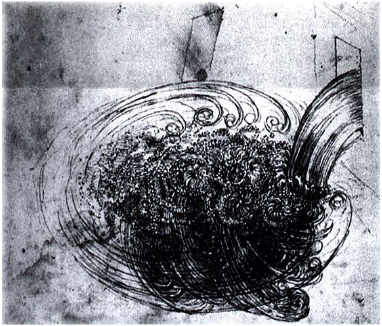
What is Turbulence?

Fabian Waleffe

Depts of Mathematics and Engineering Physics
University of Wisconsin, Madison

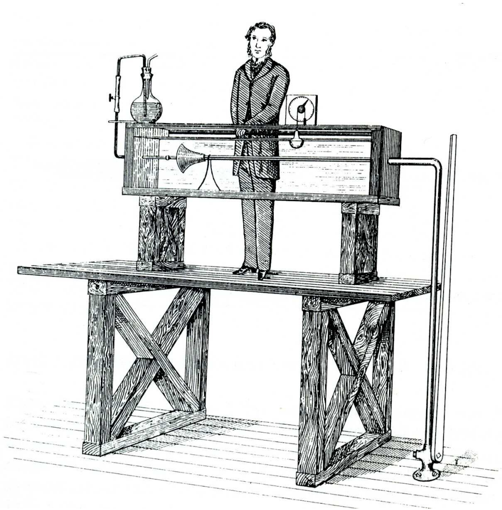
it's all around, . . . and inside us!

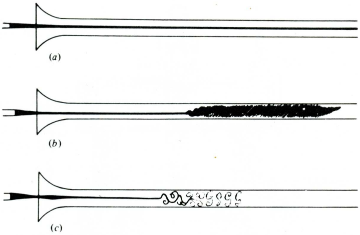


Leonardo da Vinci (c. 1500)

- River flow, pipe flow, flow from a faucet, . . .
- Clouds, smoke, . . .
- Wakes behind boats, golf balls, bikers, cars, airplanes . . .
- Nuclear fusion, stars, . . .
- Blood flow in heart and large arteries ($R \approx 20,000$ in aorta!)

Reynolds 1883: turbulence in pipe flow





Navier-Stokes equations

3D velocity field $\mathbf{v}(\mathbf{x}, t)$:

$$\boldsymbol{\nabla} \cdot \mathbf{v} = 0$$

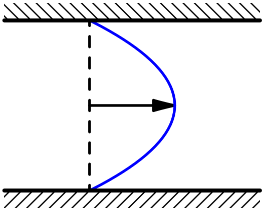
$$\partial_t \mathbf{v} + \boldsymbol{\nabla} \cdot (\mathbf{v}\mathbf{v}) + \boldsymbol{\nabla} p = \frac{1}{R} \nabla^2 \mathbf{v}$$

Control parameter: *Reynolds number*

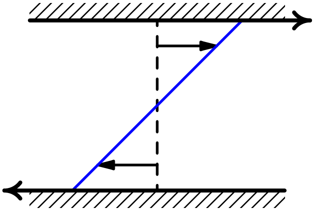
$$R \equiv \text{non-dimensional } \textit{velocity}$$

$$\frac{1}{R} \equiv \text{non-dimensional } \textit{viscosity}$$

Simple geometries, simple flow?



Pipes, Channels



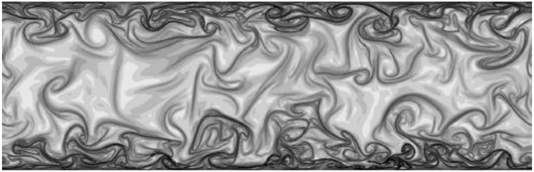
plane Couette

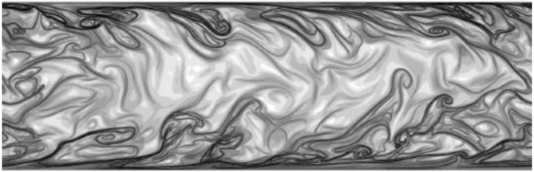
Laminar solutions: $\mathbf{v} = (1 - y^2)\hat{\mathbf{x}}$ $\mathbf{v} = y\hat{\mathbf{x}}$

Linearly stable yet: $R \lesssim 2,000$ $R \lesssim 350$

Turbulence in channel flow

Front





Side

Top

Green, M. A., Rowley, C. W. & Haller, G.
Detection of Lagrangian coherent structures in three-dimensional turbulence,
J. Fluid Mech., **572**, 2007, 111-120.

Classical statistical approach to Turbulence

$\mathbf{v}(\mathbf{x}, t)$ 'random' $\Rightarrow$ *ensemble average* $\langle \mathbf{v} \rangle$

$$\boldsymbol{\nabla} \cdot \langle \mathbf{v} \rangle = 0$$

$$\partial_t \langle \mathbf{v} \rangle + \boldsymbol{\nabla} \cdot (\langle \mathbf{v} \rangle \langle \mathbf{v} \rangle) + \boldsymbol{\nabla} \langle p \rangle = \frac{1}{R} \nabla^2 \langle \mathbf{v} \rangle - \boldsymbol{\nabla} \cdot \langle \mathbf{v}\mathbf{v} \rangle$$

'Closure problem:' *Reynolds stress* $\langle \mathbf{v}\mathbf{v} \rangle = ?!$

$\langle \mathbf{vv} \rangle$ modeling

- [Prandtl-von Karman]
  Turbulence = collisions of eddies?

$$\rightarrow \text{eddy viscosity: } \langle \mathbf{vv} \rangle \approx -\nu_T \left(\nabla \langle \mathbf{v} \rangle + \nabla \langle \mathbf{v} \rangle^T \right)$$

ν_T? mixing length, Smagorinsky, ...

$\langle \mathbf{vv} \rangle$ modeling

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- [Richardson-Kolmogorov]
  Turbulence = Cascade of energy from large to small scales?

  scale-similarity, inertial range, return-to-isotropy, ...

  $\rightarrow$ K-Epsilon ν_T, Dynamic model, RANS, LES, ...

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- Walls?! *Ouch!*

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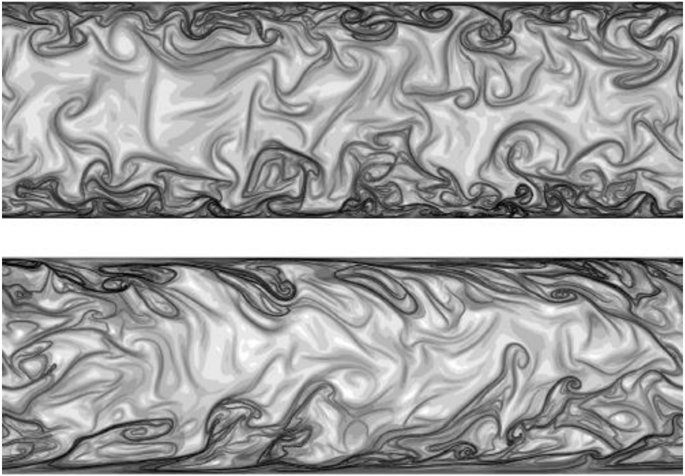
- Walls?! *Ouch!*
- **v** random? *coherent structures!*

Turbulence is controlled by boundary conditions

(with apologies to Kolmogorov and Clay Institute)



What are coherent structures?

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How do they fit with classical models of turbulence?

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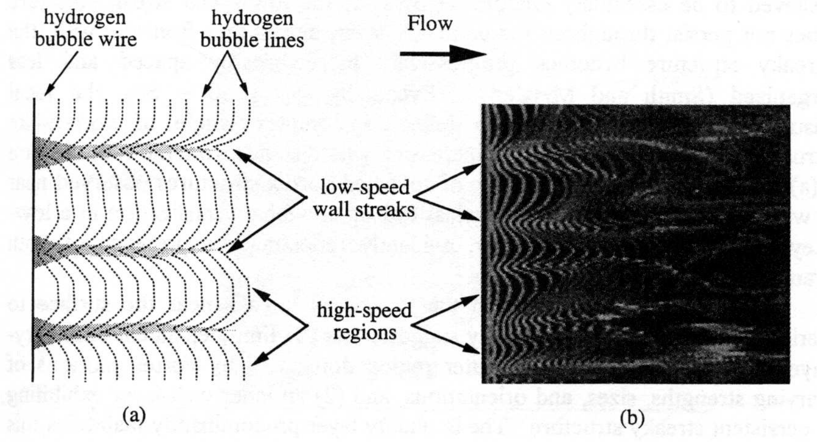
?

About 50 years of *a posteriori*, qualitative studies

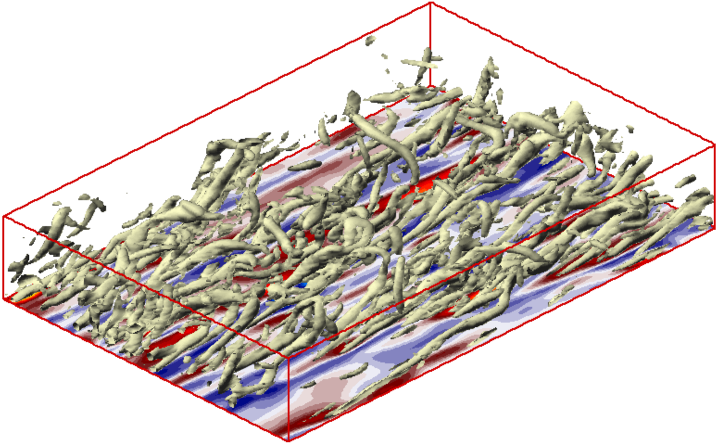
Streaks with 100^{+} z-spacing



Kline, Reynolds, Schraub & Runstadler, JFM 1967

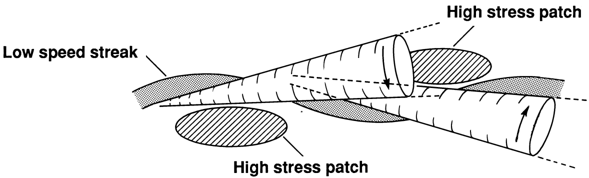
(diagram from Smith & Walker, 1997)

Near wall coherent structures



John Kim, `http://www.turbulence.ucla.edu/`

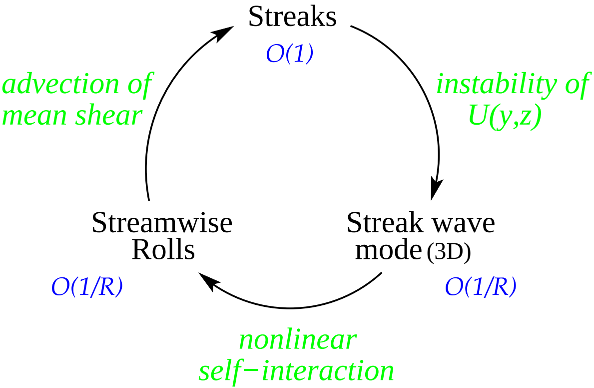
Characteristic near-wall coherent structure



Derek Stretch, CTR Stanford, 1990

Streaks + staggered quasi-streamwise vortices: why?

Self-Sustaining Process (SSP)



Streaks

O(1)

instability of
U(y,z)

Streak wave
mode (3D)

O(1/R)

nonlinear
self−interaction

Streamwise
Rolls

O(1/R)

advection of
mean shear

WKH 1993, HKW 1995, W 1995, 1997

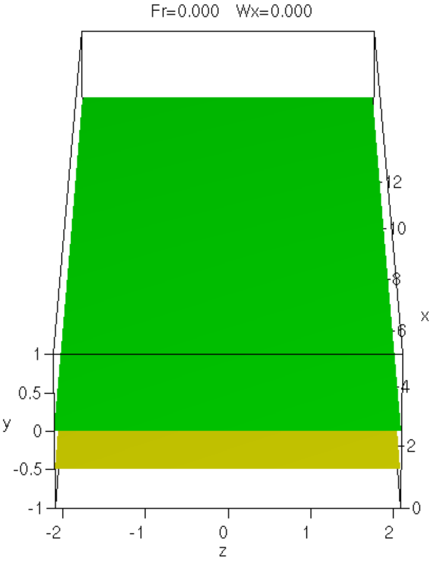
SSP theory ⟶ SSP method

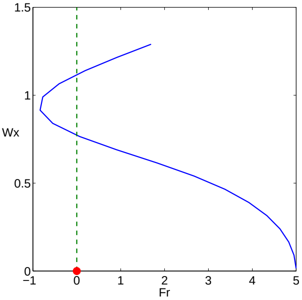
Construction of *Exact Coherent States* from SSP

(‘Full’ NSE, Newton’s method)

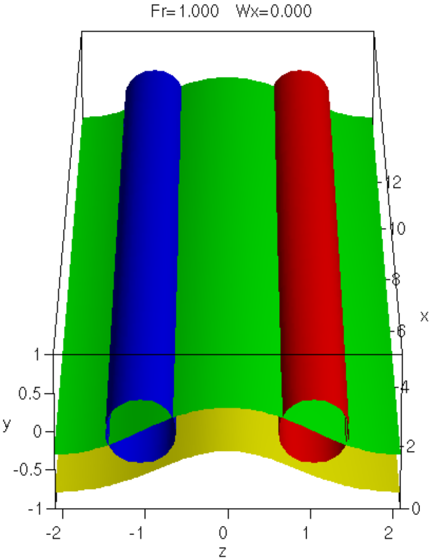
PRL 1998, JFM 2001, PoF 2003

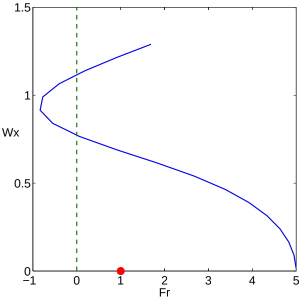
Laminar Couette flow: u=0 & u=-0.5





SSP: Streamwise Rolls create Streaks





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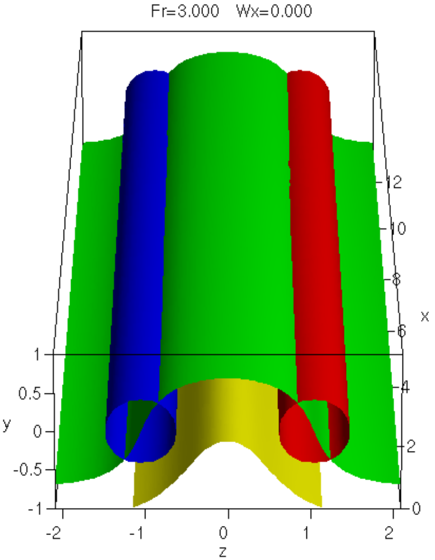

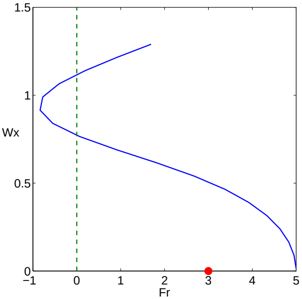

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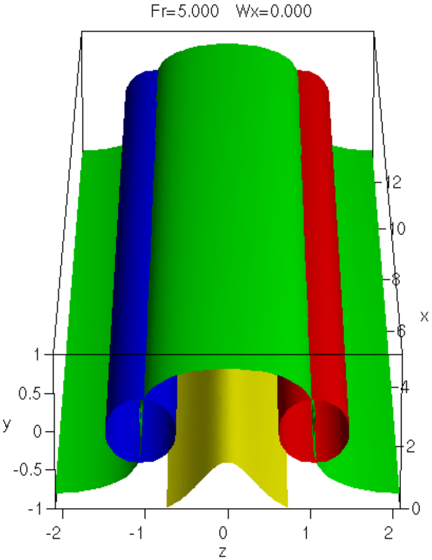
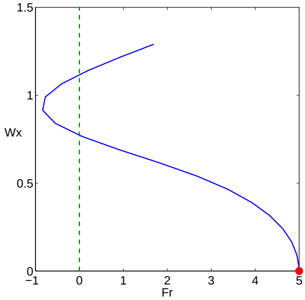
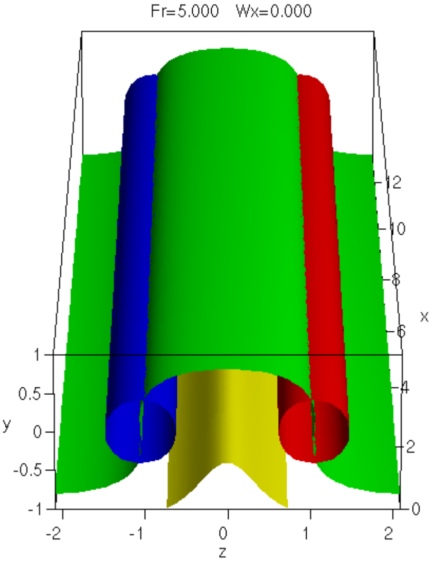
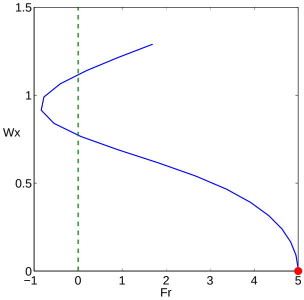




SSP in action: Subcritical Bifurcation from Streaks





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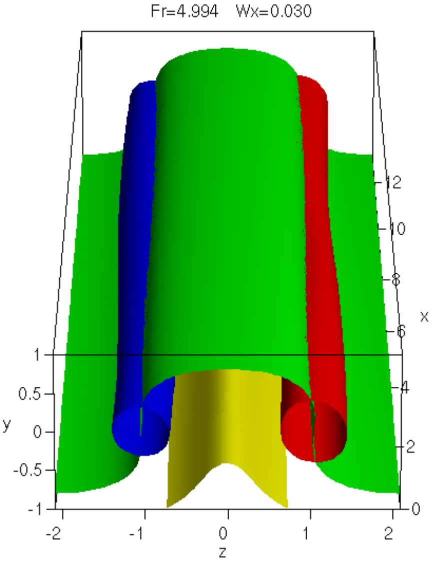

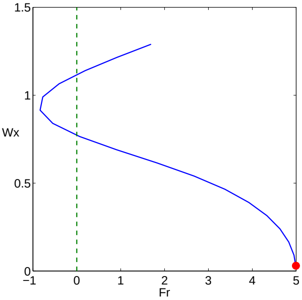

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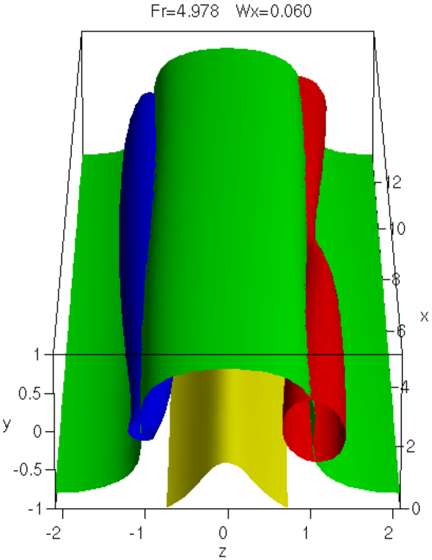

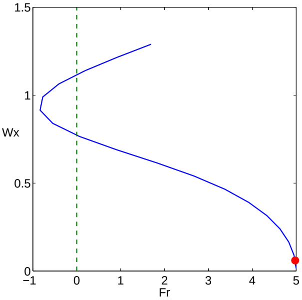

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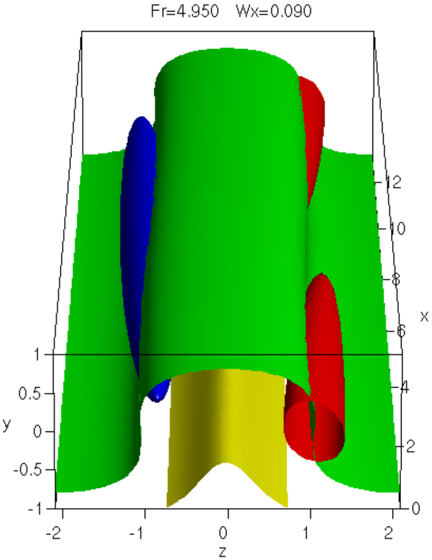

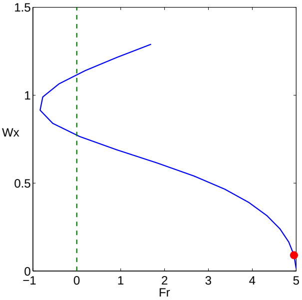

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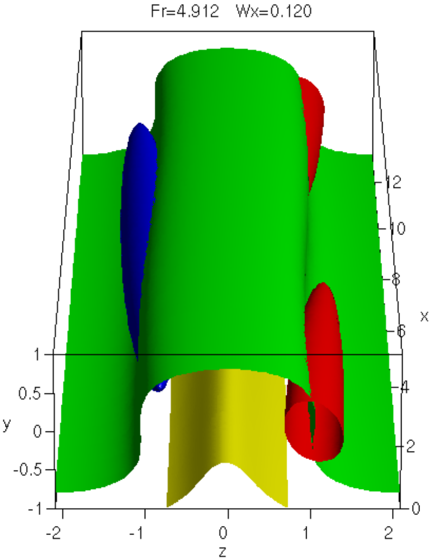

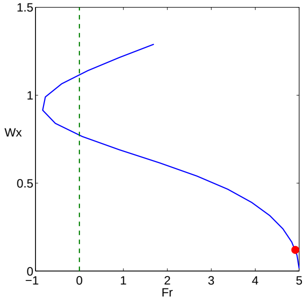

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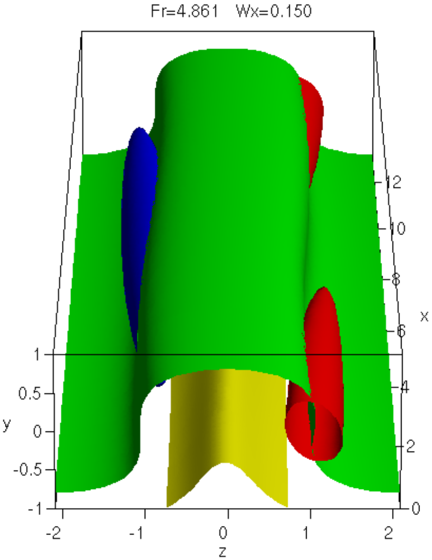

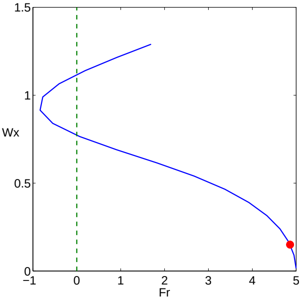

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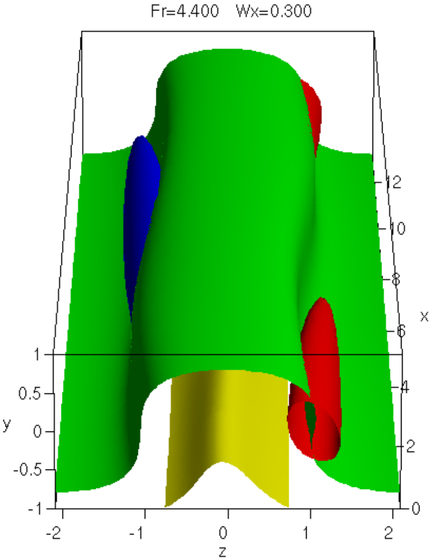

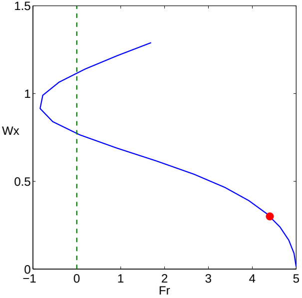

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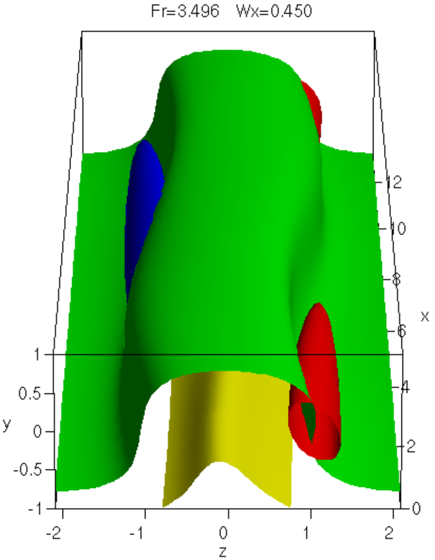

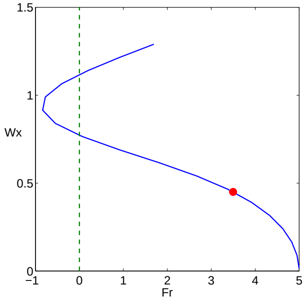

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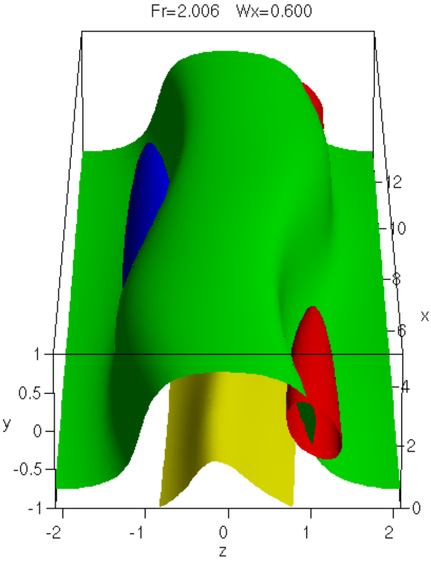
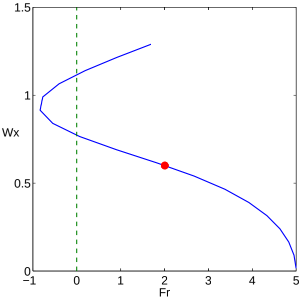
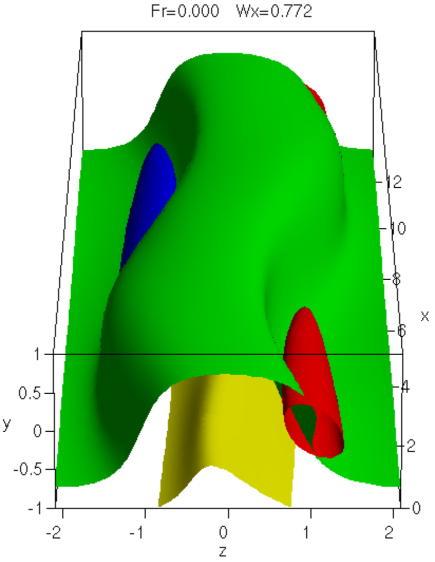
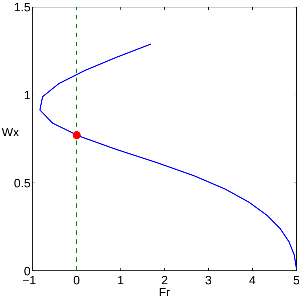




SSP: Self-Sustained! 3D Lower branch





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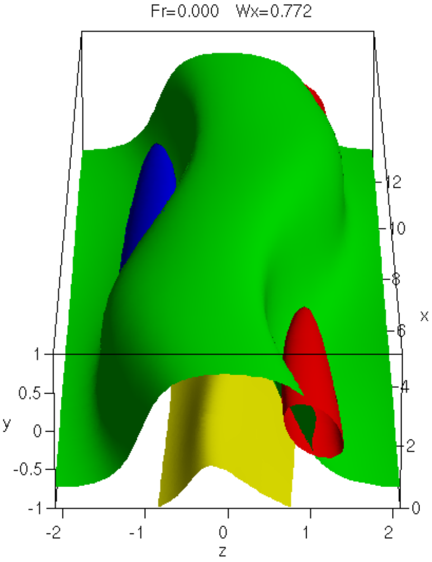

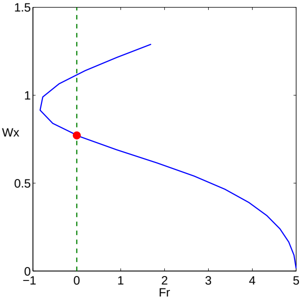

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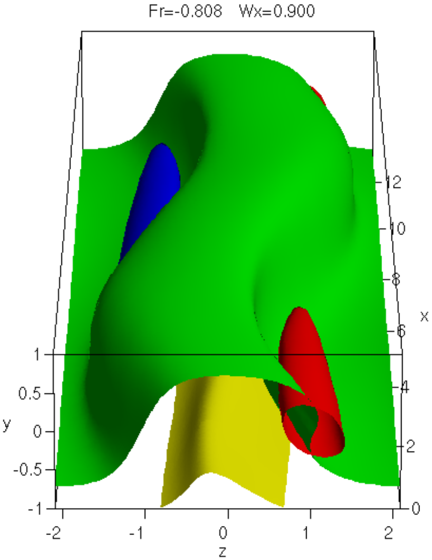

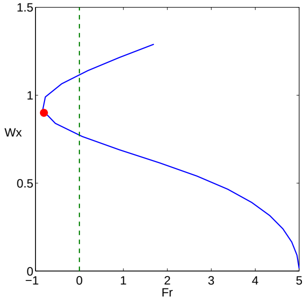

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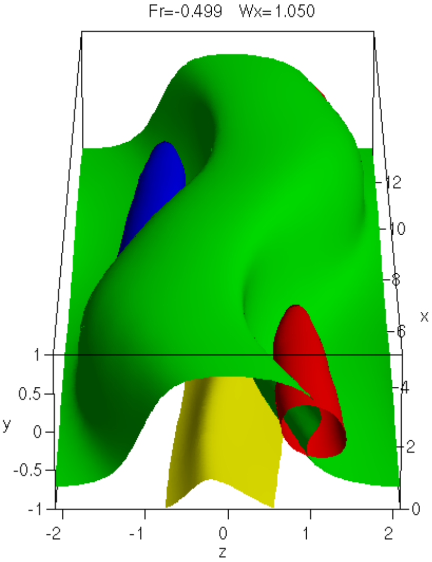
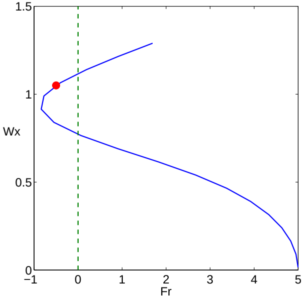
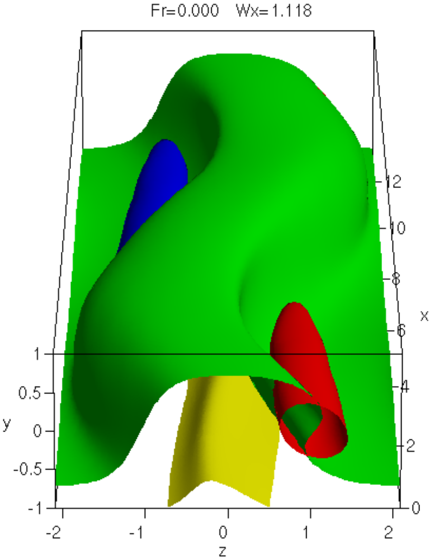
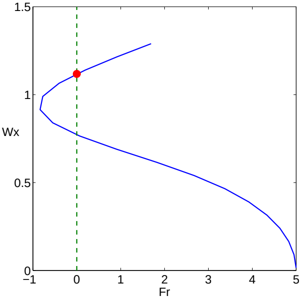




SSP: Self-Sustained! 3D Upper branch





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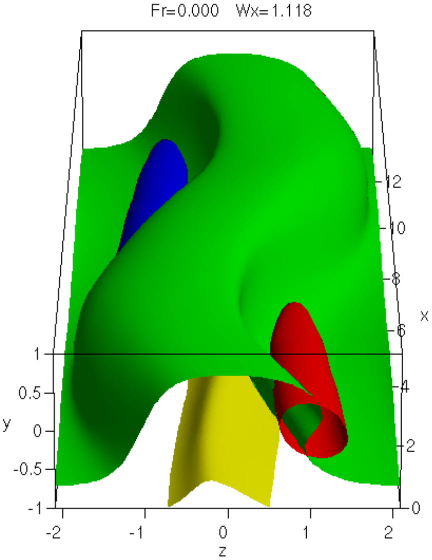
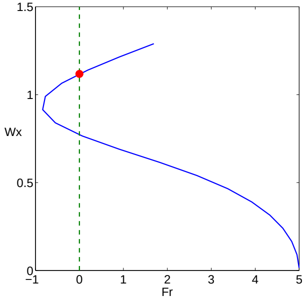




Homotopy

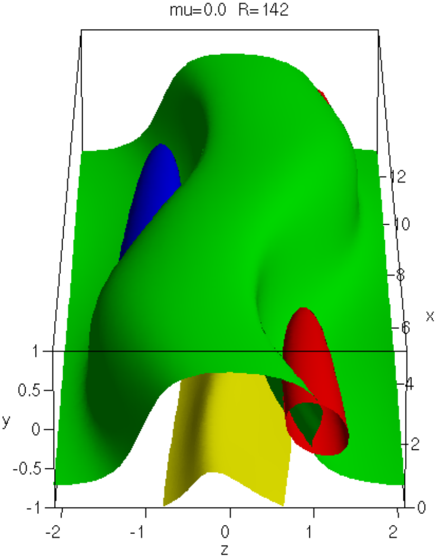
Free-Free Couette (FFC) → Rigid-Free Poiseuille (RFP)

$$\mu = 0 \to 1$$

$$BC: \quad (1-\mu)\frac{du}{dy} + \mu u = 0$$

$$Flow: \quad U_L(y) = y + \mu\left(\frac{1}{6} - \frac{y^2}{2}\right)$$

FFC ⟶ RFP



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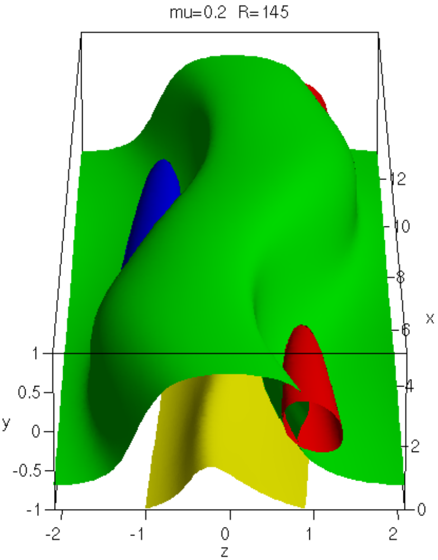
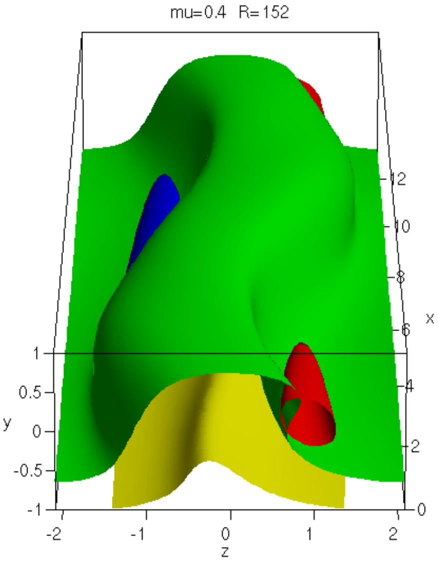


FFC $\longrightarrow$ RFP



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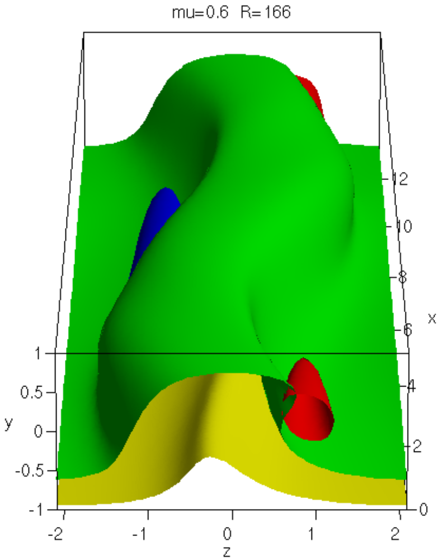

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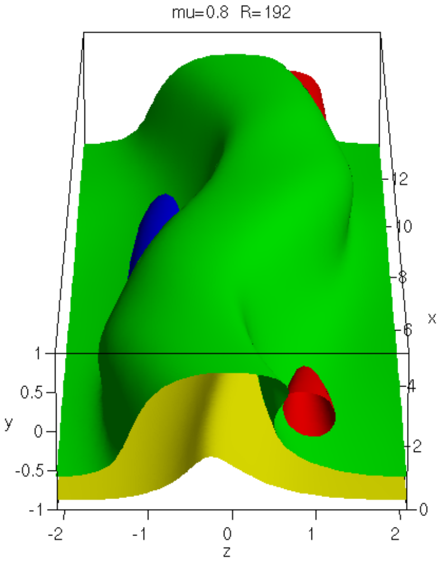
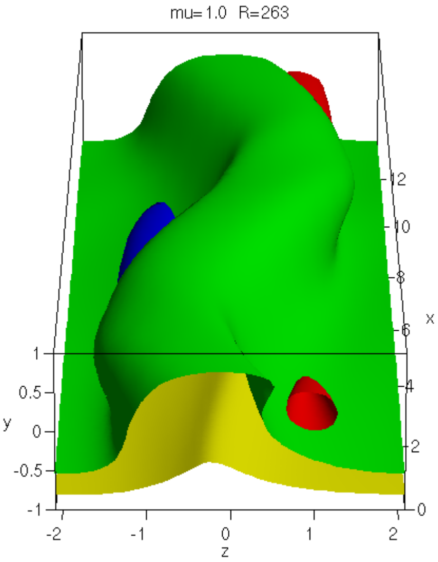


Poiseuille traveling wave!



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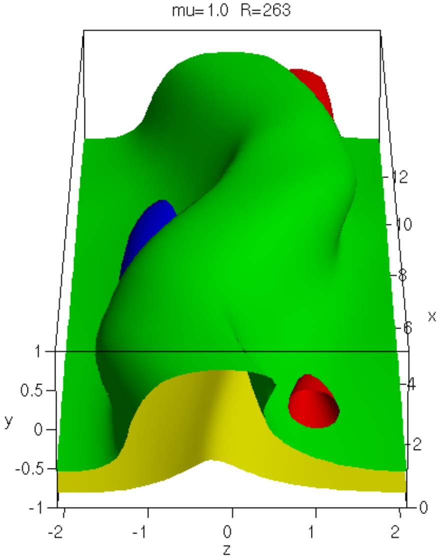
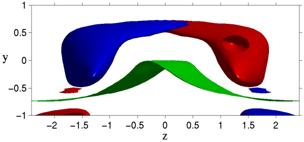


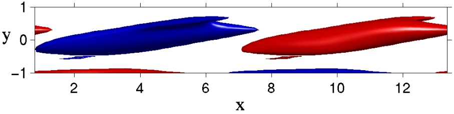
SSP, ECS: Generic

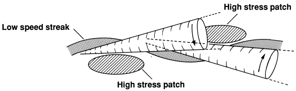
- *structurally* stable, dynamically unstable
- Plane Couette *and* Channel, free-slip, no-slip, any-slip!
- Pipe:

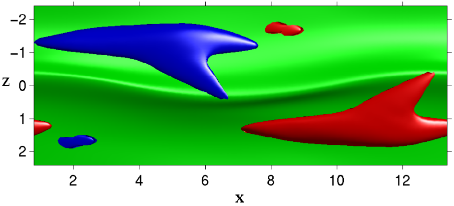
  Faisst & Eckhardt PRL 2003,
  Wedin & Kerswell JFM 2004,
  Hof *et al.* Science 2004,
  Pringle & Kerswell PRL 2007

Optimum Traveling Wave: 100^+ !



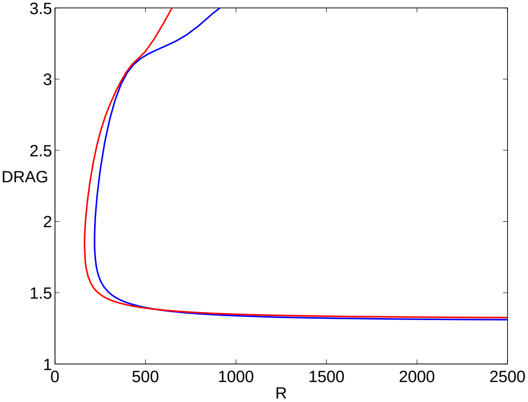






$\boxed{\min R_\tau = 2h^+ = 44 \text{ for } L_x^+ = 274,\ L_z^+ = 105}$ just right!

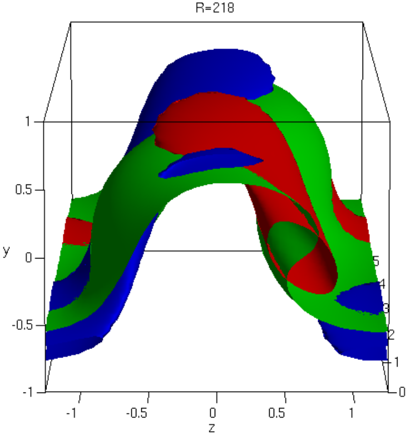
‘Out-of-the-blue-sky’ (saddle-node)



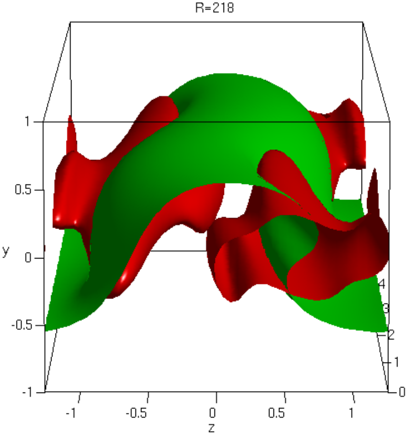
Lower branch does NOT bifurcate from laminar flow!

RRC $(\alpha, \gamma) =$ (1, 2), (1.14, 2.5), up to R≈ 60 000 + asymptotics

Vortex visualization

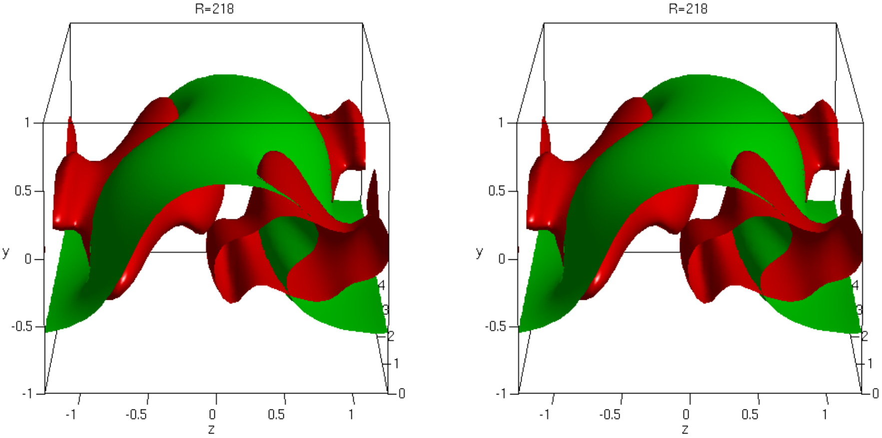


ω_x



$2\mathrm{Q} = \nabla^2 p = \Omega_{ij}\Omega_{ij} - S_{ij}S_{ij}$

Upper and Lower branches



$0.6\ \mathrm{max}(Q),\ (\alpha, \gamma) = (1.14, 2.5).$

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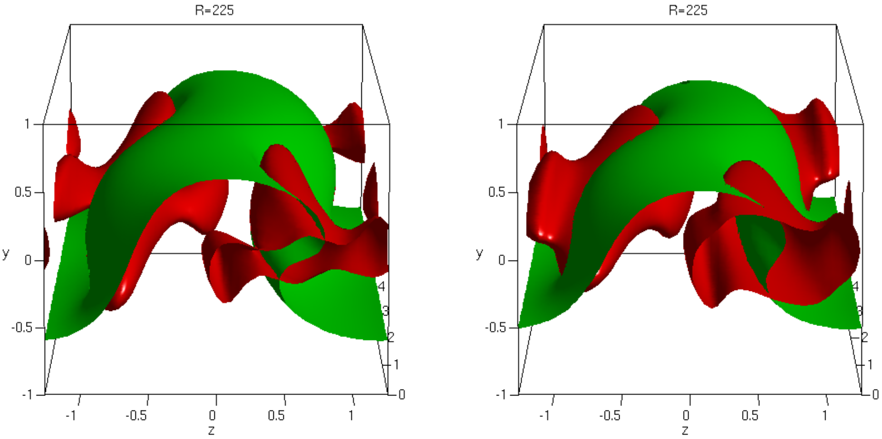

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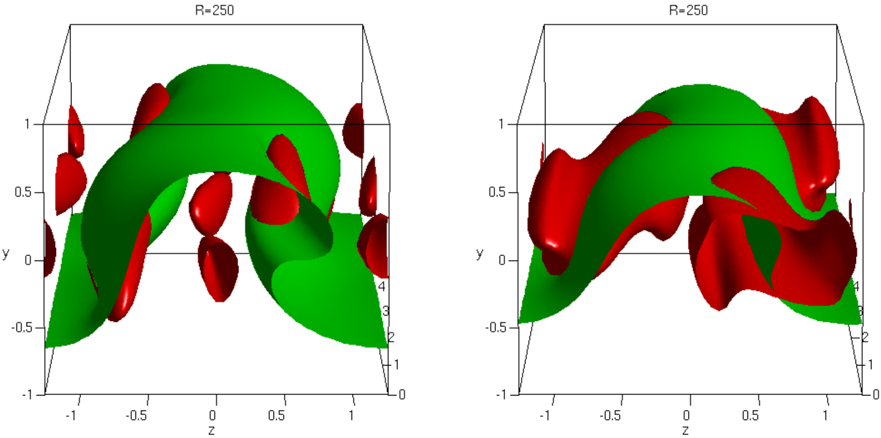

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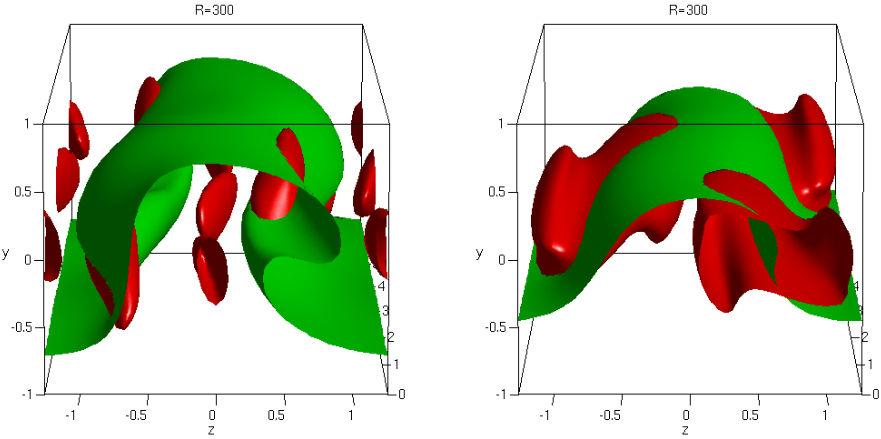

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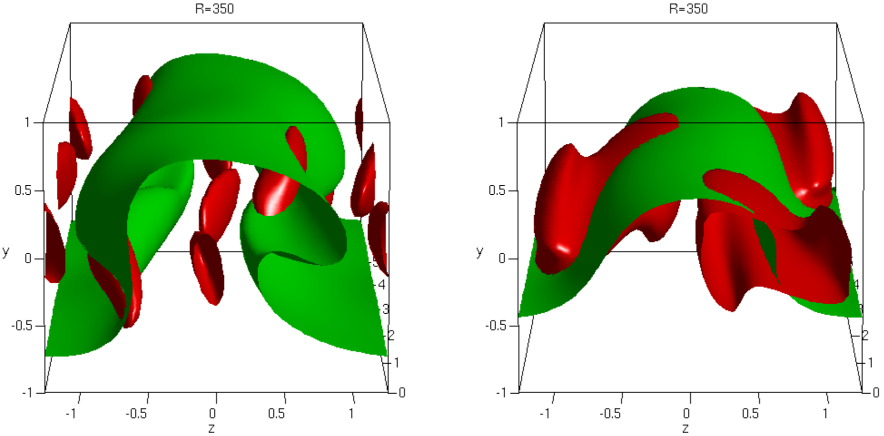

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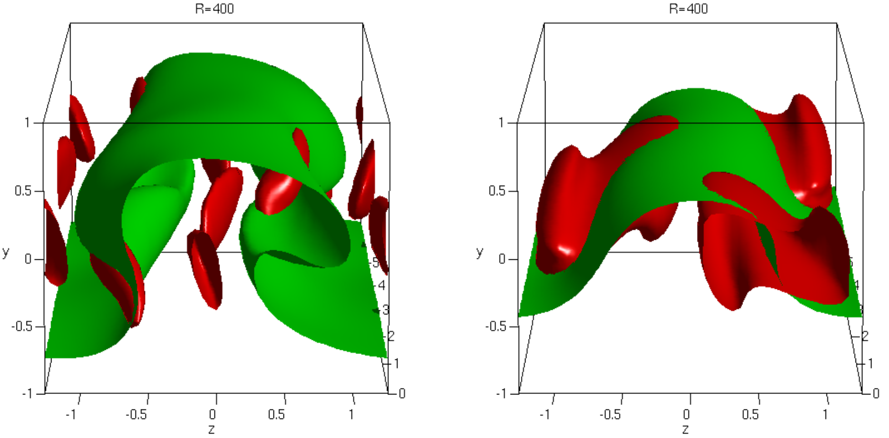

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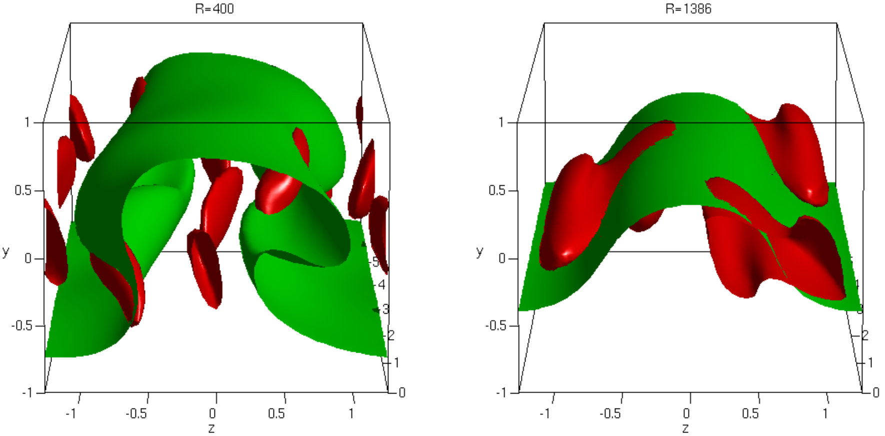

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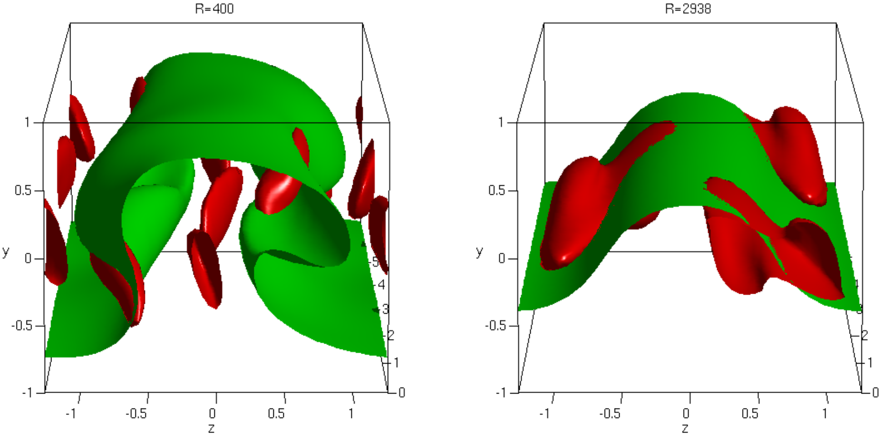

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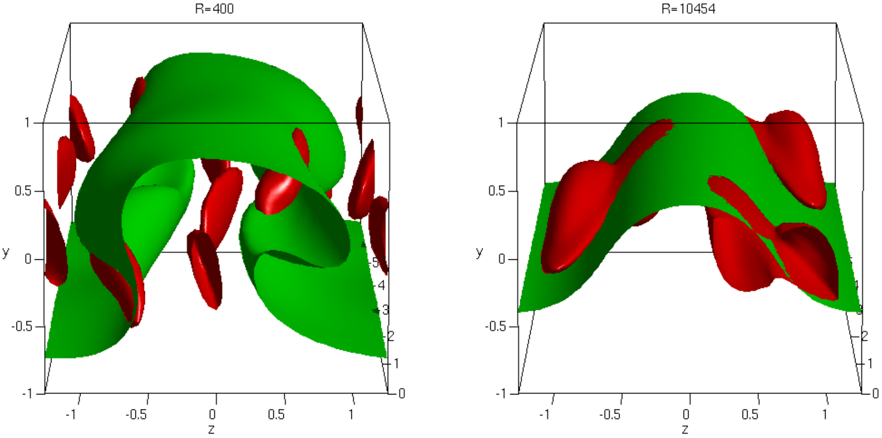

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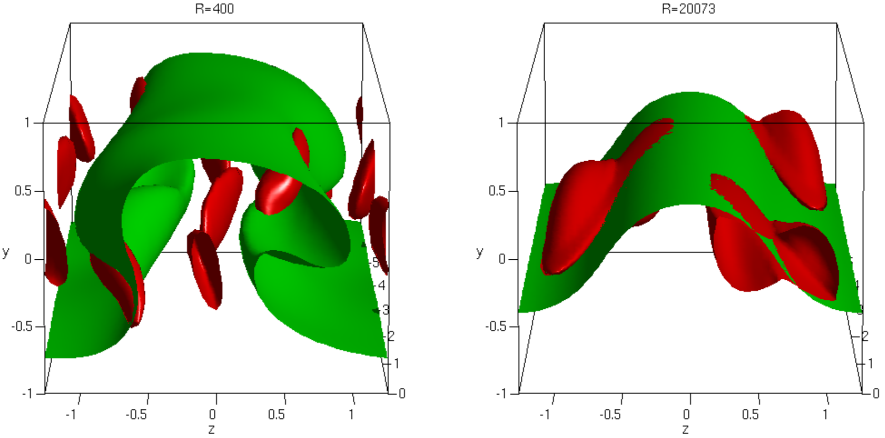

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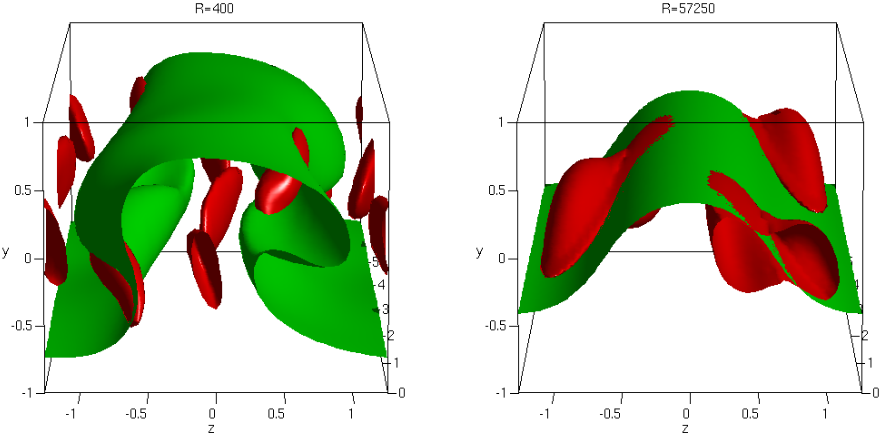

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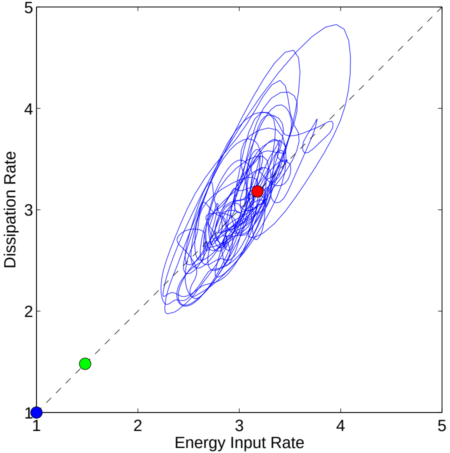
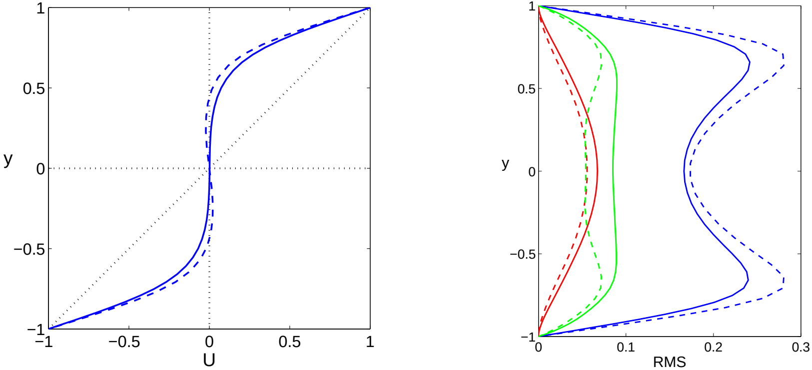
Upper and Lower branches no-slip Couette



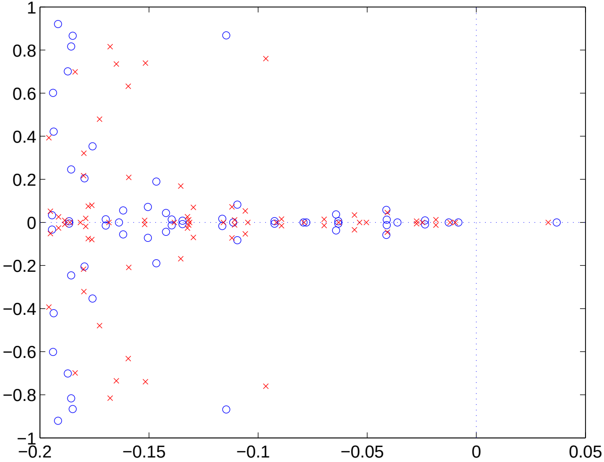
Steady State & 'turbulent' (by Jue Wang & John Gibson) in RRC, $R = 400$, (α, γ)=(0.95,1.67)

Upper branches ⟷ Turbulence (no-slip Couette)



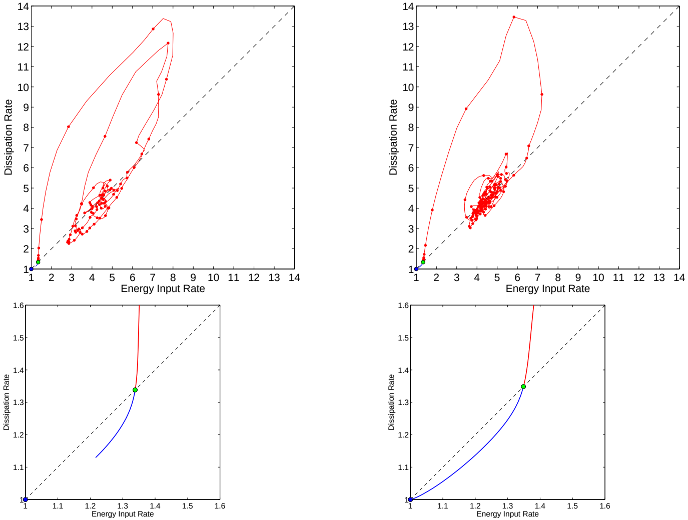
solid: Turbulent (avg t=2000), dash: fixed point
Mean and RMS velocity profiles

LB eigenvalues, $(\alpha, \gamma) = (1.14, 2.5)$, $(1, 2)$, $R = 1000$

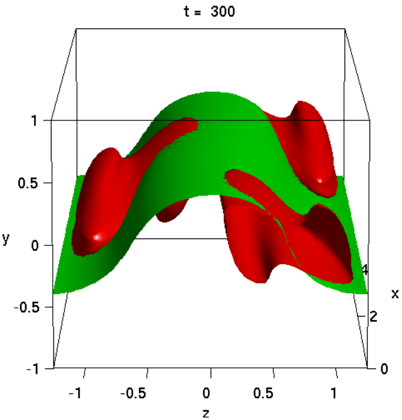


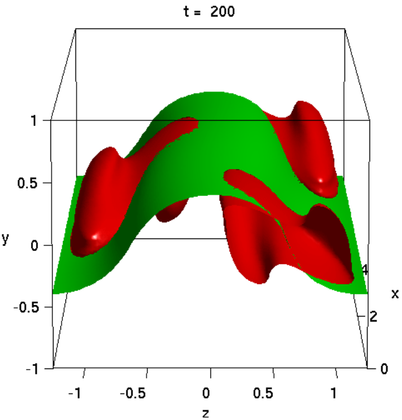
Only 1 unstable eig!

LB $\longleftrightarrow$ Transition $(\alpha, \gamma) = (1.14, 2.5),\ (1, 2),\ R = 1000$



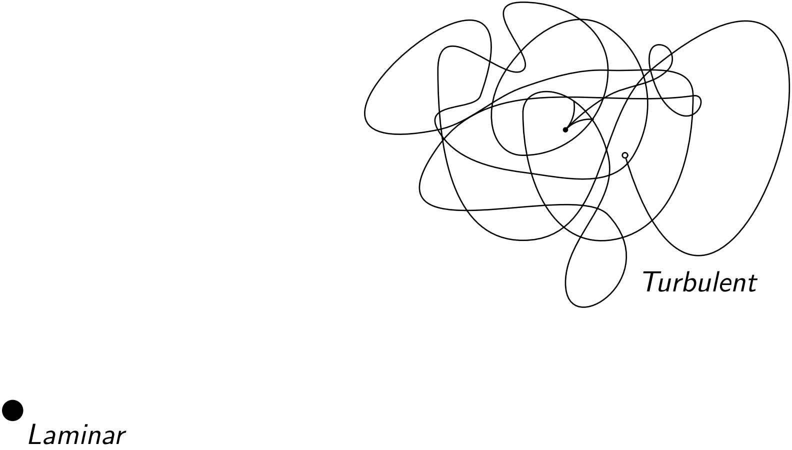
Lower branch $R = 1000$



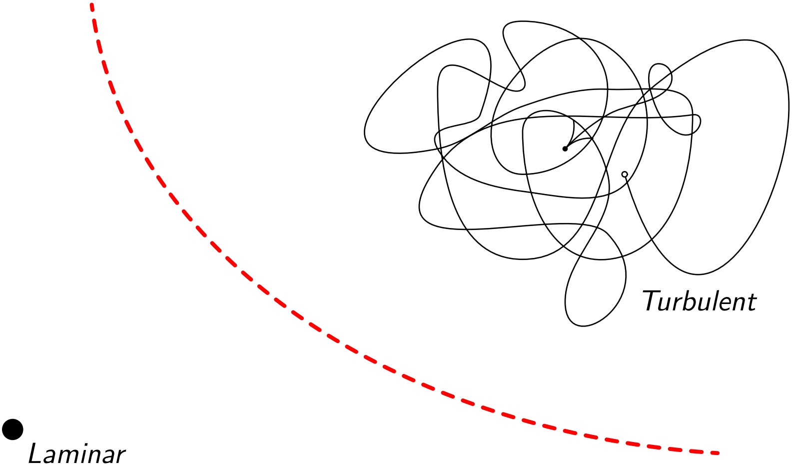


0.6 max(Q), $R = 1000$, $(\alpha, \gamma) = (1.14, 2.5)$.

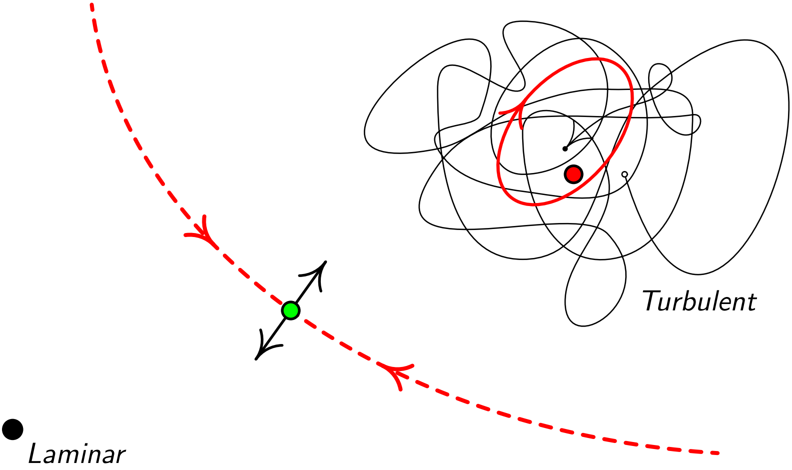
Two states of fluid flow



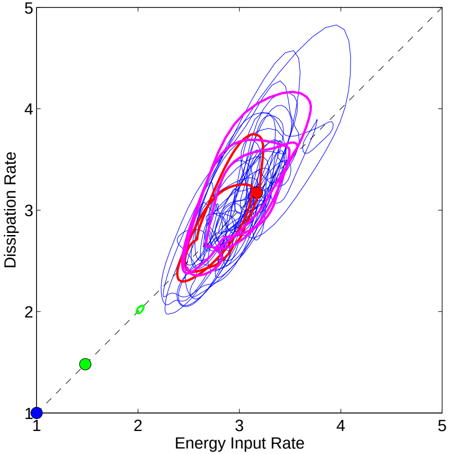
Separatrix, transition threshold



Unstable Coherent States!

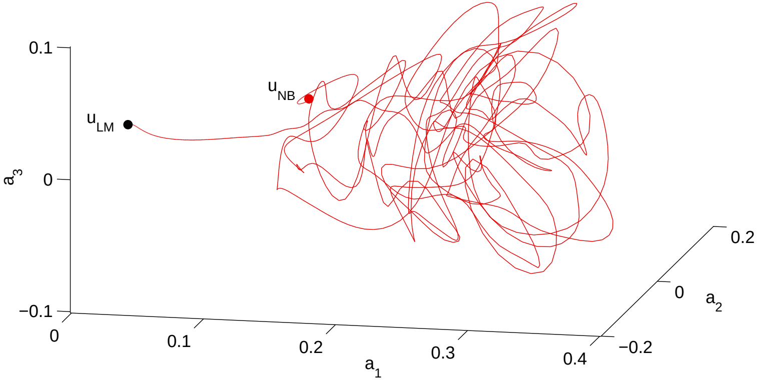


PCF data (*R* = 400)



Periodic solutions in HKW (1.14, 1.67) by *Viswanath, JFM 2007* & *Gibson (TBA)*

Visualizing State Space (10^5 dof's)



RRC, R=400, *Gibson, Halcrow, Cvitanovic*, *JFM to appear* `arxiv.org/0705.3957`

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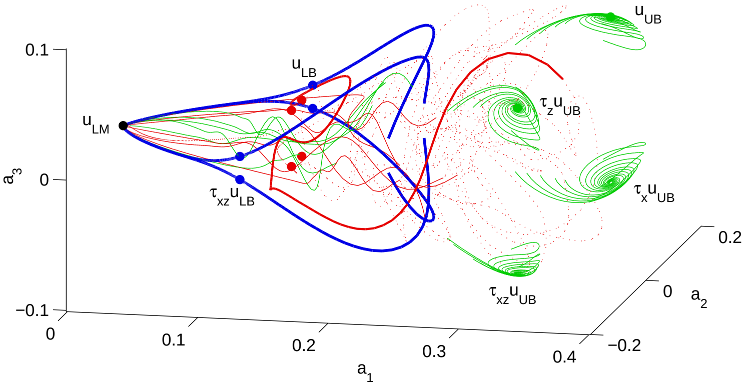

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- Asymptotic theory of lower branch solutions? Proof?