Tracer Advection using Characteristic Discontinuous Galerkin

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Overview

- Generalize of Prather's moment method (JGR 1986) to unsplit advection on general mesh topologies
- Take advantage of existing Lagrange-remap algorithms (Lipscomb & Ringler, MWR 2005)
- Resulting method: Characteristic Discontinuous Galerkin (CDG), which is based on space-time discontinuous Galerkin
- Ultimate goal: Minimize spurious diapycnal mixing (e.g., Griffies et al 2000)
  - Here, our approach is to increase the order-of-accuracy

Outline

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Some Past Work on Explicit DG

- Explicit Runge-Kutta DG:
  - Cockburn & Shu 1989
  - Levy, Nair, Tufo 2007
  - Giraldo & Warburton 2008
  - Giraldo & Restelli 2008
  - ...
- Explicit Space-time DG:
  - Lowrie 1996
  - Falk & Richter 1999
  - Palaniappan, Haber, Jerrard 2004

Tracer Advection

- Given $\vec{u}(\vec{x}, t)$, solve

$$\partial_t \rho + \nabla \cdot (\rho \vec{u}) = 0 \,, \tag{1a}$$
$$\partial_t (\rho T) + \nabla \cdot (\rho T \vec{u}) = 0 \,. \tag{1b}$$

- Implies

$$\frac{DT}{Dt} = 0 \,, \qquad \frac{D}{Dt} \equiv \partial_t + \vec{u} \cdot \nabla \,.$$

- To ensure conservation, we discretize the system (1).

Some manipulations...

- Begin with

$$\partial_t(\rho T) + \nabla \cdot (\rho T \vec{u}) = 0\,.$$

- Multiply by a smooth function $\phi_{k,i}(\vec{x}, t)$ and rearrange:

$$\partial_t(\phi_{k,i}\rho T) + \nabla \cdot (\phi_{k,i}\rho T \vec{u}) = \rho T \frac{D\phi_{k,i}}{Dt}\,.$$

- Weak form over a control volume (element) $\Omega_k \times [t^n, t^{n+1}]$:

$$\int_{\Omega_k} \left[(\phi_{k,i}\rho T)^{n+1} - (\phi_{k,i}\rho T)^{n} \right] d\Omega + \int_{t^n}^{t^{n+1}} \oint_{\partial\Omega_k} \phi_{k,i}\rho T \vec{u} \cdot \vec{n}\, ds dt = \int_{t^n}^{t^{n+1}} \int_{\Omega_k} \rho T \frac{D\phi_{k,i}}{Dt}\, d\Omega dt\,.$$

Space-Time DG

- Over each $\Omega_k \times (t^n, t^{n+1}]$, expand solution as

$$(\rho T)(\vec{x}, t) = \sum_{j=1}^{N} c_{k,j}^{n+1} \phi_{k,j}(\vec{x}, t) , \quad \vec{x} \in \Omega_k \, , \; t \in (t^n, t^{n+1}] \, .$$

- Discontinuous at element boundaries
- Alternatively, expand ρ and T separately
- For each $\phi_{k,i}(\vec{x}, t)$, $i = 1..N$, solve weak form:

$$\int_{\Omega_k} \left[(\phi_{k,i} \rho T)^{n+1} - (\phi_{k,i} \rho T)^n \right] d\Omega + \int_{t^n}^{t^{n+1}} \oint_{\partial\Omega_k} \phi_{k,i} \rho T \vec{u} \cdot \vec{n} \, ds dt = \int_{t^n}^{t^{n+1}} \int_{\Omega_k} \rho T \frac{D\phi_{k,i}}{Dt} \, d\Omega dt \, .$$

  which gives an equation for each $\{c_{k,j}^{n+1}\}_{j=1}^{N}$
- Boundary terms upwinded based on space-time characteristics

Semi-Discrete DG

- On each element Ω_k, expand solution as

$$(\rho T)(\vec{x}, t) = \sum_{j=1}^{N} c_{k,j}(t)\beta_{k,j}(\vec{x})\,, \quad \vec{x} \in \Omega_k\,.$$

- For each $\beta_{k,i}(\vec{x})$, $i = 1..N$, write the tracer advection equation as

$$\int_{\Omega_k} \beta_{k,i}\partial_t(\rho T)\, d\Omega + \oint_{\partial\Omega_k} \beta_{k,i}\rho T\vec{u}\cdot\vec{n}\, ds = \int_{\Omega_k} \rho T\vec{u}\cdot\nabla\beta_{k,i}\, d\Omega\,.$$

  which gives an equation for each $\{c_{k,j}(t)\}_{j=1}^{N}$.
- Evolve $c_{k,j}(t)$ using Runge–Kutta (RKDG).
- Basis polynomials of order-p: CFL $< 1/(2p+1)$ for small p, "stages = order."

Space-Time DG vs. RKDG

- Advantages of Space-Time DG:
  - Can obtain same order-of-accuracy in both space and time
  - Independent of order-of-accuracy, explicit methods are stable for CFL $\equiv (|\vec{u}|\Delta t/\Delta x)_{\mathrm{max}} < 1$
- Disadvantages:
  - Complicated to code
  - Computational cost generally higher
  - More unknowns per element than semi-discrete methods
  - Enforcement of positivity or monotonicity less clear

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- Answer: For the most part, yes.

Outline

1. Review of Discontinuous Galerkin
2. Characteristic Discontinuous Galerkin (CDG)
3. Results

More manipulations...

Replace

$$\partial_t(\phi_{k,i}\rho T) + \nabla \cdot (\phi_{k,i}\rho T\vec{u}) = \rho T \frac{D\phi_{k,i}}{Dt}\,,$$

with the system

$$\partial_t(\phi_{k,i}\rho T) + \nabla \cdot (\phi_{k,i}\rho T\vec{u}) = 0\,, \tag{2a}$$

$$\frac{D\phi_{k,i}}{Dt} = 0\,. \tag{2b}$$

- For $\vec{u} = \text{const.}$, eq. (2b) $\Rightarrow \phi_{k,i}(\vec{x}, t) \equiv \mathcal{F}(\vec{x} - \vec{u}t)$
- Because we seek $\frac{DT}{Dt} = 0$, eq. (2b) might seem redundant. However,
  - (2a) maintains conservation
  - (2b) is local to each element and can be solved once for all tracers

Characteristic Discontinuous Galerkin (CDG)

- For a polygon Ω_k with faces $\partial\Omega_{k,f}$,

$$\int\limits_{\Omega_k} \left[(\phi_{k,i}\rho T)^{n+1} - (\phi_{k,i}\rho T)^n \right] d\Omega + \sum_f \int\limits_{t^n}^{t^{n+1}} \int\limits_{\partial\Omega_{k,f}} \phi_{k,i}\rho T \vec{u}\cdot\vec{n}\, dsdt = 0\,.$$

- In this study, we solve the equivalent form

$$\int\limits_{\Omega_k} \left[(\phi_{k,i}\rho T)^{n+1} - (\phi_{k,i}\rho T)^n \right] d\Omega + \sum_f \int\limits_{\Omega'_{k,f}} \left(\phi_{k,i}\rho T\right)^n d\Omega = 0\,,$$

  where $(\Omega'_{k,f}, t^n)$ is the Lagrangian pre-image of the face $\partial\Omega_{k,f} \times [t^n, t^{n+1}]$.
- Need to define $\phi_{k,i}(\vec{x}, t)$

Solving $D\phi_{k,i}/Dt = 0$

- For each time-level n, on element Ω_k, let

$$(\rho T)(\vec{x}, t^n) = \sum_{j=1}^{N} c_{k,j}^n \beta_{k,j}(\vec{x}) \,, \quad \vec{x} \in \Omega_k \,.$$

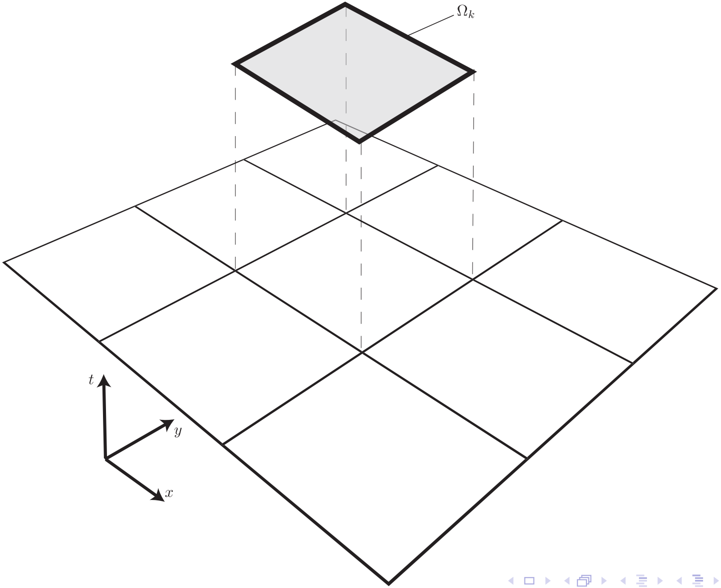
- For a given time interval $t^n \le t \le t^{n+1}$, we have $\phi_{k,i}(\vec{x}, t) = \beta_{k,i}(\vec{\Gamma}(\vec{x}, t))$, where

$$\begin{aligned}\vec{\Gamma}(\vec{x}, t) &= \vec{x} + \int_{t}^{t^{n+1}} \vec{u}(\vec{\Gamma}(\vec{x}, \xi), \xi)\, d\xi \\ &= \vec{x} + (t^{n+1} - t)\vec{u} \,, \qquad \text{for } \vec{u} = \text{const.}\end{aligned}$$

- Integration of characteristics needed once for ALL tracers.

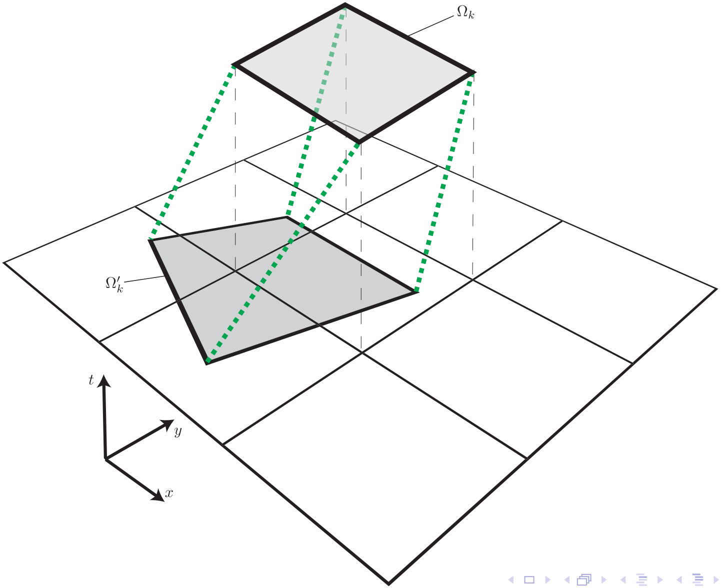
CDG on a Cartesian Mesh

Quest: Find polynomial representation of solution in center cell at new time level.



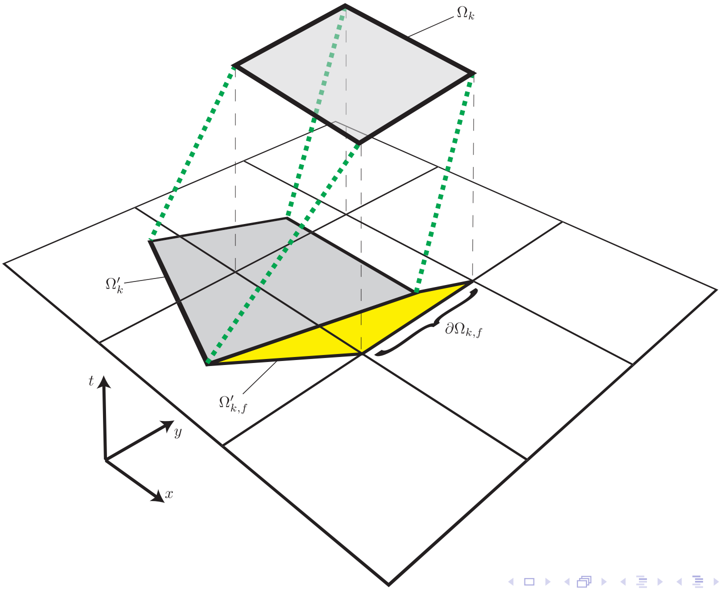
"Semi-Lagrangian" Step

Trace characteristics at each node from t^{n+1} to t^n (use RK4)



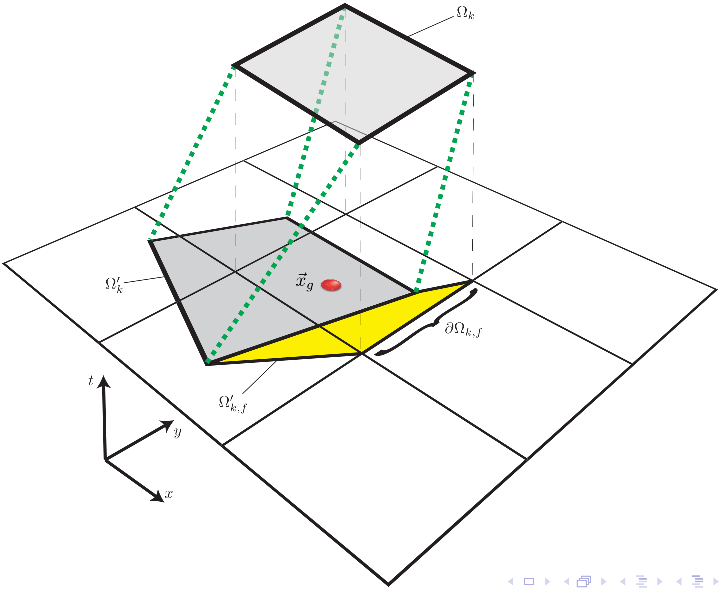
Find Lagrangian pre-image for each face...

...and break into triangles; see Lipscomb & Ringler (MWR 2005)



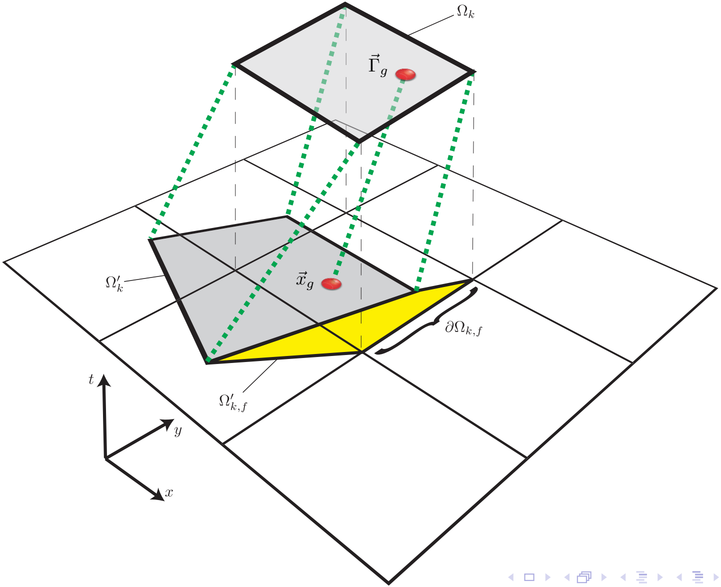
Evaluate each integral with quadrature

Below is an example quadrature point, $\vec{x}_g$



At each quadrature point, trace characteristics...

... from t^n to t^{n+1} to determine $\phi(\vec{x}_g, t^n) = \beta(\vec{\Gamma}_g)$



Local Linear System for CDG

- CDG solves

$$\int_{\Omega_k} \left[(\phi_{k,i}\rho T)^{n+1} - (\phi_{k,i}\rho T)^{n} \right] d\Omega + \sum_f \int_{\Omega'_{k,f}} \left(\phi_{k,i}\rho T \right)^n d\Omega = 0 \,.$$

- Reduces to a local $N \times N$ system on each element-k:

$$M_{i,j} c_{k,j}^{n+1} = f_{k,i}^{n} \,.$$

- Because $\phi_{k,i}(\vec{x}, t^{n+1}) \equiv \beta_{k,i}(\vec{x})$,

$$M_{i,j} = \int_{\Omega_k} \beta_{k,i}(\vec{x})\beta_{k,j}(\vec{x})\, d\Omega \,.$$

- Same form as each stage of RKDG.

CDG ⇒ L^2-minimization

- CDG solves

$$\int_{\Omega_k} \left[(\phi_{k,i}\rho T)^{n+1} - (\phi_{k,i}\rho T)^{n} \right] \, d\Omega + \sum_f \int_{\Omega'_{k,f}} \left(\phi_{k,i}\rho T \right)^n \, d\Omega = 0 \,.$$

- If exact integration is used, then this may be written as

$$\int_{\Omega_k} \left(\phi_{k,i}\rho T \right)^{n+1} \, d\Omega = \int_{\Omega'_k} \left(\phi_{k,i}\rho T \right)^{n} \, d\Omega \,,$$

where (Ω'_k, t^n) is the Lagrangian pre-image of (Ω_k, t^{n+1})

- Equivalent to L^2-minimization:

$$\min_{c_{k,i}^{n+1}} \int_{\Omega'_k} \left[(\rho T)(\vec{\Gamma}(\vec{x}, t^{n+1}))J(\vec{x}) - (\rho T)(\vec{x}, t^n) \right]^2 \, d\Omega \,,$$

where $J = |d\Omega_k / d\Omega'_k|$.

Properties of CDG(p)

- CDG(p) uses a polynomial basis of order-p, with $p \geq 0$.
- Requires a method for integration on each Lagrangian pre-image and evaluation of characteristic trajectories.
  - For general mesh topologies, can use incremental remap method of Lipscomb & Ringler (MWR 2005)
  - $\Rightarrow$ stable for CFL < 1
- Locally conservative
- At a fixed CFL, error is typically $O(\Delta x^{p+1})$ in space *and* time
  - But "quasi-accurate:" If pre-image is non-polygonal, then current remap limits overall accuracy to $O(\Delta x^2)$.
- Parallelizes well with a *single* communication per Δt
  - RKDG communicates at each RK stage, but only data at face quadrature points

CDG(p): Relationship to Other Methods

- In 1-D with mass coordinates (or ρ, $\vec{u}$ constant):
  - CDG(0) is equivalent to first-order upwind
  - CDG(1) is equivalent to:
    - ★ Van Leer's Scheme III (JCP 1977, "exact evolution with L^2-projection")
    - ★ Russell & Lerner's method (JAM 1981)
  - CDG(2) is equivalent to:
    - ★ Van Leer's Scheme IV (JCP 1977)
    - ★ Prather's method (JGR 1986)
    - ★ Piecewise-Parabolic Boltzmann (PPB) (Woodward 1986)
- Can be viewed as the following extensions to Prather's method:
  - Any $p \geq 0$ (Prather: $p = 2$)
  - General mesh topologies (Prather: Cartesian)
  - Dimensionally unsplit (Prather: split)
  - Triangle or diamond basis truncation (Prather: triangle)

Outline

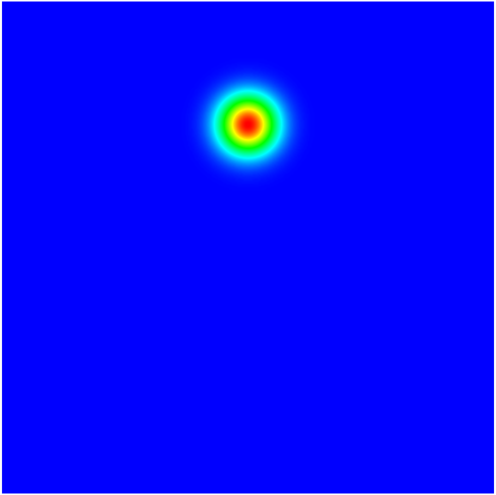
Common Properties of Test Cases

- ρ constant
- 2-D unit square, doubly periodic
- Cartesian mesh, $\Delta x = \Delta y$
- CFL $= 0.8$
- CDG(p) used tensor-product Legendre polynomials with triangle truncation

Solid-Body Rotation of a Gaussian Bump

Gaussian bump rotates about center of domain.

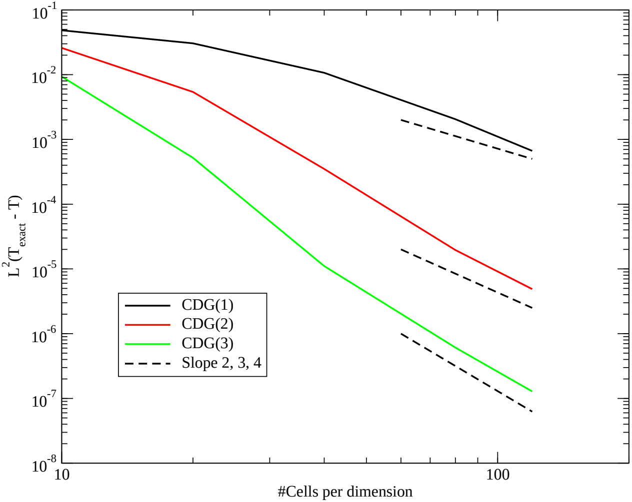
$t = 0$ and $t = 1$



Errors for Solid-Body Rotation of a Gaussian Bump

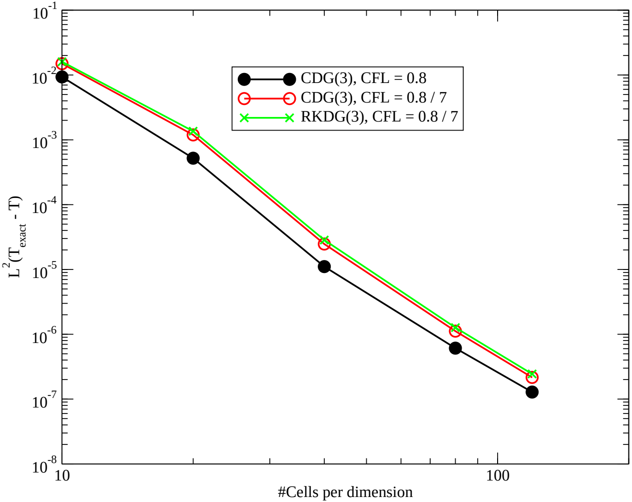
After 1 rotation

In this case, each cell's Lagrangian pre-image is a polygon.



CDG vs. RKDG (RKDG using RK4 in time)

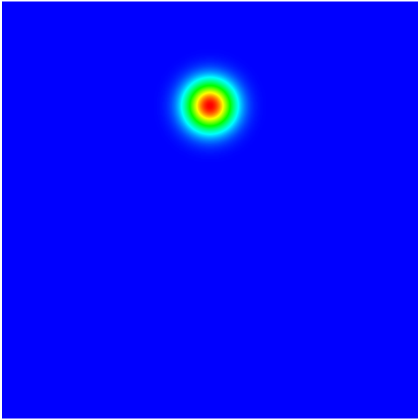
CDG(3) CFL limit is 7 times that of RKDG(3)



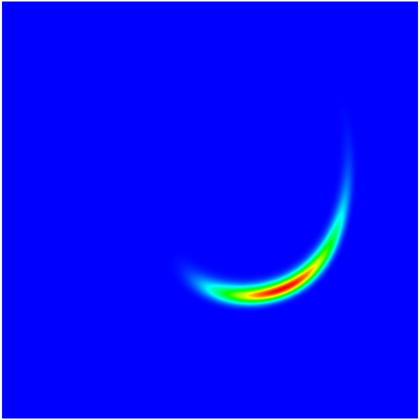
Deformation of a Gaussian Bump

Stream function: $\psi(x, y, t) = \cos(\pi t/2)\sin^2(\pi x)\sin^2(\pi y)/\pi$. Compute errors at $t = 2$.

$t = 0$ and $t = 2$



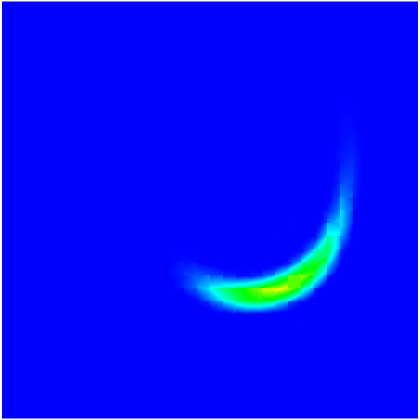
$t = 1$



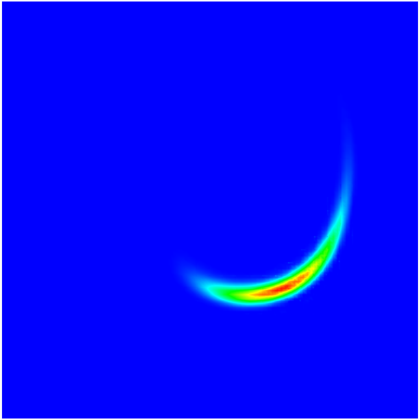
Sample Results at $t = 1$

32×32 Mesh, exact $T_{max} = 1$. Both methods used the same Δt (CFL $= 0.8$)

CDG(1), $T_{max} = 0.7997$



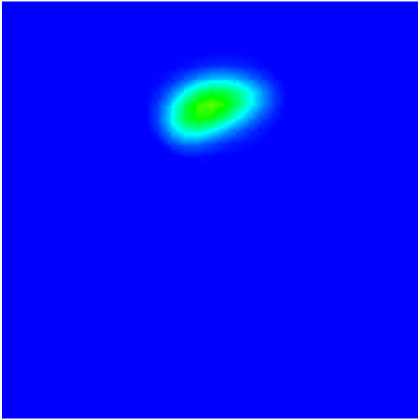
CDG(3), $T_{max} = 1.0170$



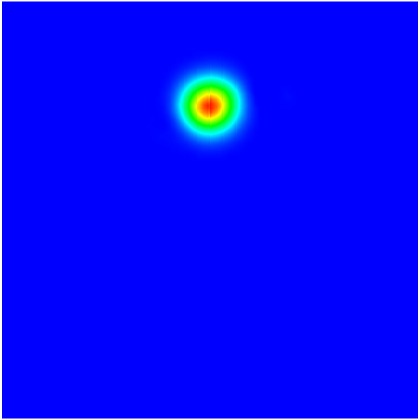
Sample Results at $t = 2$

32×32 Mesh, exact $T_{\text{max}} = 1$. Approximately 4 cells across initial Gaussian.

CDG(1), $T_{\text{max}} = 0.6080$



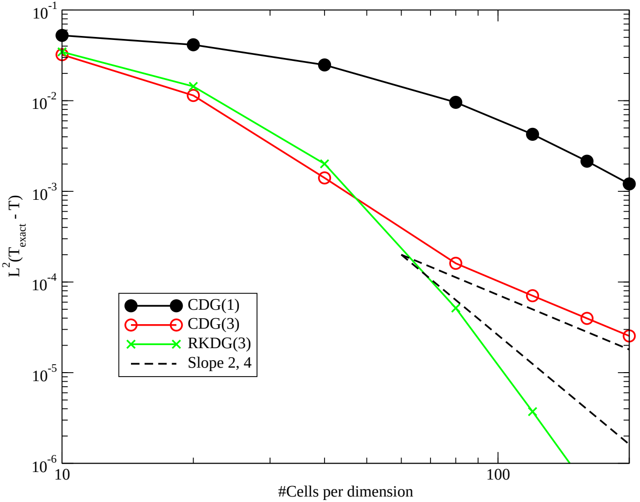
CDG(3), $T_{\text{max}} = 0.9872$



CDG(2), $T_{\text{max}} = 0.8685$

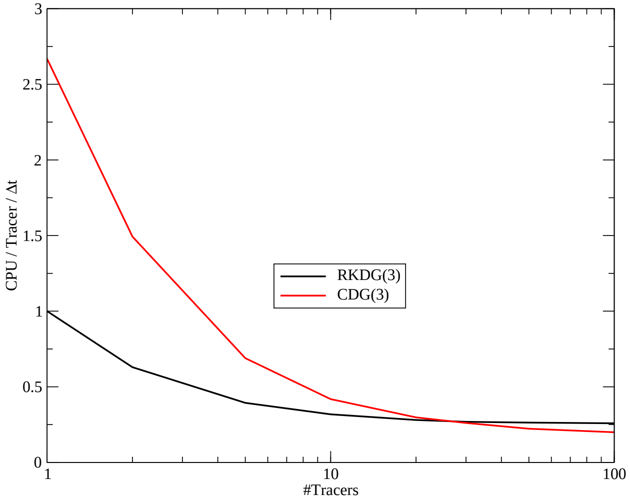
Errors for Deformation of a Gaussian Bump

Lagrangian pre-image non-polygonal ⇒ CDG accuracy limited to 2nd-order. RKDG maintains accuracy.



Scaling of CPU Time with Number of Tracers

Results normalized by RKDG(3) time for 1 tracer



Summary and Future Work

Summary:

- At a fixed CFL, CDG(p) with incremental remap is
  - stable for CFL < 1
  - $O(\Delta x^{p+1})$ accurate in space and time whenever pre-image is a polygon; otherwise, $O(\Delta x^2)$
- Majority of computational work independent of number of tracers
- Van Leer IV, Prather, PPB, ... $\Rightarrow$ CDG(2)

Future work:

- Monotonicity, positivity
- Couple with fluid models
- Other meshes (triangulations, Voronoi)
- Other geometries (*e.g.*, on the sphere)