# A large system of bubbles in Stokes flow

Thorwald Stein
Mathematics Institute
University of Warwick

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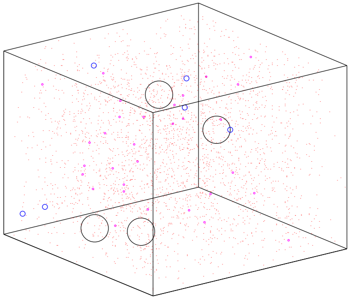
# Typical "Turbulence" Setup

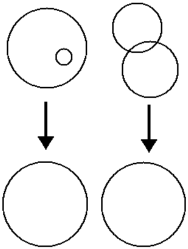
- 3D Periodic box.
- Forcing particles of size $m_0$.
- Particles move in $z$-direction at terminal velocity.
- No interaction with fluid (Stokes flow)
- Removing particles of size $M \gg m_0$.



# Collision Rules

- Spherical particles.
- Particles can cross trajectories.

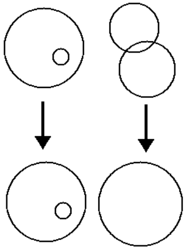
1. *Free merging*: Any two particles merge when they touch.



# Collision Rules

- Spherical particles.
- Particles can cross trajectories.

1. *Free merging*: Any two particles merge when they touch.
2. *Forced locality*: Particles only merge if they are of similar size.

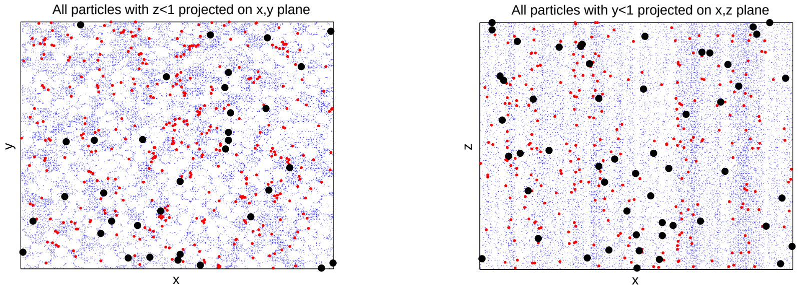


# Computational Setup

At any given time step, the following events occur:

- All particles in the system move up according to their terminal velocity.
- New particles of size $m_0$ and $2m_0$ are created at random positions in the domain.
- All particles are checked for intersections with others, and merged according to the collision rules.
- Merged particles that are now larger than $M$ are removed from the system.

# Long Time Behaviour




Black dots are particles larger than $1000m_0$; blue dots are smaller than $10m_0$; red dots have $10m_0 < m < 1000m_0$.

# Smoluchowski Coagulation Equation

$$\partial_t n_m + u\partial_z n_m = \int_0^\infty \int_0^\infty R_{12}^m - R_{2m}^1 - R_{m1}^2 \, dm_1 dm_2$$

$$R_{12}^m = \frac{1}{2} K_{12} n_1 n_2 \delta(m - m_1 - m_2)$$

$$\begin{aligned} K_{12} &= C_{eff}(m_1, m_2)|u_1 - u_2|(r_1 + r_2)^2 \\ &= C_{eff}(m_1, m_2) G |m_1^{2/3} - m_2^{2/3}|(m_1^{1/3} + m_2^{1/3})^2 \end{aligned}$$

- Collision efficiency $C_{eff}(m_1, m_2) \equiv 1$ for free merging
- Constant $G$ involving density, viscosity, and gravity

# Scaling Solution

- Smoluchowski: Assume locality of interactions
- Assume $n \sim m^{\nu}$
- Note $K \sim m^{4/3}$ for $m_1 \approx m_2$

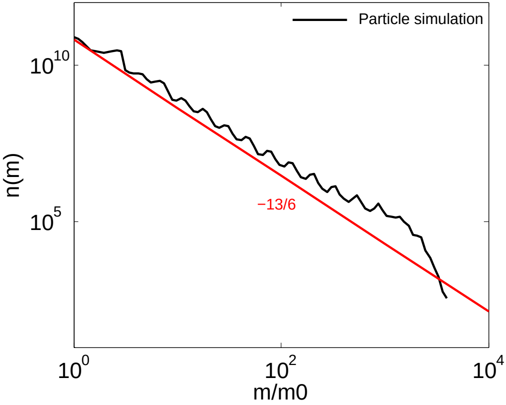
$$\int_m \frac{dn}{dt} m' dm' \sim m^2 \times \left(m^2 K n^2 m^{-1}\right)$$

$$\Rightarrow \int_m \frac{dn}{dt} m' dm' \sim m^{2\nu + 13/3}$$

$$\Rightarrow n \sim m^{-13/6}$$

# Scaling Solution

- DNS: Use $C_{eff} = 1$ for $\frac{1}{2} < \frac{m_1}{m_2} < 2$, 0 otherwise



# Nonlocal Solution

- Rewrite Smoluchowski Coagulation Equation as follows:

$$\frac{dn(m)}{dt} = \int_{m_0}^{m/2} K(m', m-m')n(m')n(m-m')dm' - \int_{m_0}^{M} K(m', m)n(m')n(m)dm'$$

- Assume nonlocal interactions dominate, then contribution near $m_0$:

$$\int_{m_0} K(m', m-m')n(m')n(m-m') - K(m', m)n(m')n(m)dm' \approx -\int_{m_0} n(m')\partial_m \left(m^{4/3}n(m)\right) dm' = -c_1\partial_m \left(m^{4/3}n(m)\right)$$

# Nonlocal Solution

- Then contribution near $M$:

$$-\int^{M} K(m', m)n(m')n(m)dm' \approx -n(m)\int^{M} m'^{4/3} n(m')dm'$$
$$= -c_2 n(m)$$

- Kinetic equation becomes an ODE:

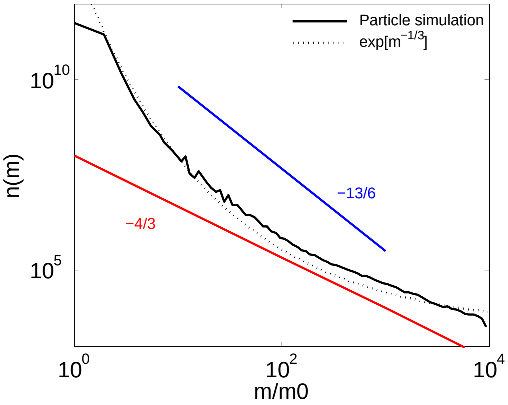
$$\frac{dn}{dt} \approx -c_1 \partial_m \left(m^{4/3} n\right) - c_2 n \qquad (1)$$

- With solution:

$$n(m) \sim m^{-4/3} \exp\left(c' m^{-1/3}\right) \qquad (2)$$

# Nonlocal Solution

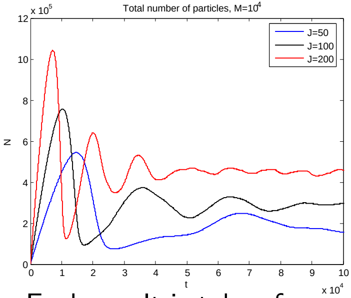
- DNS: Use $C_{eff} \equiv 1$

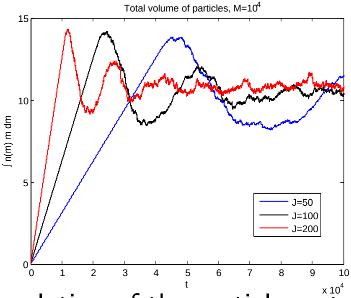


# Gelation

- Recall kernel $K_{12} \sim |m_1^{2/3} - m_2^{2/3}|(m_1^{1/3} + m_2^{1/3})^2$
- Homogeneity degree: $K(\zeta m_1, \zeta m_2) = \zeta^{4/3} K_{12}$
- Traditional system undergoes *instantaneous gelation*: There exists a finite time singularity with gel time $t_c = 0$
- Particle system with finite source and sink hints at gelation at $t_c > 0$

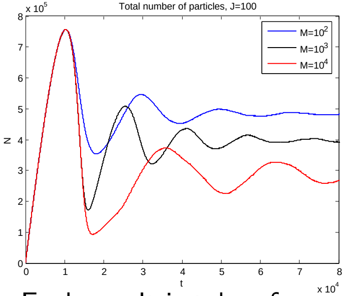
# Changing Flux of $m_0$

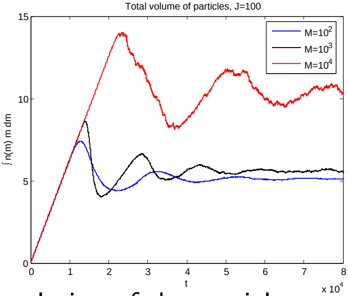



Each result is taken from a single simulation of the particle system.

# Changing Cutoff at $M$

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# An Interesting Particle System

- Surprising agreement with Smoluchowski Coagulation Equation and Mean Field numerics
- No clear indication of “instantaneous” gelation
- Instead, fluctuations in total volume and number of particles
- Fluctuations depend on flux at the source ($J$) and large particle cutoff $M$