Adjoint Sensitivity Analysis for Two Layered Shallow Water Equations

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Organization of the talk

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- Adjoint Sensitivity Analysis for two layered shallow water equations
  - Shallow water approximation for two layered sea
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Motivation and Literature Survey

Why an Adjoint Equation?

Diffusion equation defined in the domain $x \in (0, L)$ is

$$D \frac{\partial^2 C}{\partial x^2} + f = 0, \ C(0) = C(L) = 0$$

Objective function:

$$J = \int_0^L r(x) C(x) \, dx$$

where $r = \delta(x - x_T)$.

Adjoint Equation

$$D \frac{\partial^2 C^*}{\partial x^2} + r = 0$$

Objective function:

$$J = \int_0^L f(x) C^*(x) \, dx$$

where $f = M \delta(x - x_s)$. 
Derivation

\[
\begin{align*}
\int_0^L C^* \left[ D \frac{\partial^2 C}{\partial x^2} + f \right] \, dx &= D \left[ C^* \frac{\partial C}{\partial x} \right]_0^L - D \left[ \frac{\partial C^*}{\partial x} C \right]_0^L \\
&\quad + \int_0^L \left[ CD \frac{\partial^2 C^*}{\partial x^2} + C^* f \right] \, dx = 0
\end{align*}
\]

\[
J = \int_0^L \left[ CD \frac{\partial^2 C^*}{\partial x^2} + C^* f \right] \, dx + \int_0^L rC \, dx
\]

\[
J = \int_0^L C \left[ D \frac{\partial^2 C^*}{\partial x^2} + r \right] \, dx + \int_0^L fC^* \, dx
\]
Shallow Water Equations

\[ \frac{\partial \eta}{\partial t} + \frac{\partial (Hu' + \bar{U}\eta)}{\partial x} = 0 \]

\[ \frac{\partial u'}{\partial t} + \frac{\partial (\bar{U}u' + g\eta)}{\partial x} = 0 \]

Objective Function

\[ J = \int_{0}^{T} \int_{0}^{L} r(\eta, u'; x, t) dx dt \]

\( t = \text{time; } x = \text{distance along the channel; } \eta = \text{flow perturbation; } u' = \text{discharge per unit width; } r = \text{user-defined measuring function} \)

\[ r = \frac{1}{2} [\eta(x_0, t) - \bar{\eta}(x_0)]^2 \delta(x - x_0) \]
Direct Sensitivity Method

- The basic problem is solved
- A reference value of the objective function $J_0$ is computed
- A flow variable is perturbed at the boundary at the selected perturbation time
- The associated objective function value is computed again
- This procedure is repeated for sufficiently many selections of time
- This yields the temporal evolution of the sensitivities
Adjoint Sensitivity Method
Derivation of 1D Adjoint Equations

- **Adjoint variables**: $\phi(x, t)$ and $\psi(x, t)$
- Multiply the continuity equation by $\phi(x, t)$ and the momentum equation by $\psi(x, t)$
- The sum of these two products is then integrated over space and time
- Flow control is desired
- Variation of the objective function with respect to $\eta$ and $u'$ is calculated
- Add these two variations
- Assign initial condition for the adjoint problem
- Assign boundary values for the adjoint variables
Sensitivity Equations

- Sensitivities required for subcritical flow control are given by \( \frac{\delta J}{\delta u'(0,t)} \) and \( \frac{\delta J}{\delta \eta(L,t)} \)

- Sensitivities required for supercritical flow control are given by \( \frac{\delta J}{\delta u'(0,t)} \) and \( \frac{\delta J}{\delta \eta(0,t)} \)

- To derive expressions for sensitivities in terms of the adjoint variables

- Initial condition for the basic problem
  \( \eta(x, 0) = \eta_0(x), u'(x, 0) = u'_0(x), \delta \eta(x, 0) = 0, \delta u'(x, 0) = 0 \)

- Initial conditions in the adjoint problem
  \( \phi(x, T) = 0 = \psi(x, T) \)

- Selection of boundary conditions for the adjoint problem is motivated by the boundary conditions of the basic problem

- Expressions are then derived for sensitivities to these boundary values
Shallow water equations and its Adjoint

\[
\frac{\partial \eta}{\partial t} + \bar{U} \frac{\partial \eta}{\partial x} + \bar{V} \frac{\partial \eta}{\partial y} + H \frac{\partial u'}{\partial x} + H \frac{\partial v'}{\partial y} = 0 \\
\frac{\partial u'}{\partial t} + \bar{U} \frac{\partial u'}{\partial x} + \bar{V} \frac{\partial u'}{\partial y} + g \frac{\partial \eta}{\partial x} = 0 \\
\frac{\partial v'}{\partial t} + \bar{U} \frac{\partial v'}{\partial x} + \bar{V} \frac{\partial v'}{\partial y} + g \frac{\partial \eta}{\partial y} = 0
\]

Adjoint

\[
\frac{\partial \phi}{\partial \tau} - \bar{U} \frac{\partial \phi}{\partial x} - \bar{V} \frac{\partial \phi}{\partial y} - g \frac{\partial \psi_1}{\partial x} - g \frac{\partial \psi_2}{\partial y} + \frac{\partial r}{\partial \eta} = 0 \\
\frac{\partial \psi_1}{\partial \tau} - \bar{U} \frac{\partial \psi_1}{\partial x} - \bar{V} \frac{\partial \psi_1}{\partial y} - H \frac{\partial \phi}{\partial x} + \frac{\partial r}{\partial u'} = 0 \\
\frac{\partial \psi_2}{\partial \tau} - \bar{U} \frac{\partial \psi_2}{\partial x} - \bar{V} \frac{\partial \psi_2}{\partial y} - H \frac{\partial \phi}{\partial y} + \frac{\partial r}{\partial v'} = 0
\]
Nonreflective Outflow Boundary

Adjoint boundary conditions:

\[ \psi(0, t) = 0 \] and

\[ \phi(L, t) + 2 \frac{u'(L, t)}{\eta(L, t)} \psi(L, t) = 0 \]

Sensitivities for the subcritical flow then are:

\[
\frac{\delta J}{\delta u'(0, t_p)} = \phi(0, t_p)
\]

\[
\frac{\delta J}{\delta \eta(L, t_p)} = \left[ \left( \frac{u'(L, t_p)}{\eta(L, t_p)} \right)^2 - g \eta(L, t_p) \right] \psi(L, t_p)
\]
A two layered sea of constant depth
Two layered Shallow water model

The two layered shallow water model is given by,

\[
\frac{\partial \eta_1}{\partial t} + \frac{\partial}{\partial x} [(\eta_1 - \eta_0)u_1] = 0
\]

\[
\frac{\partial u_1}{\partial t} + \frac{\partial}{\partial x} \left[ \frac{u_1^2}{2} + ga\eta_1 + gb\eta_2 \right] = 0
\]

for the lower layer and

\[
\frac{\partial \eta_2}{\partial t} + \frac{\partial}{\partial x} [(\eta_1 - \eta_0)u_1 + (\eta_2 - \eta_1)u_2] = 0
\]

\[
\frac{\partial u_2}{\partial t} + \frac{\partial}{\partial x} \left[ \frac{u_2^2}{2} + gb\eta_2 \right] = 0
\]

for the upper layer where \( a = \frac{(\rho_1 - \rho_2)}{\rho_0} \) and \( b = \frac{\rho_2}{\rho_0} \) and \( u_1 = u_1(x, t) \) and \( u_2 = u_2(x, t) \) are the velocity components, \( \eta_1 = \eta_1(x, t) \) and \( \eta_2 = \eta_2(x, t) \) are perturbations of the height field for lower and upper layers respectively. The problem is defined in limited domain ie., \( 0 \leq x \leq L \).
Barotropic and Baroclinic Modes

The linearized two layered shallow water model in normal form is given by,

\[
\frac{\partial \zeta}{\partial t} + \left( \frac{h + h'}{h} \right) \frac{\partial U}{\partial x} = 0
\]

\[
\frac{\partial U}{\partial t} + gh \frac{\partial \zeta}{\partial x} = 0
\]

\[
\frac{\partial \zeta'}{\partial t} + \frac{\partial U'}{\partial x} = 0
\]

\[
\frac{\partial U'}{\partial t} + \left( \frac{g\epsilon hh'}{h + h'} \right) \frac{\partial \zeta'}{\partial x} = 0
\]

This is in barotropic and baroclinic modes. Here, \( \epsilon = \frac{\rho' - \rho}{\rho} \).

Initial conditions for the Direct Problem:

\[
\zeta(x, 0) = \zeta_0(x); \quad \zeta'(x, 0) = \zeta'_0(x); \quad U(x, 0) = U_0(x); \quad U'(x, 0) = U'_0(x)
\]

\[
\delta \zeta(x, 0) = 0; \quad \delta \zeta'(x, 0) = 0; \quad \delta U(x, 0) = 0; \quad \delta U'(x, 0) = 0
\]
Sensitivities using Direct Sensitivity Method

Direct sensitivity of lower layer height variable with non-periodic BCs

\[-\log(\text{sensitivity of the height variable in the lower level})\]

\[\text{Direct sensitivity of lower layer height variable with non-periodic BCs}\]
Sensitivities using Direct Sensitivity Method

Direct sensitivity of lower layer velocity field with non-periodic BCs

\[ \log(\text{sensitivity of the velocity field in the lower level}) \]

size of the perturbation times

Direct sensitivity of lower layer velocity field with non-periodic BCs

\[ \log(\text{sensitivity of the velocity field in the lower level}) \]

size of the perturbation times
Sensitivities using Direct Sensitivity Method

Direct sensitivity of upper layer height field with non-periodic BCs

-\log(\text{sensitivity of the height field in the upper layer})

size of the perturbation times
Sensitivities using Direct Sensitivity Method

Direct sensitivity of upper layer velocity field with non-periodic BCs

- log(sensitivity of the velocity field in the upper layer)

size of the perturbation times
The Adjoint

The adjoint for the normal form of two layered shallow water equations (in barotropic and baroclinic modes) is given by,

\[
\frac{\partial \phi}{\partial \tau} - gh \frac{\partial \psi}{\partial x} + \frac{\partial r}{\partial \zeta} = 0
\]

\[
\frac{\partial \psi}{\partial \tau} - \left( \frac{h + h'}{h} \right) \frac{\partial \phi}{\partial x} + \frac{\partial r}{\partial U} = 0
\]

\[
\frac{\partial \phi'}{\partial \tau} - gh' \frac{\partial \psi'}{\partial x} + \frac{\partial r}{\partial \zeta'} = 0
\]

\[
\frac{\partial \psi'}{\partial \tau} - \left( \frac{h + h'}{h'} \right) + \frac{\partial r}{\partial U'} = 0
\]

Initial conditions for the Adjoint Problem:

\[
\phi(x, T) = 0; \quad \phi'(x, T) = 0; \quad \psi(x, T) = 0; \quad \psi'(x, T) = 0
\]
Standard Adjoint Problem Formalism

Boundary Conditions:

\[ \phi(L, t) = \zeta(L, t); \; \psi(0, t) = U(0, t); \; \phi'(L, t) = \zeta'(L, t); \; \psi'(0, t) = U'(0, t) \]

\[ \delta J = \int_0^T \left[ a \phi(L, t) \delta U + b \psi(L, t) \delta \zeta + c \phi'(L, t) \delta U' + d \psi'(L, t) \delta \zeta' \right] dt \]

Sensitivities:

\[ \frac{\delta J}{\delta U(0, t_p)} = -a \phi(0, t_p); \quad \frac{\delta J}{\delta U'(0, t_p)} = -c \phi'(0, t_p) \]

\[ \frac{\delta J}{\delta \zeta(L, t_p)} = b \psi(L, t_p); \quad \frac{\delta J}{\delta \zeta'(L, t_p)} = d \psi'(L, t_p) \]
Sensitivities using Adjoint Sensitivity Method
Sensitivities using Adjoint Sensitivity Method
Sensitivities using Adjoint Sensitivity Method
Sensitivities using Adjoint Sensitivity Method
Transformation of Direct Equations

\[ \frac{\partial W}{\partial t} + A \frac{\partial W}{\partial x} = 0 \]

where

\[ W = \begin{pmatrix} \zeta \\ U \\ \zeta' \\ U' \end{pmatrix}, \quad A = \begin{bmatrix} 0 & a & 0 & 0 \\ b & 0 & 0 & 0 \\ 0 & 0 & 0 & c \\ 0 & 0 & d & 0 \end{bmatrix} \]

with \( a = \frac{(h+h')}{h}, \ b = gh, \ c = 1 \) and \( d = \frac{g\epsilon hh'}{h+h'} \).

The eigenvalues of \( A \) are given by,
\[ \lambda_1 = -\sqrt{ab}, \ \lambda_2 = \sqrt{ab}, \ \lambda_3 = -\sqrt{cd}, \ \lambda_4 = \sqrt{cd}. \]
Transformation of Direct Equations

The matrix of right and left eigenvectors are then given by,

\[ RE = \begin{pmatrix} \sqrt{\frac{a}{b}} & 1 & 0 & 0 \\ \sqrt{\frac{a}{b}} & 1 & 0 & 0 \\ 0 & 0 & -\sqrt{\frac{c}{d}} & 1 \\ 0 & 0 & \sqrt{\frac{c}{d}} & 1 \end{pmatrix} \]

\[ LE = \frac{1}{2} \begin{pmatrix} -\sqrt{\frac{b}{a}} & \sqrt{\frac{b}{a}} & 0 & 0 \\ \sqrt{\frac{b}{a}} & 1 & 1 & 0 \\ 0 & 0 & -\sqrt{\frac{d}{c}} & \sqrt{\frac{d}{c}} \\ 0 & 0 & 1 & 1 \end{pmatrix} \]
Characteristics for Direct Equations

\[ C^1 : \frac{d\zeta}{dt} - \frac{dU}{dt} = 0 \quad \text{along} \quad \frac{dx}{dt} = -\sqrt{ab} \]

\[ C^2 : \frac{d\zeta}{dt} + \frac{dU}{dt} = 0 \quad \text{along} \quad \frac{dx}{dt} = \sqrt{ab} \]

\[ C^3 : \frac{d\zeta'}{dt} - \frac{dU'}{dt} = 0 \quad \text{along} \quad \frac{dx}{dt} = -\sqrt{cd} \]

\[ C^4 : \frac{d\zeta'}{dt} + \frac{dU'}{dt} = 0 \quad \text{along} \quad \frac{dx}{dt} = \sqrt{cd} \]
Transformation of Adjoint Equations

\[ \Phi_\tau + \tilde{A} \Phi_x + \tilde{R} = 0 \]

where

\[ \Phi = \begin{pmatrix} \phi \\ \psi \\ \phi' \\ \psi' \end{pmatrix}, \quad A = \begin{bmatrix} 0 & -b & 0 & 0 \\ -a & 0 & 0 & 0 \\ 0 & 0 & 0 & -d \\ 0 & 0 & -c & 0 \end{bmatrix}, \quad \tilde{R} = \begin{pmatrix} \frac{\partial r}{\partial \zeta} \\ \frac{\partial r}{\partial U} \\ \frac{\partial \zeta'}{\partial r} \\ \frac{\partial U'}{\partial r} \end{pmatrix} \]

with \( a = \frac{(h + h')}{h}, b = gh, c = 1 \) and \( d = \frac{-g\epsilon hh'}{h + h'} \). The eigenvalues of \( \tilde{A} \) are given as,

\[ \lambda_1 = -\sqrt{g(h + h')}, \lambda_2 = \sqrt{g(h + h')}, \lambda_3 = -\sqrt{\frac{g\epsilon hh'}{h + h'}}, \lambda_4 = \sqrt{\frac{g\epsilon hh'}{h + h'}}. \]
Transformation of Adjoint Equations

The matrix of right and left eigenvectors are then given as,

\[
RE = \begin{pmatrix}
\sqrt{\frac{b}{a}} & 1 & 0 & 0 \\
-\sqrt{\frac{b}{a}} & 1 & 0 & 0 \\
0 & 0 & \sqrt{\frac{d}{c}} & 1 \\
0 & 0 & -\sqrt{\frac{d}{c}} & 1
\end{pmatrix}
\]

and

\[
LE = \frac{1}{2} \begin{pmatrix}
\sqrt{\frac{a}{b}} & -\sqrt{\frac{a}{b}} & 0 & 0 \\
1 & 1 & 0 & 0 \\
0 & 0 & \sqrt{\frac{c}{d}} & -\sqrt{\frac{c}{d}} \\
0 & 0 & 1 & 1
\end{pmatrix}
\]
Characteristics for Adjoint Equations

\[ C^1 : \frac{d\phi}{d\tau} - \frac{d\psi}{d\tau} + \frac{\partial r}{\partial \zeta} - \frac{\partial r}{\partial U} = 0 \quad \text{along} \quad \frac{dx}{d\tau} = -\sqrt{ab} \]

\[ C^2 : \frac{d\phi}{d\tau} + \frac{d\psi}{d\tau} + \frac{\partial r}{\partial \zeta} + \frac{\partial r}{\partial U} = 0 \quad \text{along} \quad \frac{dx}{d\tau} = \sqrt{ab} \]

\[ C^3 : \frac{d\phi'}{d\tau} - \frac{d\psi'}{d\tau} + \frac{\partial r}{\partial \zeta'} + \frac{\partial r}{\partial U'} = 0 \quad \text{along} \quad \frac{dx}{d\tau} = -\sqrt{cd} \]

\[ C^4 : \frac{d\phi'}{d\tau} + \frac{d\psi'}{d\tau} + \frac{\partial r}{\partial \zeta'} + \frac{\partial r}{\partial U'} = 0 \quad \text{along} \quad \frac{dx}{d\tau} = \sqrt{cd} \]
Characteristic Adjoint Formalism

To obtain sensitivity expressions for $U$ and $U'$ at upstream and downstream, we substitute the above characteristics in $\delta J$, we get

\[
\frac{\delta J}{\delta U(0, t_p)} = -\frac{h + h'}{h} \phi(0, t_p) + gh \psi(0, t_p)
\]

\[
\frac{\delta J}{\delta U'(0, t_p)} = -\phi'(0, t_p) + \frac{g \epsilon hh'}{h + h'} \psi'(0, t_p)
\]

\[
\frac{\delta J}{\delta U(L, t_p)} = \frac{h + h'}{h} \phi(L, t_p) + gh \psi(L, t_p)
\]

\[
\frac{\delta J}{\delta U'(L, t_p)} = \phi'(L, t_p) + \frac{g \epsilon hh'}{h + h'} \psi'(L, t_p)
\]
Characteristic Adjoint Formalism

Similarly by substituting the above characteristics, we can also obtain the sensitivity expressions for height field at both the upstream and downstream as,

\[
\frac{\delta J}{\delta \zeta(0, t_p)} = \frac{h + h'}{h} \phi(0, t_p) - gh\psi(0, t_p)
\]

\[
\frac{\delta J}{\delta \zeta'(0, t_p)} = \phi'(0, t_p) - \frac{g\epsilon h'h}{h + h'} \psi'(0, t_p)
\]

and

\[
\frac{\delta J}{\delta \zeta(L, t_p)} = \frac{h + h'}{h} \phi(L, t_p) + gh\psi(L, t_p)
\]

\[
\frac{\delta J}{\delta \zeta'(L, t_p)} = \phi'(L, t_p) + \frac{g\epsilon hh'}{h + h'} \psi'(L, t_p)
\]
Summary

Characteristic adjoint sensitivity analysis for transient motion in a two-layered sea

- To determine how a specific condition is dependent on many parameters which influence it
- Adjoint sensitivity method is more efficient than Direct sensitivity method
- Fast and accurate tool for computing the sensitivities
- Characteristic adjoint sensitivity method gives better results than Standard adjoint sensitivity method
- Boundary conditions derived for linear approximation can be used for the nonlinear problem
- Numerical work in progress