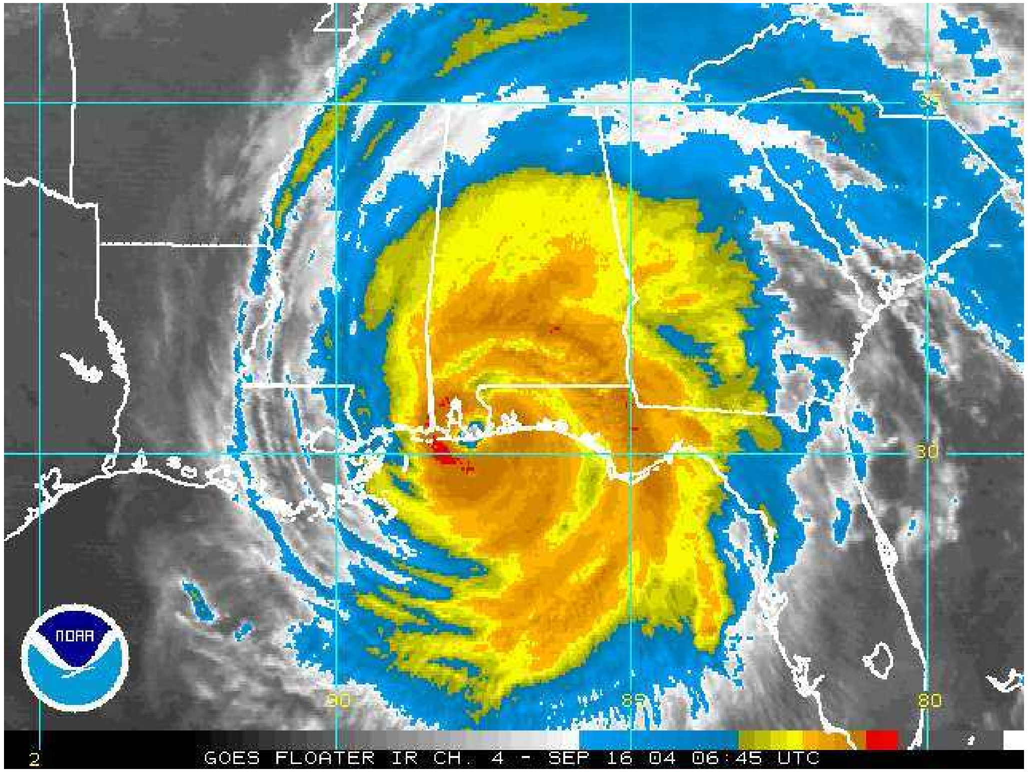


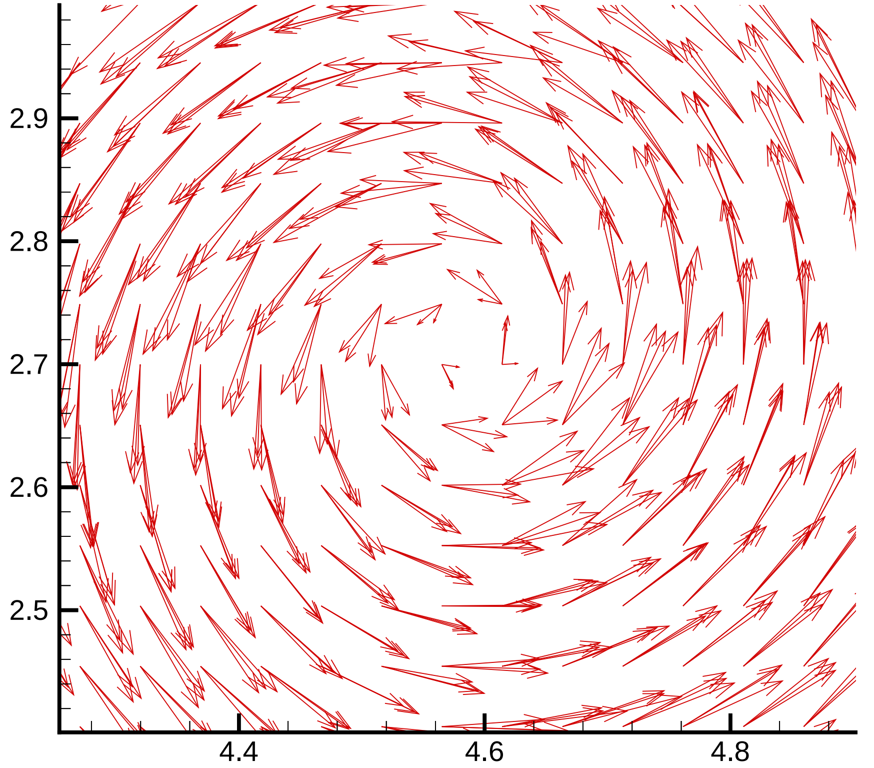
Solid: $E(k)$; Dashed: $E(k_h, k_z = 0)$; Red line: k^{-3}

Cyclonic vortices in the 128^3 simulation



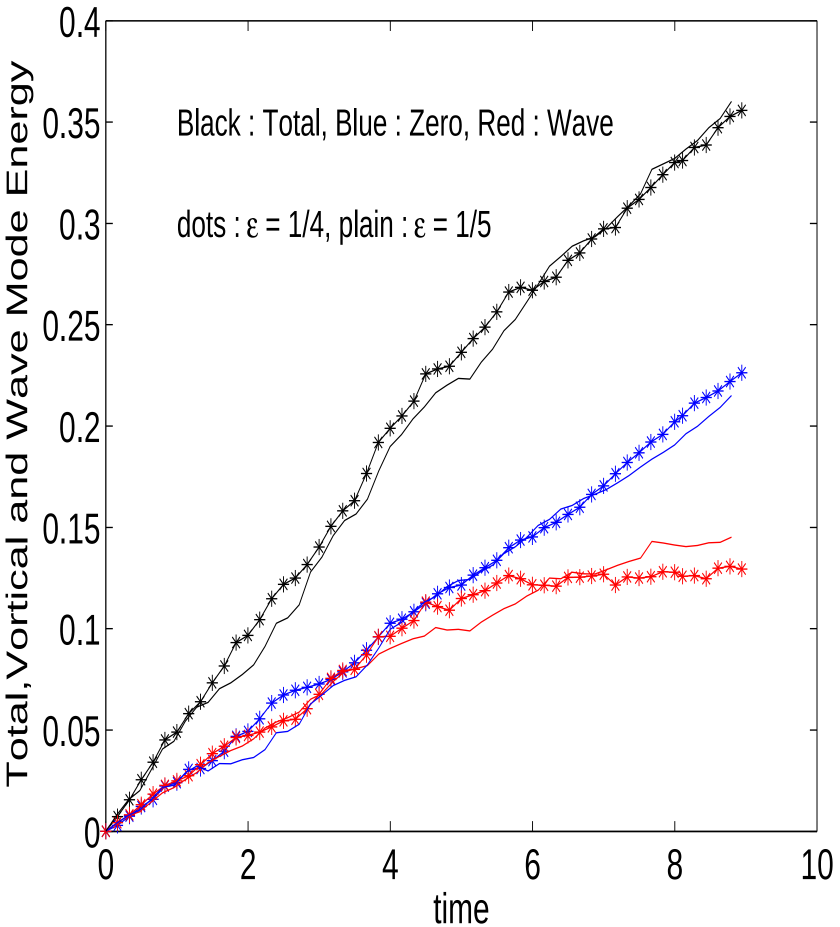
Just for fun





Left: Hurrican Ivan. Right: Velocity vectors in 3 planes from a 128^3 simulation with random forcing at small scales. In both cases, large-scale, cyclonic vortices are fueled by smaller-scale fluctuations.

Large-scale forcing: asymmetry with respect to $f/N = 1$



f/N = 1/4, 1/5

f/N = 4, 5

Bartello 1995, Sukhatme & Smith 2008

Part III. PDE Intermediate Models

Intermediate models add more physics to QG

Improving upon 2DQG: Allen, Barth & Newberger 1990; Spall & McWilliams 1992; Yavneh & McWilliams 1994; Warn, Bokhove, Shepherd & Vallis 1995; Vallis 1996

Improving upon 3DQG: Allen 1991, 1993; Muraki, Snyder & Rotunno 1999; Muraki & Hakim 2001

Many previous intermediate models are perturbative in nature with small parameter $\epsilon = Ro$

A hierarchy of Intermediate Models

- Derived by adding subsets of wave-vortical mode interactions to QG
- Non-perturbative
- Include near-resonant triads
- Provides a framework for understanding the coupling between balanced and unbalanced flow components

  e.g. Kuo, Allen, Polvani 99; Ford, McIntyre, Norton 00; Majda 03

Symbolically

The Full Equations:

$$0 \quad | \; 00 \;\oplus\; 0+ \;\oplus\; 0- \;\oplus\; ++ \;\oplus\; +- \;\oplus\; -- \tag{1}$$

$$+ \quad | \; 00 \;\oplus\; 0+ \;\oplus\; 0- \;\oplus\; ++ \;\oplus\; +- \oplus\; -- \tag{2}$$

$$- \quad | \; 00 \;\oplus\; 0+ \;\oplus\; 0- \;\oplus\; ++ \;\oplus\; +- \oplus\; -- \tag{3}$$

QG (vortical mode interactions only):

$$0 \quad |00$$

Two NEW Models

PPG (add to QG interactions involving exactly 1 wave):

$$0 \quad | \; 00 \;\oplus\; 0+ \;\oplus\; 0- \qquad (ppg1)$$

$$+ \quad | \; 00 \qquad \text{and} \qquad - \quad | \; 00 \qquad (ppg2,3)$$

P2G (add to PPG interactions involving exactly 2 waves):

$$0 \quad | \; 00 \;\oplus\; 0+ \;\oplus\; 0- \;\oplus\; ++ \;\oplus\; +- \;\oplus\; -- \qquad (p2g1)$$

$$+ \quad | \; 00 \;\oplus\; 0+ \;\oplus\; 0- \qquad (p2g2)$$

$$- \quad | \; 00 \;\oplus\; 0+ \;\oplus\; 0- \qquad (p2g3)$$

Illustration using the Rotating Shallow Water Equations

$$\frac{D\mathbf{u}_H}{Dt} + f\hat{\mathbf{z}} \times \mathbf{u}_H = -g\boldsymbol{\nabla} h$$

$$\frac{Dh}{Dt} + (H + h)(\boldsymbol{\nabla} \cdot \mathbf{u}_H) = 0$$

$$\mathbf{u}_H = u\hat{\mathbf{x}} + v\hat{\mathbf{y}}$$

$$f = 2\Omega, \qquad Ro = \frac{U}{fL}, \qquad Fr = \frac{U}{(gH)^{1/2}}$$

Analogous eigenmode representation for nonlinear flows

Linear eigenmodes $\boldsymbol{\phi}_s(\mathbf{k})$, $s = \pm, 0$ form an orthogonal basis

$$[\mathbf{u}_H, h]^T(\mathbf{x}, t) = \sum_{\mathbf{k}} \sum_s b_s(t;\mathbf{k}) \boldsymbol{\phi}_s(\mathbf{k}) \exp\left[i \left(\mathbf{k} \cdot \mathbf{x} - \sigma_s(\mathbf{k}) t \right) \right]$$

and the equations become

$$\frac{\partial}{\partial t} b_{s_\mathbf{k}} = \sum_{\triangle} \sum_{s_\mathbf{p}, s_\mathbf{q}} C_{\mathbf{kpq}}^{s_\mathbf{k} s_\mathbf{p} s_\mathbf{q}} \; b_{s_\mathbf{p}}^* \; b_{s_\mathbf{q}}^* \exp\left[i \left(\sigma_{s_\mathbf{k}} + \sigma_{s_\mathbf{p}} + \sigma_{s_\mathbf{q}} \right) t \right]$$

$$\sigma^0 = 0, \quad \sigma^\pm = \pm\sqrt{f^2 + c^2 k^2)}, \quad c = \sqrt{gH}$$

Find linear modes from skew Hermitian form of equations

QG and PPG Rotating Shallow Water (RSW) Equations

QG: $\partial Q/\partial t + J(\Psi, Q) = 0$

$$\text{PPG}: \qquad \frac{\partial \nabla^2 \chi}{\partial t} - \nabla^2 V = 2J\left(\frac{\partial A}{\partial x}, \frac{\partial A}{\partial y}\right), \quad (1)$$

$$\frac{\partial Q}{\partial t} + J(\Psi, Q) + \nabla\chi \cdot \nabla Q + \langle u \rangle \frac{\partial Q}{\partial x} + \langle v \rangle \frac{\partial Q}{\partial y} + Q\nabla^2\chi = 0, \quad (2)$$

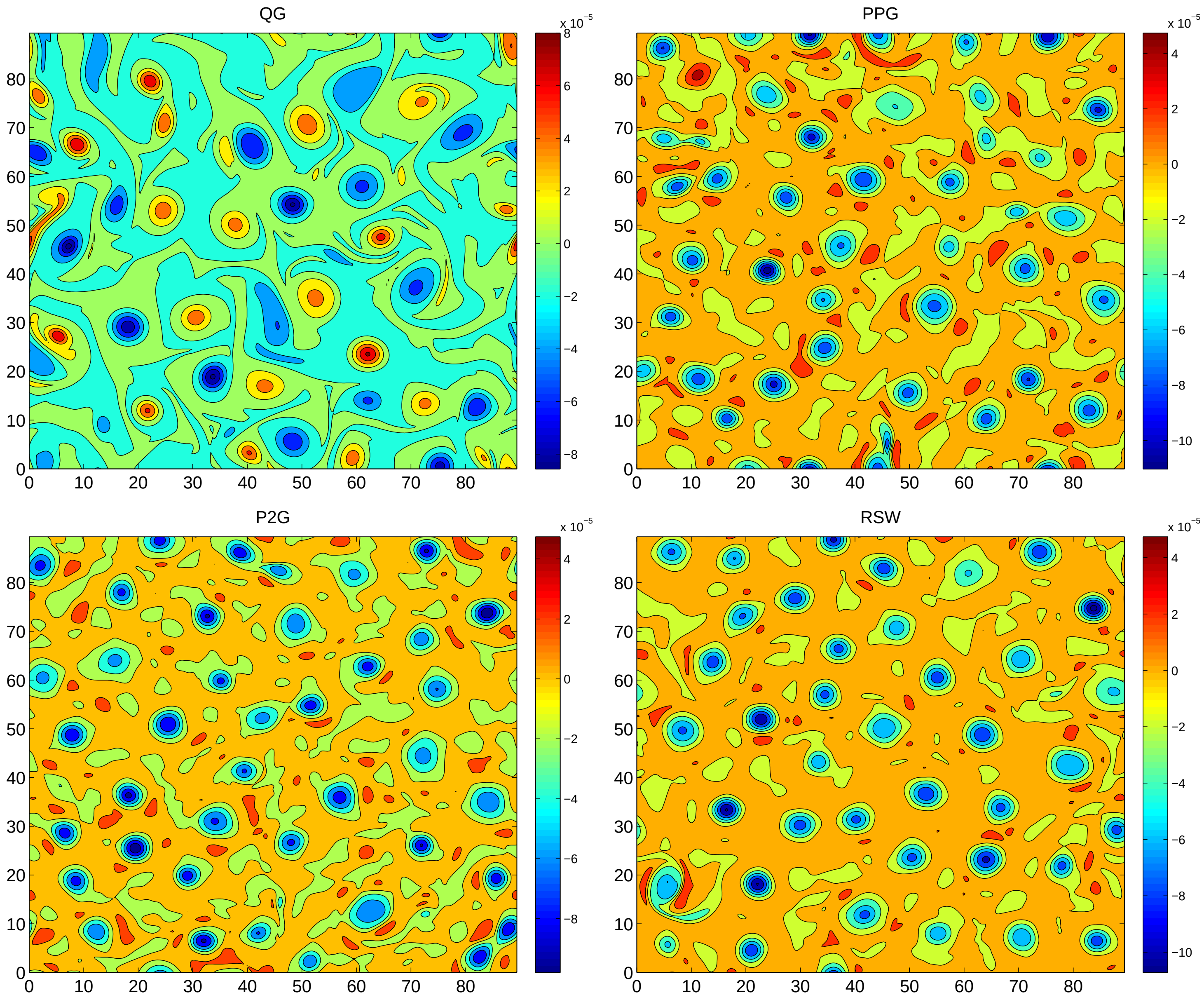
$$\frac{\partial \nabla^2 V}{\partial t} - c^2\nabla^4\chi + f^2\nabla^2\chi = fJ(A, Q) \quad (3)$$

$Q = (\nabla^2 - \frac{f^2}{gH})\Psi, \quad u = \chi_x - \Psi_y, \quad v = \chi_y + \Psi_y, \quad \nabla^2\chi = u_x + v_y$

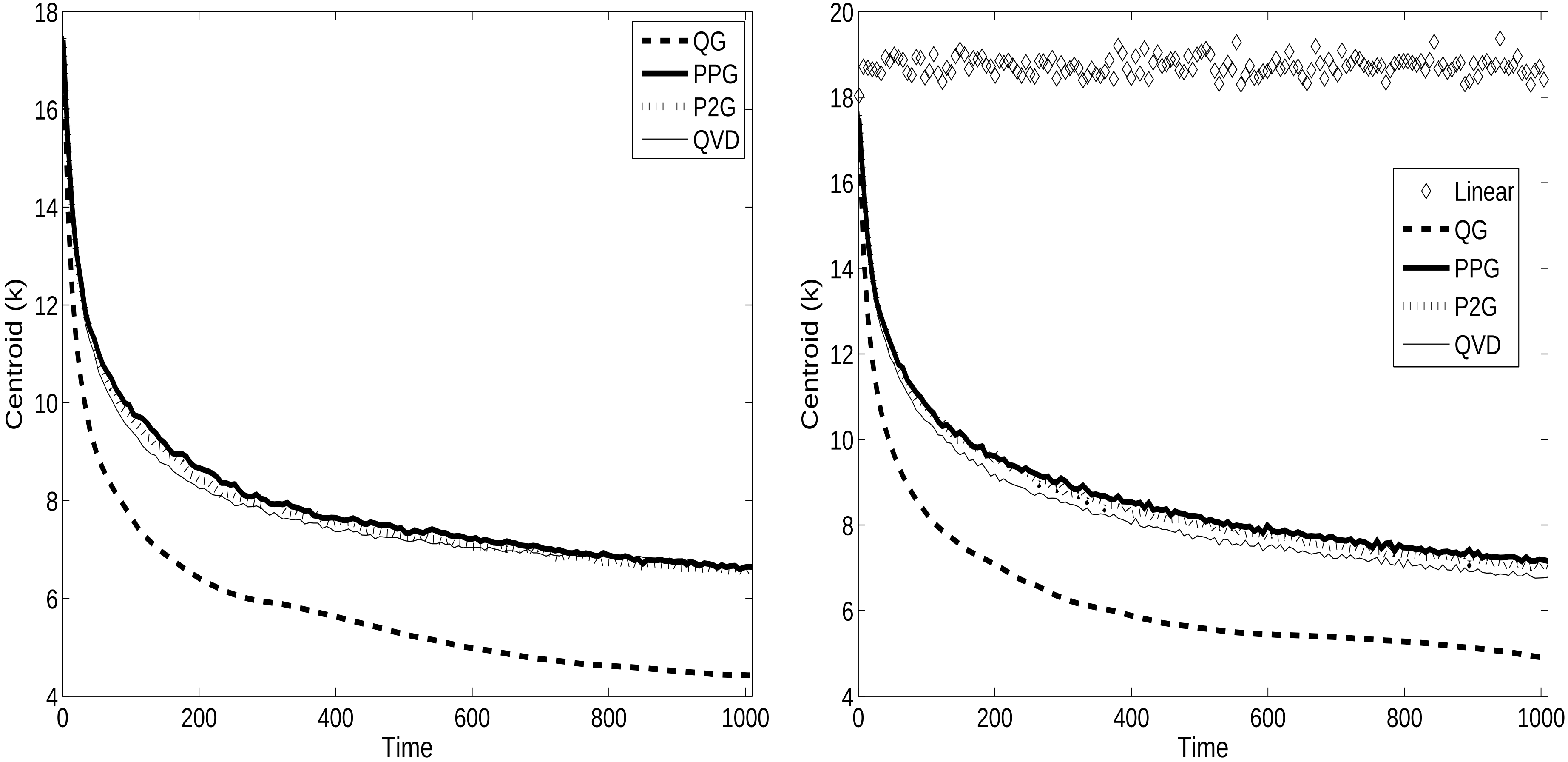
$\nabla^2 V = \nabla^2(f\Psi - gh)$ is a measure of geostrophic imbalance (Vallis 96); also called geostrophic departure (Warn 95); ageostrophic vorticity (Mohebalhojeh & Dritschel 01)

$A \equiv (f^2 - c^2\nabla^2)^{-1}c^2 Q$

RSW decay, Ro=0.4, Fr = 0.25, divergence-free unbalanced i.c.



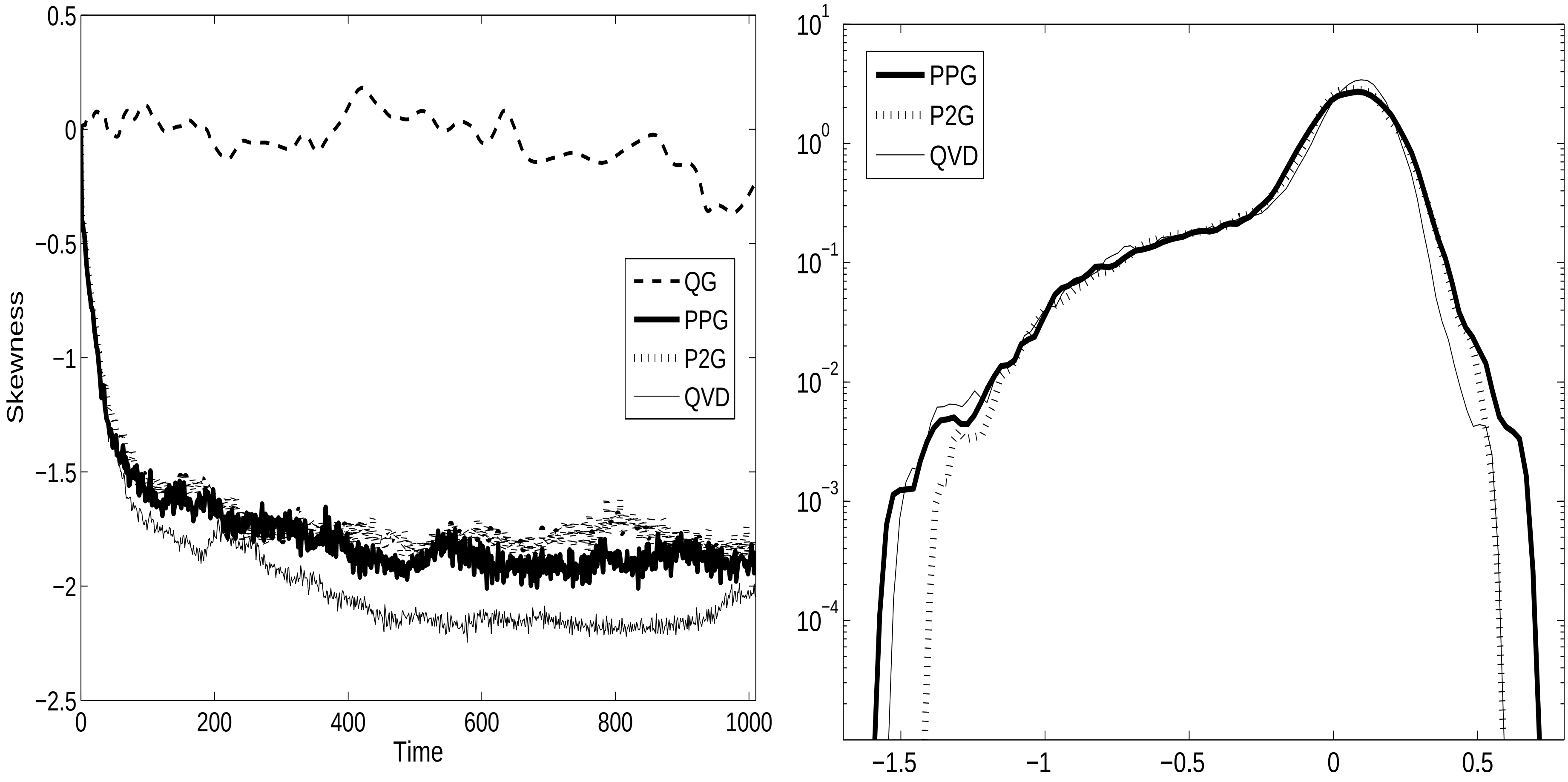
Centroid in RSW decay; divergence-free unbalanced i.c.



Ro = 0.4, Fr = 0.25 Ro = 0.25, Fr = 0.2

$$\text{Cent}(k) = (\sum_{\mathbf{k}} k(|u_{\mathbf{k}}|^2 + |v_{\mathbf{k}}|^2) / \sum_{\mathbf{k}}(|u_{\mathbf{k}}|^2 + |v_{\mathbf{k}}|^2)$$

Vorticity skewness in RSW decay; divergence-free unbalanced i.c.



Ro = 0.4, Fr = 0.25

Conclusions

There exists a hierarchy of reduced models "in between" QG and Boussinesq involving wave-vortical interactions

Wave-vortical interactions lead to non-QG behaviors away from the 3DQG regime $Fr \approx Ro \to 0$

A path to understand wave-vortical interactions :

- Restrict the wave-space sum to include any subset of different interactions
- Inverse transform to derive a PDE in physical space which can be used with any physical bcs
- Use numerical simulations to compare the reduced PDE to the full equations