A Reduced Model for Rotational Flows

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Background

For 3-D rotating flows,it's an interesting problem to see how energy transferring into large-scale zero-frequency modes and the corresponding generation of anisotropic structures,such as large-scale zonal flows and vortices.Newell(1969) showed that near resonances are important on a time scale of $O(1/R)$.Smith and Lee showed numerically that the near resonances reproduced all of the important characteristics of the full simulation,but it's not a PDE.Our goal is to generate a PDE reduced model that including all the interactions with zonal flow.

Governing Equation for 3-D Rotational Flow

for the purely-rotation equation:

$$\begin{cases} \vec{u}_t + \vec{u}\cdot\nabla\vec{u} + 2\Omega\hat{z}\times\vec{u} & = -\nabla p + \nu\nabla^2\vec{u} \\ \nabla\cdot\vec{u} & = 0 \end{cases}$$

The linear,inviscid limit has wave solutions of the form:

$$\overline{u}(x,t;\overrightarrow{k}) = \vec{h}_{s_k} exp[i(\overrightarrow{k}\cdot\overline{x} - \sigma(\overrightarrow{k})t)]$$

we have the frequency $\sigma_{s_k} = s_k 2\Omega\frac{k_z}{k}$.With eigenvector

$$\vec{h}_{s_k} = \begin{pmatrix} \frac{k_x k_z}{k k_h} & +i s_k \frac{k_y}{k_h} \\ \frac{k_y k_z}{k k_h} & -i s_k \frac{k_x}{k_h} \\ & -\frac{k_h}{k} \end{pmatrix}$$

Governing Equation

consider decomposition:

$$\vec{u} = \sum_k (b_k^+ \vec{h}_k^+ + b_k^- \vec{h}_k^-) * e^{i*(\vec{k}\cdot\vec{x} - \sigma(\overrightarrow{k}))}$$

we will have the equation in Fourier Space:

$$\begin{aligned}\frac{\partial b_{s_k}}{\partial t} &= \frac{1}{4} \sum_{\vec{k}=\vec{p}+\vec{q}} \sum_{s_p, s_q} (s_p p - s_q q) C_{\vec{k},\vec{p},\vec{q}}^{s_k, s_p, s_q} \cdot b_{s_p} * b_{s_q} \\ &* exp(i(\sigma_{s_k}(\overrightarrow{k}) + \sigma_{s_p}(\overrightarrow{p}) + \sigma_{s_q}(\overrightarrow{q}))t)\end{aligned}$$

where

$$C_{\vec{k},\vec{p},\vec{q}}^{s_k, s_p, s_q} = (\vec{h}_{s_p} \times \vec{h}_{s_q}) \cdot \vec{h}_{s_k}^*$$

Resonant triad interactions means

$$\overrightarrow{k} + \overrightarrow{p} + \overrightarrow{q} = 0 \qquad \sigma_{s_k}(\overrightarrow{k}) + \sigma_{s_p}(\overrightarrow{p}) + \sigma_{s_q}(\overrightarrow{q}) = 0$$

Background

- Waleffe(1993)showed all interactions transfer energy toward smaller values of frequency.But resonant interactions cannot transfer energy directly to 2-d flow(zero frequency flow) since the $C_{kpq} = 0$
- Smith and Lee showed numerically that the near resonances reproduced all of the important characteristics of the full simulation,but it's not a PDE which will make the simulation slow.
- Babin generated a PDE form of equation for the exact resonant reduced model,but since exact resonant cannot transfer energy to 2-d,this is not a good model in studying how energy transferring into 2-d.
- My goal here is to derive a reduced model that one wave in the triad interactions has zero-frequency(2-d flow).This reduced model will be in a PDE form,then the numerical simulation will save some time.Also ,we want to show this model will reproduce most of the important characteristics of the full model,such as energy transferring into large-scale zero-frequency modes and the generation of anisotropic structures.

Deriving Reduced Model

Now we'll use the solution in the form

$$\vec{u} = \sum_k (a_k^+(t)\vec{h}_k^+ + a_k^-(t)\vec{h}_k^-)$$

Here $a(t;k) = b(t;k) * exp(-i\sigma(\overrightarrow{k})t)$
Now the equation in Fourier Space becomes:

$$\frac{\partial a_{s_k}}{\partial t} - \sigma_{s_k} k_z a_{s_k} = \frac{1}{4} \sum_{\vec{k}=\vec{p}+\vec{q}} \sum_{s_p, s_q} (s_p p - s_q q) C_{\vec{k},\vec{p},\vec{q}}^{s_k, s_p, s_q} \cdot a_{s_p} * a_{s_q}$$

Deriving Reduced Model

Use the eigenvectors:

$$\begin{cases} u & = \sum_{\vec{k}} \left(\frac{k_x k_z}{k k_h} + i\frac{k_y}{k_h}\right) a_k^+ + \left(\frac{k_x k_z}{k k_h} - i\frac{k_y}{k_h}\right) a_k^- \\ v & = \sum_{\vec{k}} \left(\frac{k_y k_z}{k k_h} - i\frac{k_x}{k_h}\right) a_k^+ + \left(\frac{k_y k_z}{k k_h} + i\frac{k_x}{k_h}\right) a_k^- \\ w & = \sum_{\vec{k}} -\frac{k_h}{k}(a_k^+ + a_k^-) \end{cases}$$

use stream functions:

$$u = \chi_x - \psi_y \, , v = \chi_y + \psi_x$$

then

$$\begin{cases} \psi & = \sum_{\vec{k}} -\frac{1}{k_h}(a_k^+ - a_k^-) \\ \chi & = \sum_{\vec{k}} -i\frac{k_z}{k k_h}(a_k^+ + a_k^-) \\ w & = \sum_{\vec{k}} -\frac{k_h}{k}(a_k^+ + a_k^-) \end{cases}$$

Deriving Reduced Model

Since what we are interested in now is how the energy is transferred to 2-D flow for 3-D rotational flows.In Fourier space,2-d flow means the case $k_z = 0$ which means $\sigma_k = 0$,so 2-d flow is also called slow wave mode.Now for the slow modes(we'll use ^{o} to denote that):

$$\vec{h}^0_{s_k} = \begin{pmatrix} i s_k \frac{k_y}{k_h} \\ -i s_k \frac{k_x}{k_h} \\ -\frac{k_h}{k} \end{pmatrix}$$

$$\begin{cases} \overline{u} & = \sum_{\vec{k}} i \frac{k_y}{k_h} a_k^+ - i \frac{k_y}{k_h} a_k^- \\ \overline{v} & = \sum_{\vec{k}} -i \frac{k_x}{k_h} a_k^+ + i \frac{k_x}{k_h} a_k^- \\ \overline{w} & = -\sum_{\vec{k}} \frac{k_h}{k} (a_k^+ + a_k^-) \end{cases}$$

stream functions:

$$\overline{u} = -\psi_y \ , \overline{v} = \psi_x$$

Deriving Reduced Model

Now all the interactions we'll have :

$$\begin{array}{c|cccccc} 0 & 00 & ++ & +- & -- & & \\ + & ++ & +- & -- & 0+ & 0- \\ - & ++ & +- & -- & 0+ & 0- \end{array}$$

Since in the triad interactions,if one wave is 2-d,say,$k_z = 0$,then p_z and q_z should be both zero or non-zero.So there is no (0,0,+) type of interactions. Since we want to see how energy is transferred into 2-d flow,we'll derive the equation that contains at least one slow mode.We what we have now :

$$\begin{array}{c|cccc} 0 & 00 & ++ & +- & -- \\ + & 0+ & 0- & & \\ - & 0+ & 0- & & \end{array}$$

Now we'll call slow wave mode n,and fast wave mode w.

Equation for (n,w,w)

first calculate C.

$$\vec{h}_{s_p} \times \vec{h}_{s_q} = \begin{vmatrix} i & j & k \\ \frac{p_x p_z}{p p_h} + i s_p \frac{p_y}{p_h} & \frac{p_y p_z}{p p_h} - i s_p \frac{p_x}{p_h} & -\frac{p_h}{p} \\ \frac{q_x q_z}{q q_h} + i s_q \frac{q_y}{q_h} & \frac{q_y q_z}{q q_h} - i s_q \frac{q_x}{q_h} & -\frac{q_h}{q} \end{vmatrix}$$

Remind $\psi = \sum_{\vec{k}} -\frac{1}{k_h}(a_k^+ - a_k^-)$,to get $a_k^{0,+} - a_k^{0,-}$,need to calculate

$$(\vec{h}_{s_p} \times \vec{h}_{s_q}) \cdot (h_{s_k}^{\vec{0,+}} - h_{s_k}^{\vec{0,-}})^* = 2(\vec{h}_{s_p} \times \vec{h}_{s_q}) \cdot (-i\frac{k_y}{k_h}, i\frac{kx}{k_h}, 0)$$

Equation for (n,w,w)

$$(\vec{h}_{s_p} \times \vec{h}_{s_q}) \cdot (-i\frac{k_y}{k_h}, i\frac{kx}{k_h}, 0) = s_p \frac{q_h}{qp_h k_h}(\vec{p} \times \vec{q}) \cdot \hat{z} + s_q \frac{p_h}{pq_h k_h}(\vec{p} \times \vec{q}) \cdot \hat{z} + i\frac{q_h p_z (p_y k_y + p_x k_x)}{pqp_h k_h} - i\frac{p_h q_z (k_y q_y + q_x k_x)}{pqq_h k_h}$$

(Since $p \times q = p \times k = k \times q$)
use $k_x = p_x + q_x,\ k_y = p_y + q_y,\ k_z = 0, p_z + q_z = 0$

$$\begin{aligned}
&\partial_t (a_k^{0,+} - a_k^{0,-}) \\
= \ & \frac{1}{2}\sum_{\vec{k}=\vec{p}+\vec{q}}\sum_{s_p,s_q}(s_p p - s_q q)(\vec{h}_{s_p} \times \vec{h}_{s_q}) \cdot (-i\frac{k_y}{k_h}, i\frac{kx}{k_h}, 0) \cdot a_{s_p} * a_{s_q} \\
= \ & \sum_{\vec{k}=\vec{p}+\vec{q}}\sum_{s_p,s_q} s_p p(\vec{h}_{s_p} \times \vec{h}_{s_q}) \cdot (-i\frac{k_y}{k_h}, i\frac{kx}{k_h}, 0) \cdot a_{s_p} * a_{s_q} \\
= \ & \sum_{\vec{k}=\vec{p}+\vec{q}}[\frac{pq_h}{p_h q k_h}(\vec{p} \times \vec{q}) \cdot \hat{z}(a_p^+ + a_p^-)(a_q^+ + a_q^-) \\
& + \frac{p_h}{q_h k_h}(\vec{p} \times \vec{q}) \cdot \hat{z}(a_p^+ - a_p^-)(a_q^+ - a_q^-) \\
& + [i\frac{q_h p_z}{qp_h k_h}(p_x q_x + p_y q_y + p_h^2) - i\frac{p_h q_z}{qq_h k_h}(p_x q_x + p_y q_y + q_h^2)] \\
& (a_p^+ - a_p^-)(a_q^+ + a_q^-)]
\end{aligned}$$

Equation for (n,w,w)

multiply k_h both sides

$$
\begin{aligned}
& -k_h^2 \cdot [-\tfrac{1}{k_h}\partial_t(a_k^{0,+} - a_k^{0,-})] \\
= \quad & \textstyle\sum_{\vec{k}=\vec{p}+\vec{q}} [\frac{p q_h}{p_h q}(\vec{p}\times\vec{q})\cdot\hat{z}(a_p^+ + a_p^-)(a_q^+ + a_q^-) + \frac{p_h}{q_h}(\vec{p}\times\vec{q})\cdot\hat{z}(a_p^+ - a_p^-)(a_q^+ - a_q^- \\
& +[i\tfrac{q_h p_z}{q p_h}(p_x q_x + p_y q_y + p_h^2) - i\tfrac{p_h q_z}{q q_h}(p_x q_x + p_y q_y + q_h^2)](a_p^+ - a_p^-)(a_q^+ + a_q^-)]
\end{aligned}
$$

take this to real space:

$$
\begin{aligned}
\partial_t \nabla_H^2 \overline{\psi} = \quad & -J(\nabla^2\nabla_H^{-2} w, w) + J(\nabla_H^2\psi,\psi) + \nabla_H^2\partial_x\psi\cdot\partial_x\partial_z\nabla_H^{-2} w \\
& +\nabla_H^2\partial_y\psi\cdot\partial_y\partial_z\nabla_H^{-2} w + \nabla_H^2\psi\cdot\partial_z w - \partial_z\partial_x\psi\cdot\partial_x w - \partial_z\partial_y\psi\cdot\partial_y w \\
& -\nabla_H^2\partial_z\psi\cdot w
\end{aligned}
$$

to get $a_k^{0,+} + a_k^{0,-}$,need to calculate

$$
(\vec{h}_{s_p}\times\vec{h}_{s_q})\cdot(h_{s_k}^{\vec{0},+} + h_{s_k}^{\vec{0},-})^* = 2(\vec{h}_{s_p}\times\vec{h}_{s_q})\cdot(0,0,-\frac{k_h}{k})
$$

Equation for (n,w,w)

Similarly, we can get the equation for $\overline{w}$

$$\begin{aligned}\partial_t \overline{w} = \quad & \partial_x \nabla^2 \nabla_H^{-2} w \cdot \partial_z \partial_x \nabla_H^{-2} w + \partial_y \nabla^2 \nabla_H^{-2} w \cdot \partial_z \partial_y \nabla_H^{-2} w + \partial_x \partial_z \psi \cdot \partial_x \psi \\ & + \partial_y \partial_z \psi \cdot \partial_y \psi - J(\partial_z \psi, \partial_z \nabla_H^{-2}) + J(\nabla^2 \nabla_H^{-2} w, \nabla_H^{-2})\end{aligned}$$

Similarly,we can get the equation for (w,n,w):

$$\begin{aligned}\partial_t \nabla_H^2 \psi + 2\Omega \partial_z w \quad & = \tfrac{1}{2}(-J(\overline{w}, w) + J(\nabla_H^2 \overline{\psi}, \psi) + \partial_x \nabla_H^2 \overline{\psi} \cdot \partial_z \partial_x \nabla_H^{-2} w \\ & + \partial_y \nabla_H^2 \overline{\psi} \cdot \partial_z \partial_y \nabla_H^{-2} w + \nabla_H^2 \overline{\psi} \partial_z w \\ & - J(\nabla^2 \nabla_H^{-2} w, \overline{w}) + J(\nabla_H^2 \psi, \overline{\psi}) - \partial_z \partial_x \psi \cdot \partial_x \overline{w} \\ & - \partial_z \partial_y \psi \cdot \partial_y \overline{w} - \partial_z \nabla_H^2 \psi \cdot \overline{w})\end{aligned}$$

Set of equations:

Now we'll take the equations to (u,v,w) form:

$$
\begin{aligned}
\partial_t \overline{u} \quad & + \overline{u u_x + v u_y + w u_z} = -\overline{p_x} \\
\partial_t \overline{v} \quad & + \overline{u v_x + v v_y + w v_z} = -\overline{p_y} \\
\partial_t \overline{w} \quad & + \overline{u w_x + v w_y + w w_z} = \overline{p_z} \\
\partial_t u + 2\Omega v \quad & + \tfrac{1}{2}(\overline{u} u_x + \overline{v} u_y + \overline{w} u_z + u \overline{u}_x + v \overline{u}_y) = -p_x \\
\partial_t v - 2\Omega u \quad & + \tfrac{1}{2}(\overline{u} v_x + \overline{v} v_y + \overline{w} v_z + u \overline{v}_x + v \overline{v}_y) = -p_y \\
\partial_t \nabla_H^2 w \quad & + \tfrac{1}{2}(\overline{u} w_x + \overline{v} w_y + \overline{w} w_z + u \overline{w}_x + v \overline{w}_y) = -p_z
\end{aligned}
$$

This set of equations is basically doubled the number of 3-d FFTs of original N-S equation and involved some 2-d FFT.So we can say this is doable

Equation for (n,n,n)

For the case where all three are slow-wave mode,we have:

$$\partial_t \nabla_H^2 \Psi + J(\Psi, \nabla_H^2 \Psi) = 0$$

$$\partial_t w + J(w, \Psi) = 0$$

This is just the 2-D 3-C equation.And this showed our method to derive the equation works.

2-d

For 2-d,we’ll use beta-plane equations and use the same idea to generate a PDE reduced model.But in 3-d,three non-wave interaction is 2d-3c,but in beta-plane,the coupling coefficient for three non-wave interaction is zero.So in this case,we will consider near-zonal wave interaction instead of exact zonal interaction.

Background

β-plane equation is a simple two-dimensional system that describes the flow of a thin layer of homogeneous fluid on the surface of a rotating sphere.
Chekhlov,Orszag,Galperin,Sukoriansky and Starosesky showed the energy spectrum exhibits the scaling $E(k) \propto \beta^2 k^{-5}$.Lee and Smith showed the near-resonant triad interactions capture the main features of the main features of time-developing flow on the full β-plane.
Now our goal here is to write down a PDE for a reduced model and see how energy goes to zonal parts.

Governing equations

The β-plane equation for the stream functions is:

$$\partial_t \nabla^2 \Psi + \beta \partial_x \Psi + J(\Psi, \nabla^2 \Psi) = \nu \nabla^4 \Psi$$

Ψ:Stream function of the flow

β is the linear variation of the Coriolis parameter.($f = f_o + \beta y$)

$Rh = U/(\beta L^2)$ is the Rhines number

If $\Psi = \sum_{\overrightarrow{k}} b(k;t) e^{i(\overrightarrow{k} \cdot \overrightarrow{x} - \sigma(\overrightarrow{k})t)}$

$$k^2 \partial_t b(\overrightarrow{k}) = \sum_{\overrightarrow{k} + \overrightarrow{p} + \overrightarrow{q} = 0} C_{kpq} b^*(\overrightarrow{p}) b^*(\overrightarrow{q}) e^{i(\sigma(\overrightarrow{k}) + \sigma(\overrightarrow{p}) + \sigma(\overrightarrow{q}))t}$$

with dispersion relation: $\sigma(\overrightarrow{k}) = -\beta \frac{k_x}{k^2}$

$$C_{kpq} = \frac{1}{2}(q^2 - p^2)(\overline{p} \times \overline{q}) \cdot \overline{z}$$

For general case,we have:

$C_{kpq} + C_{pqk} + C_{qpk} = 0 \qquad C_{kpq} = C_{kqp}$

$k^2 C_{kpq} + p^2 C_{pqk} + q^2 C_{qkp} = 0$

These showed conservation of energy and enstrophy by triad interactions.

Zonal Flow

Zonal Flow is all about the movement and exchange of information between east and west.In Fourier space, zonal flow means $k_x = 0$ and implies $\sigma(\overrightarrow{k}) = 0$.β-plane with isotropic forcing and hyperviscosity at small scales show strong transfer into large-scale zonal flow.

Two important triad interactions

- Resonant triad interaction:here $\sigma(\overrightarrow{k}) + \sigma(\overrightarrow{p}) + \sigma(\overrightarrow{q}) = 0$
- Triad interactions including zonal flow.
- Unfortunately, there is no resonant triad interaction to zonal flows.

Interactions including zonal flows

Let's call the zonal flow non-wave mode (n). From $k_x + p_x + q_x = 0$, so the only triad interactions that make sense are

- Three non-wave modes interaction
- One non-wave mode interacts with two wave modes.
- Three wave modes interaction.

For the three non-wave interaction , since $(\overline{p} \times \overline{q}) \cdot \overline{z} = 0$, we have

$$C_{kpq} = C_{pqk} = C_{qkp} = 0$$

near-zonal and outside zonal

Now we'll think about the interaction including near-zonal.Near-zonal is defined as the modes including $k_x = 0$ and a sector which is defined by all the modes such that $\frac{k_x}{k_y} \sim O(Rh)$.Since the condition for a mode near-zonal is a linear relation between k_x and k_y,we can write a pde for this reduced model.

Equations for reduced model

We'll call the near-zonal "S",and the modes outside the near-zonal "P".Now all the interactions we'll have are (S,P,P),(S,S,P),(S,S,S) and (P,P,P).Since there is no triad that can make (S,S,P),we won't consider this one.
Now if we consider (S,S,S),the equation in Fourier space will be:

$$\sum_{k\in S} k^2 \partial_t b(\overrightarrow{k}) = \sum_{\overrightarrow{k}+\overrightarrow{p}+\overrightarrow{q}=0} \sum_{\overrightarrow{p}+\overrightarrow{q}\in S} \sum_{\overrightarrow{p}\in S,\overrightarrow{q}\in S} C_{kpq} b^*(\overrightarrow{p}) b^*(\overrightarrow{q}) e^{i(\sigma(\overrightarrow{k})+\sigma(\overrightarrow{p}$$

Now denote $_p$ and $_s$ are the operator constraint in near-zonal and outside-zonal.The case we'll consider will be (S,S+P,S+P).

$$\partial_t \nabla_s^2 \Psi + s + \beta \partial_{xs} \Psi_s + J_s(\Psi, \nabla^2 \Psi) = \nu \nabla_s^4 \Psi_s$$

$$\partial_t \nabla^2 \Psi + \beta \partial_x \Psi + \frac{1}{2}(J(\Psi_s, \nabla^2 \Psi) + J(\Psi, \nabla^2 \Psi_s)) = \nu \nabla^4 \Psi$$

Numerical Simulation

For the numerical simulation ,we'll use

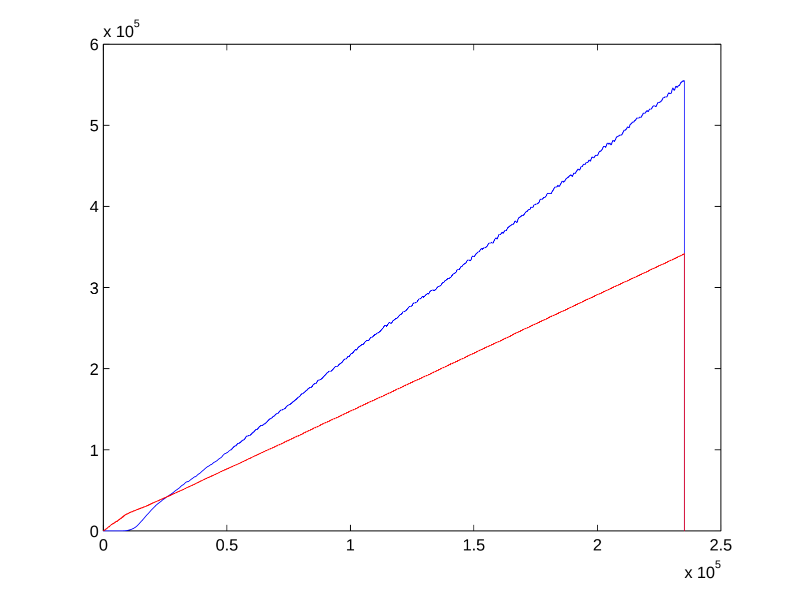
- Consider the equations in $2\pi \times 2\pi$box
- We will start with zero initial condition
- We'll add random white noise forcing in the range $73 < k < 77$ for 384^2 Fourier Modes.

$$F(k) = \epsilon_f \frac{exp(-0.5(k - k_f)^2)}{(2\pi)^{1/2}}$$

- We'll add hyperviscosity at small scales.

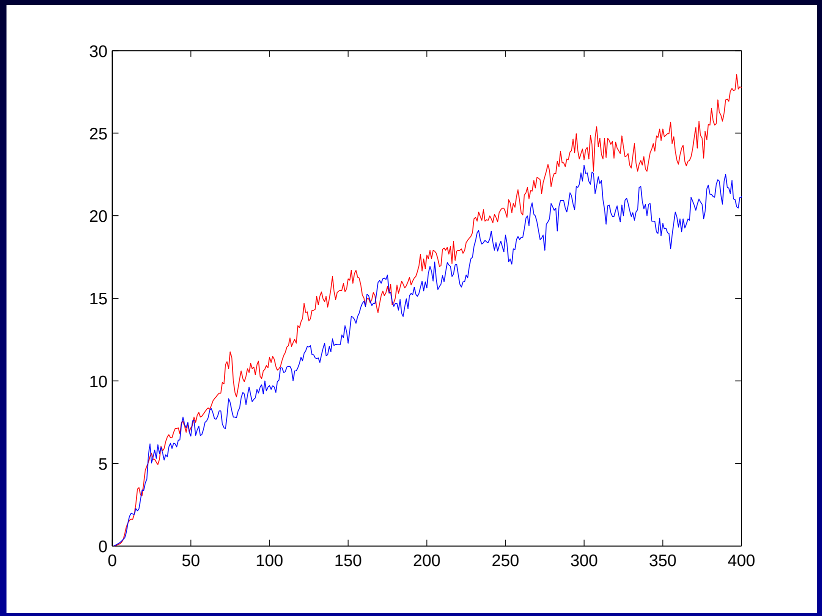
plots

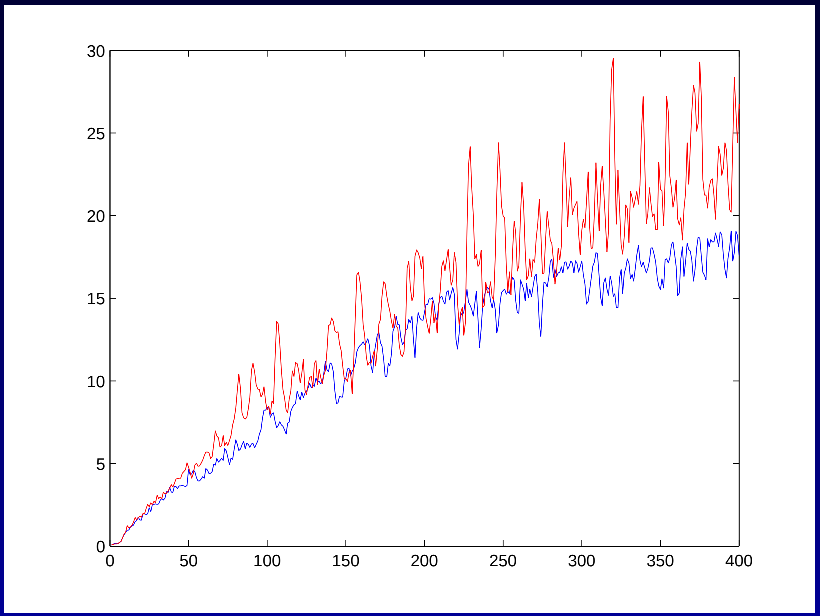
At the beginning,we start with the sector with size$\pi/6$.Define $\delta = (sizeofsector)/(Rh)$,Now we'll see the case for (S,S+P,S+P) for Rh=0.3,and $\delta = 1.5$(Rh=0.5 for western Atlantic)



This is the total energy vs time for reduced model(blue) and full model.From the plot,we can see the reduced model can gain more energy than full model.But for all later plots,we'll consider the one-time plot at the same time lever,say 350 turn-over time.

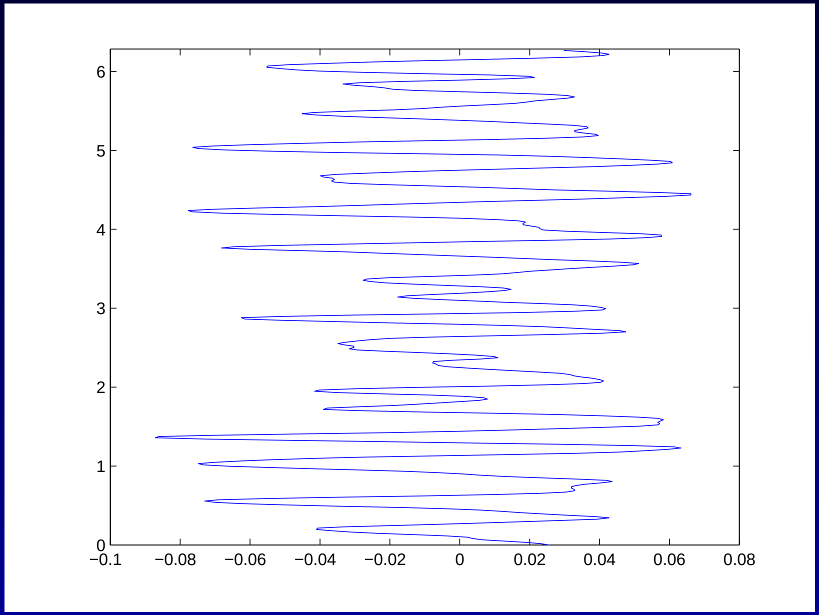
plots

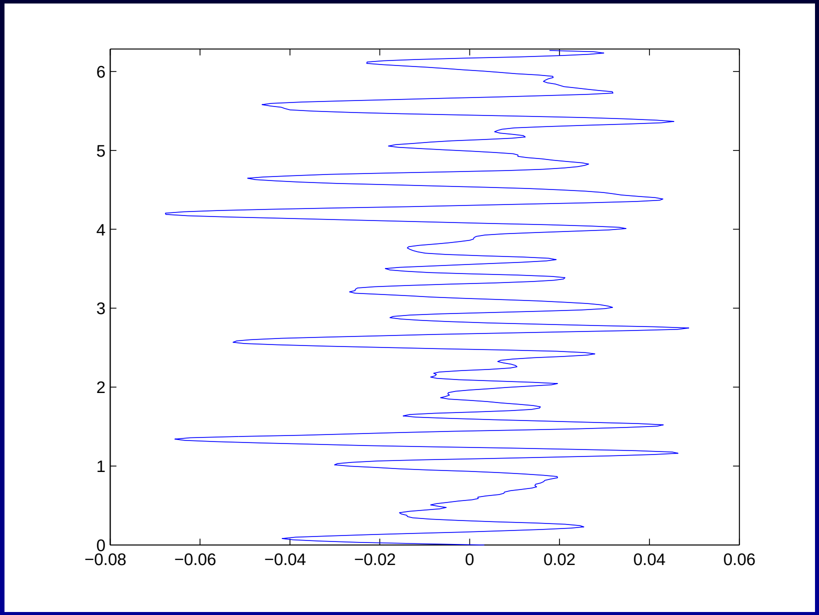




Energy of maximum zonal averaged velocity VS time.West:Red,East:East.Left:reduced model.Right:full simulation.

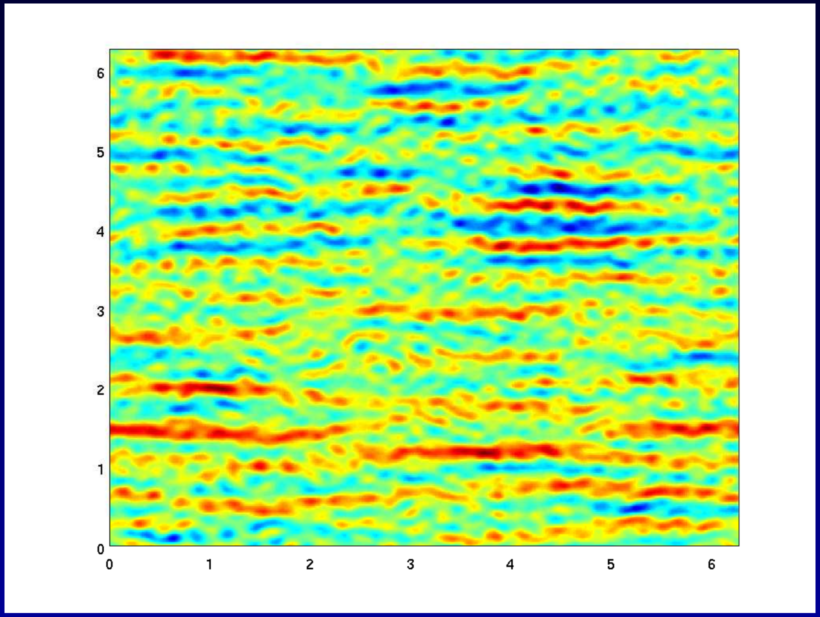
plots

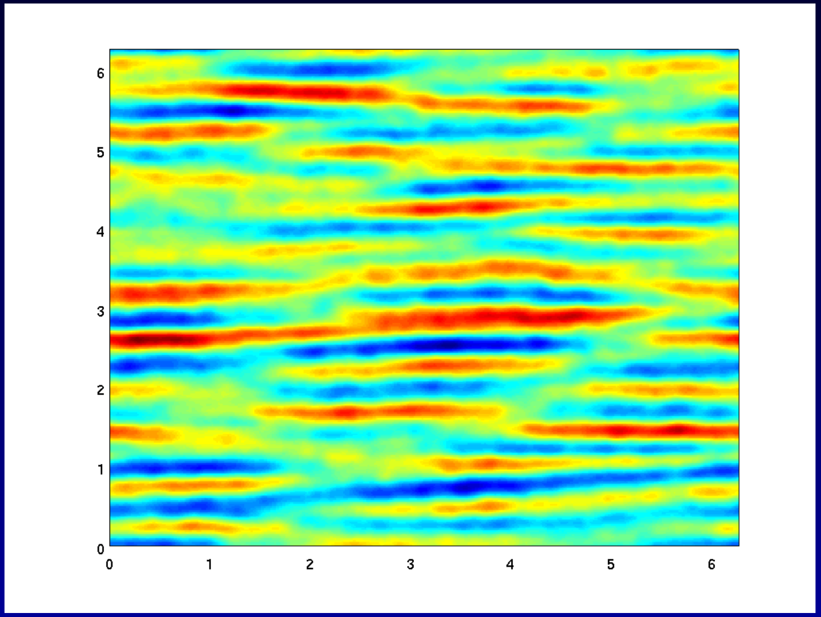




Zonal averaged velocity at t=350.

plots





This plot is the contour of the stream function at t=350.

Conclusion

As shown in the plots,for (S,S+P,S+P),its large scale behavior is same as the full model with forcing.And looking at the zonally averaged velocity,we can see the westward velocity is larger than the eastward velocity which is an important behavior for beta-plane equation

Thank you!