A Vortex/Radial Basis Function Algorithm for the Barotropic Vorticity Equation on a Rotating Sphere

Vortex methods are highly adaptive because degrees of freedom are located only where there is vorticity. Point vortices have been replaced by vortex blobs, typically Gaussians of radius, in Cartesian geometry, but persisted on the sphere because an analytical solution for the Poisson equation was known only for point forcing. Recently, we found an analytical solution for the Poisson equation on the sphere with Gaussian forcing [?]. This allowed us to develop an efficient Gaussian vortex method for solving the barotropic vorticity equation on the sphere. Traditionally, vortex methods avoid the cost of inverting an interpolation matrix by cheating. Our model employs full RBF interpolation and is therefore spectrally-accurate until unresolvable fine scales develop. One goal is to compare the merits of interpolation versus quasi-interpolation. The model uses a panel technique to adaptively add and subtract Gaussian vortices so as to maintain resolution even when thin filaments have developed. Our intermediate-term goal is to extend the model to the shallow water equations.

Radial basis functions (RBFs) are attractive because they are theoretically a “meshless” scheme: grid points and RBF centers can be clustered around the peaks of solitary waves and thinned in the voids between.
A Vortex/Radial Basis Function Algorithm for the Barotropic Vorticity Equation on a Rotating Sphere

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Radial Basis Function (RBF) Approximation

\[ f(\vec{x}) \approx \sum_{j=1}^{N} \lambda_j \phi \left( \| \vec{x} - \vec{c}_j \|_2 \right) \quad \vec{x} \in \mathbb{R}^d \quad (1) \]

\( \phi(r) \) is the RBF
\( \vec{c}_j, j = 1, \ldots N \) are “centers”
\( \lambda_j \) are “coefficients”

Usually found by interpolation at \( \vec{x}_k \) that may or may not coincide with the centers.

Under mild conditions on \( \phi \), interpolation is provably solvable even when the interpolation points and centers are scattered randomly over an irregularly-shaped domain.

Widely used for scattered interpolation (point clouds) in computer graphics.
Table 1: Typical RBF $\phi$

<table>
<thead>
<tr>
<th>Definition</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi(r) \equiv \sqrt{r^2 + c^2}$</td>
<td>Multiquadrics</td>
</tr>
<tr>
<td>$\phi(r) \equiv \frac{1}{\sqrt{r^2+c^2}}$</td>
<td>Inverse Multiquadrics</td>
</tr>
<tr>
<td>$\phi(r) \equiv r^2 \log(r)$</td>
<td>Thin plate splines</td>
</tr>
<tr>
<td>$\phi(r) \equiv \exp(-\epsilon^2 r^2)$</td>
<td>Gaussians</td>
</tr>
<tr>
<td>$\phi(r) \equiv r^3$</td>
<td>Cubic</td>
</tr>
<tr>
<td>$\phi(r) \equiv r^5$</td>
<td>Quintic</td>
</tr>
<tr>
<td>$\phi(r) \equiv (1 - r)^m p(r)$</td>
<td>Wendland functions</td>
</tr>
<tr>
<td>$(1 - r)_+ \equiv \begin{cases} 1 - r, &amp; 0 \leq r \leq 1, \ 0, &amp; r &gt; 1 \end{cases}$</td>
<td>[p is a polynomial]</td>
</tr>
</tbody>
</table>

Our work focused on **GAUSSIAN RBFs**:

$$\phi(r) \equiv \exp(-\epsilon^2 r^2)$$

on an **UNBOUNDED** interval.

Define $h$ as the grid spacing (or average grid spacing).

Absolute width parameter $\epsilon$ is insignificant.

Key parameter is the **RELATIVE** width parameter (relative to GRID SPACING):

$$\phi(x; \alpha) \equiv \exp\left(-\frac{\alpha^2}{h^2} x^2\right)$$
Vortex-RBF Hybrid for Fluid Flow on a Sphere

- Long-term goal: highly-adaptive weather forecasting and climate models using radial basis functions (RBFs)
- Short-term goal: solve the barotropic vorticity equation on the surface of a sphere

[Cartesian coordinates for expository simplicity; \( \zeta \) is the vorticity; \( u, v \) the horizontal currents, \( \beta \) is the \( y \)-derivative of the Coriolis parameter, \( \psi \) is the streamfunction, subscript \( x \) or \( y \) or \( t \) denotes differentiation with respect to that coordinate]

\[
\zeta_t + u \zeta_x + v \zeta_y + \beta v = 0 \quad \text{[Vorticity Eq.]} \\
\]

\[
u = -\psi_y, \quad v = \psi_x
\]

\[
\nabla^2 \psi = \zeta \quad \text{[Streamfunction Poisson Eq.]} \]
Vortex Methods

• RBF strategy is closely related to classical vortex methods, so a brief review is helpful.

• Point vortex methods approximate vortical flows by advecting a small number of point vortices; first triumph was Rosenhead’s (1931) desktop calculator computation of the instability of a vortex sheet.

• Defect of point vortices: smooth vortex patches are approximated by a field of \textsc{delta-function spikes}

Figure 1: Fig. 4 of Rosenhead (1931)
Vortex Blobs

- Blob methods represent vorticity by OVERLAPPING GAUSSIAN VORTICES ("blobs")
- Because Gaussians are an important species of RBFs, vortex blob methods are always an unacknowledged RBF method.
- Blob centers are advected with the current
- Poisson equation is solved ANALYTICALLY by superimposing the exact, explicit solution for a Gaussian forcing
- Blobs need not be of uniform size
- Adaptivity: blobs only where the vorticity is; in contrast to alternatives, there are NO DEGREES-of-FREEDOM in the "DEAD ZONES" of negligible vorticity.
- Blobs not previously applied to the SPHERE because analytical solution for the Poisson equation on the sphere with Gaussian forcing was UNKNOWN.
Solving the Poisson Equation $\nabla^2 \psi = \zeta$ with Gaussian Forcing on the Sphere

1: Expand Vorticity by Interpolation:

$$\zeta(\lambda, \theta) \approx \sum_{j=1}^{N} a_j^\zeta \phi_j(\lambda, \theta) \quad (2)$$

2. Solve Poisson for a Gaussian at the NORTH POLE:

$$(1 - \mu^2) \mathcal{P}_{\mu\mu} - 2\mu \mathcal{P}_{\mu} = \exp \left( -2\epsilon^2 (1 - \mu) \right) - C$$

where

$$C = \frac{1}{4\epsilon^2} \left\{ 1 - \exp(-4\epsilon^2) \right\}$$

GENERAL solution follows by SUPERPOSITION & COORDINATE ROTATION

$$\psi(\lambda, \theta) \approx \sum_{j=1}^{N} a_j^\zeta \times$$

$$\mathcal{P} (\cos(\theta) \cos(\theta_j) \ \text{+} \ \sin(\theta) \sin(\theta_j) \ \cos(\lambda - \lambda_j); \epsilon)$$
Three Solutions for $\mathcal{P}(\theta; \epsilon)$


1. Legendre series is SLOWLY CONVERGENT for narrow RBF ($\epsilon \gg 1$), but gives exact “Gauss constraint” $C$

2. Matched Asymptotics in Powers of $1/\epsilon^2$

3. Exact Solution
Matched Asymptotics for Gaussian-Forced Poisson Eq. on the Sphere

- Fourth order; for simplicity, only 2d order below

\[
P_{2,uni} = \frac{1}{4\epsilon^2} \left\{ \text{Ei}(r^2) + \log \left\{ 1 - \cos \left( \frac{r}{\epsilon} \right) \right\} \right\}
\]

\[
+ \frac{1}{\epsilon^4} \exp(-r^2) \left( \frac{1}{16} - \frac{1}{48}r^2 \right)
\]

where \( r \equiv \epsilon \theta \).

- Approximation is UNIFORMLY VALID over entire sphere.

- Perturbation parameter is \( 1/\epsilon^2 \); very fast convergence

- Outer approximation: point vortex on sphere to ALL ORDERS:

\[
P_{outer} \sim \log \{ 1 - \cos (\theta) \}
\]

- \( P_{outer} \) decays GAUSSIAN-FAST to POINT VORTEX/OUTER APPROX.
Exact Poisson Solution

\[ \psi(x = \cos(\theta)) = \frac{1}{4\epsilon^2} \left\{ 1 - \exp(-4\epsilon^2) \right\} \log(1 - x) - \frac{1}{4\epsilon^2} \exp(-4\epsilon^2) \log\left(\frac{1 + x}{1 - x}\right) + \frac{1}{4\epsilon^2} E_1 \left(2\epsilon^2 [1 - x]\right) + \frac{1}{4\epsilon^2} \exp(-4\epsilon^2) \text{Ei} \left(2\epsilon^2 [1 + x]\right) \]

\[ E_1(z) \equiv \int_1^\infty \frac{\exp(-zt)}{t} dt = \int_z^\infty \frac{\exp(-y)}{y} dy \]

\[ \text{Ei}(z) \equiv \gamma + \log(z) + \int_0^z \frac{\exp(t) - 1}{t} dt \]

Matched asymptotics has advantages:

(i) fewer special functions
(ii) no apparent singularities at south pole
(iii) summable by treecode
Summing the Poisson Solution

- One charm of the $\zeta$-as-RBF strategy, instead of an RBF series for $\psi$, is that $\psi$ is NOT spatially localized even when $\zeta$ is:
  - $\psi$ grows logarithmically away from the forcing; $u, v$ decay only as reciprocal of distance
- “Far field” of $\psi$ summable by Fast Summations (Fast Multipole Method, Fast Gauss Transform, treecodes)
- Treecodes work well (Krasny and Wang, 2008, submitted)
- Our analysis is that the Fast Gauss Transform is USELESS for “Near Field” interactions of Gaussian RBFs (Boyd, submitted to JCP)
- RBF sums (at the moment) are only “semi-fast”: fast for long-range interactions, but restricted to direct summation for short-range interactions.
Grids

• Meshless but not gridless
• For localized vortices, use problem-specific, advectively-adapted grid
• For global waves, use a Sadourny icosahedral/triangular grid

Figure 2: The icosahedral grid for \( \nu = 5 \). The twelve vertices of the icosahedron are large yellow balls. They are connected by red curves that are the edges of the spherical icosahedron. Green spheres mark the \( \nu - 1 \) grid points on the interior of each edge. The points interior to each spherical triangle, \((\nu - 2)(\nu - 1)/2\) for each face of the icosahedron, are shown as red balls.
Lagrangian Vortex/RBF Algorithm

- Potential vorticity is CONSERVED on each moving interpolation point. Relative vorticity from $\vec{\zeta}_i = \vec{\zeta}_i^{abs} - \cos(\vec{\theta}_i)$

- Calculate $a_j^{\zeta}$, RBF coeffs. of $\zeta$, by interpolation [matrix solve]

- Solve the ODE system

$$\frac{D\lambda_i}{Dt} = \frac{1}{\sin \theta_i} \sum_{j=1}^{N} a_j^{\zeta} \frac{\partial P_j}{\partial \theta}(\lambda_i, \theta_j)$$

$$\frac{D\theta_i}{Dt} = -\frac{1}{\sin \theta_i} \sum_{j=1}^{N} a_j^{\zeta} \frac{\partial P_j}{\partial \lambda}(\lambda_i, \theta_j)$$

4th order RK requires 4 interpolations/step; AB3 requires only 1 interpolation/step
Role of the Relative Inverse Width Parameter

Relative-to-grid-spacing width $\alpha = \epsilon h$ is more pertinent than absolute width $\epsilon$
Waves and Adaptation

\[ u_t + uu_x = -cu_x \quad \text{[Inviscid Burgers]} \]

where \( c \) is a constant. A vortex-like scheme:

\[
\begin{align*}
t_{j}^{(n+1)} &= u_{j}^{(n-1)} + 2\tau(-cu_{x,j}^{(n)}) \\
x_{j}^{(n+1)} &= x_{j}^{(n)} + (1/2)\tau(u_{j}^{(n+1)} + u_{j}^{(n)})
\end{align*}
\]

The crucial point is the not the mechanics of the scheme, but the consequences of different choices of the linear phase speed \( c \).
• When $c = 0$, grid automatically and dynamically adapts to concentrate grid points right where are needed.

• When $c = 1$, however, front does moves at $u(x, t) + c$. There is no systematic trend in the movement of the grid points.
Rossby-Haurwitz Waves on the Sphere

- TRAJECTORIES are CYCLOIDS

- $\zeta = (1/5) \sin(\theta) \cos(\lambda - (1/2)t)$

- ACCURATE [relative error $8.5 \times 10^{-5}$] even with no relationship between trajectories & wave motion
Vortex Merger

Figure 3: t=0

time=0 ratio=2.5877 nv=2918 ε=300
Figure 4: $t=2$

- Time: 2
- Ratio: 2.5877
- $n v = 2918$
- $\varepsilon = 300$
Zoom of Vortex Merger

Dots show peaks of each vortex blob; the blobs actually overlap

All blobs carry the same sign of vorticity.
Figure 6: Roll-up of a vortex sheet (Kelvin-Helmholtz instability) on a non-rotating sphere, computed by our Gaussian RBF vortex blob model.
Dense Matrix Blues

• The most accurate way to determine $\lambda_j$ in 

\[
\zeta(\vec{x}) \approx \sum_{j=1}^{N} \lambda_j \phi \left( \|\vec{x} - \vec{c}_j\|_2 \right) \quad \vec{x} \in \mathbb{R}^d
\]

is by SOLVING a DENSE INTERPOLATION MATRIX problem:

\[
\vec{G}\vec{\lambda} = \vec{f}
\]

\[
f_j = f(\vec{x}_j)
\]

\[
G_{jk} = \phi([\alpha/h](\vec{x}_j - \vec{x}_k))
\]

• COST

$O(N^3)$ operations by Cholesky factorization

$O(N^2)$ operations per iteration [In contrast, Fourier or Chebyshev interpolation costs $O(N \log_2(N))$]

• Vortex methods almost never invert matrices (except for Beale (1987)), but instead use QUASI-INTERPOLATION
Quasi-Interpolation

A QUASI-INTERPOLANT is an approximation in which the coefficients are equal to the values of $f(x)$ at the grid points.

$$f_{QI}(x) \equiv \sum_{j} f(x_j) \psi_j(x) \quad (3)$$

- Gaussians yield $O(h^2)$ error [$h$ is grid spacing]
- Beale and Madja (1982) showed that higher order could be obtained (in any number of dimensions) by multiplying Gaussians by polynomials of appropriate degree

Table 2:
This omits a multiplicative factor of $\alpha^d$ where $d$ is the spatial dimension. Note that $z = (\alpha/h)||x||$.

<table>
<thead>
<tr>
<th>$d$</th>
<th>$O(h^2)$</th>
<th>$O(h^4)$</th>
<th>$O(h^6)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\frac{1}{\sqrt{\pi}} \exp(-z^2)$</td>
<td>$\frac{1}{\sqrt{\pi}} \exp(-z^2)(\frac{3}{2} - z^2)$</td>
<td>$\frac{1}{\sqrt{\pi}} \exp(-z^2)(\frac{15}{8} - \frac{5}{2}z^2 + \frac{1}{2}z^4)$</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{1}{\pi} \exp(-z^2)$</td>
<td>$\frac{1}{\pi} \exp(-z^2)(2 - z^2)$</td>
<td>$\frac{1}{\pi} \exp(-z^2)(3 - 3z^2 + \frac{1}{2}z^4)$</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{1}{\pi^{3/2}} \exp(-z^2)$</td>
<td>$\frac{1}{\pi^{3/2}} \exp(-z^2)(\frac{5}{2} - z^2)$</td>
<td>$\frac{1}{\pi^{3/2}} \exp(-z^2)(\frac{35}{8} - \frac{7}{2}z^2 + \frac{1}{2}z^4)$</td>
</tr>
</tbody>
</table>
- Big accuracy penalty for quasi-interpolation
- Beale (1987) experimented with interpolation
- Goal: cheap quasi-interpolation versus costly interpolation.

Figure 7: Classical RBF interpolation (lower curve) compared with quasi-interpolation of 2d, 4th and 6th orders, all on a uniform grid. $f(x) = \exp(-x^2)$. $\alpha = 1/2$. 
**UNIFORM vs NON-UNIFORM GRIDS**

- Vortex method moves vortex centers with flow
- Initially UNIFORM grid becomes IRREGULAR
- Good news: RBFs OK
- Bad news: Accuracy penalty

Errors in Classical Interpolation: \( f=\exp(-20 \, x^2) \)  
Gaussian RBF, \( \alpha=2/3 \)

\[
x = x_{\text{uniform}} + (h/2) \cdot (1-2 \text{ random})
\]

Mapped: \( x = y + \sin(\pi \, y)/10 \)

Figure 8: Classical RBF interpolation using Gaussian
Grid-Tolerance

Theoretical study by Boyd-Bridge: perturbing a grid by (i) shifting one point or (ii) omitting one point entirely halves $\alpha$ to achieve a given saturation error.

Fig. shows the saturation error grows when one point is shifted by $sh$. 
Modified Quasi-Interpolation on a Non-Uniform Grid

Maz’ya & Schmidt show: if GRID is the IMAGE of a UNIFORM GRID by SMOOTH MAPPING, QI can be rendered high order AGAIN.

Suppose $x_j$ is the image of a uniform grid under the mapping function $m(x)$

$$x_j = m(jh)$$

$$f^Q \equiv \frac{\alpha}{\sqrt{\pi}} \sum_{j=-\infty}^{\infty} f(x_j) \phi \left( \frac{\alpha}{h \frac{dm}{dx}(jh)} \frac{x-x_j}{h \frac{dm}{dx}(jh)} \right)$$

- In higher dimensions, $\frac{dm}{dx}$ is replaced by the determinant of the Jacobian of the vector of mapping functions.
- Flow provides the mapping for vortex methods
- It is NOT necessary to explicitly construct the mapping; local finite difference approximations suffice
• Even a slight non-uniformity increases saturation error:

\[ x_j = jh + \left(\frac{1}{1000}\right) \sin(\pi jh) \]

• L. Barba and collaborators use RBF interpolation to regrid their quasi-interpolating Gaussian vortex blob method every 10 timesteps.

Figure 9: Black: modified 4th order QI. Green: unmodified QI. Red: \(4.4E4N^{-4}\), \(f=\text{inline}('\exp(-20*x.*x)','x');\) \(g\gamma=\text{inline}('x+0.001*\sin(pi*x}','x');\) \(x\map=g\gamma(x\text{unif}); x \in [-1, 1] \).
The Future

- Vortex-RBFs on the sphere is in progress
  Comparisons of quasi-interpolation of
  with standard interpolation
  Regridding: Non-uniform ⇒ uniform
  Adding/deleting vortex blobs
  Wang-Krasny treecode (fast summation)

- Extension from the barotropic vorticity equation to the shallow water equations.

- Long term strategy: RBFs as a limited area model (LAM) embedded in a global model
188. “ Sensitivity of RBF Interpolation on an Otherwise Uniform Grid with a Point Omitted or Slightly Shifted” with Lauren R. Bridge, Submitted to Applied Numerical Mathematics, (2009).