A New Preconditioning Strategy for a Spectral-element-based Magnetohydrodynamics Solver

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Outline

- Motivation
- Equations & discretization
- Maintaining continuity; adaptivity
- Explicit MHD
- Preconditioning strategy
- ORAS preconditioner
- Coarse correction
- Conclusions & future work
MHD & Hydro Turbulence: Interaction of structures with ambient turbulent fluid & with boundaries

• Phenomenological & fundamental:

Trace

Kelvin-Helmholtz rolls
In match smoke
Geophysics-Astrophysics Spectral-element Adaptive Refinement (GASpAR)

- Object-oriented framework for solving PDEs on adaptive grids
- Uses tensor product form for multi-dimensional operators (hence, matrix-matrix- BLAS-3-products)

- Equations are derived from standard interface: advection-diffusion, Navier-Stokes, MHD
- Adaptive grid mechanics independent of equations

Available at: http://www.image.ucar.edu/TNT/Software/GASpAR
MHD Equations: Implemented in GASpAR

Magnetohydrodynamics:

\[
\begin{align*}
\partial_t u + u \cdot \nabla u &= -\nabla p + j \times b + \nu \nabla^2 u \\
\partial_t b &= \nabla \times (u \times b) + \eta \nabla^2 b \\
\nabla \cdot u &= 0, \quad \nabla \cdot b &= 0
\end{align*}
\]

Elsasser (1950) form:

\[
\begin{align*}
\partial_t Z^\pm + Z^\mp \cdot \nabla Z^\pm + \nabla p - \nu^\pm \nabla^2 Z^\pm - \nu^\mp \nabla^2 Z^\mp &= 0 \\
\nabla \cdot Z^\pm &= 0
\end{align*}
\]

Via definitions:

\[ Z^\pm = u \pm b \quad \nu^\pm = \frac{1}{2} (\nu \pm \eta) \]
Discretization via SEM method (Patera 1984)

• Problem well posed using spaces:
  \[ U_\gamma := \left\{ w = \sum_{\mu=1}^d w^\mu e^\mu \mid w^\mu \in H^1(D), \forall \mu \text{ and } w = \gamma \text{ on } \partial D \right\} \]

• Discrete spaces:
  \[
  \begin{align*}
  Z^\pm & \in U^N = U_{Z(0)} \bigcap P_N, \\
  \zeta^\pm & \in U_0^N = U_0 \bigcap P_N, \\
  p, q & \in Y^{N-2} = L_2(D) \bigcap P_{N-2}
  \end{align*}
  \]
  \[ P_N = P_{N-2} \]

• Discrete problem:
  \[
  \begin{align*}
  \langle \zeta^\pm, \partial_t Z^\pm \rangle_{GL} + \langle \zeta^\pm, C^\pm Z^\pm \rangle_{GL} - \frac{1}{\rho_0} \langle p, \nabla \cdot \zeta^\pm \rangle_{GL} \\
  = -\nu^\pm \sum_{\mu=1}^d \langle \partial_\mu \zeta^\pm, \partial_\mu Z^\pm \rangle_{GL} \\
  = \langle g, \nabla \cdot Z^\pm \rangle_{GL} = 0,
  \end{align*}
  \]

\[ C^\pm := Z^\pm \cdot \vec{\nabla} \]

Expand using GL or G polynomials…
Discrete staggered grid
Conforming continuity: Associate local and global dofs

\[
u = \begin{pmatrix} u_0 \\ \vdots \\ u_{17} \end{pmatrix} = \begin{pmatrix} u_{0,1} \\ \vdots \\ u_{8,1} \\ u_{0,2} \\ \vdots \\ u_{8,2} \end{pmatrix} = A \begin{pmatrix} u_{g,0} \\ \vdots \\ u_{g,14} \end{pmatrix} \]
Nonconforming continuity (1)
Nonconforming continuity (2): Interpolation implied

\[ u = \begin{pmatrix} u_0 \\ \vdots \\ u_{26} \end{pmatrix} = \Phi \begin{pmatrix} u_{0,1} \\ \vdots \\ u_{5,3} \end{pmatrix} = A \begin{pmatrix} u_{g,0} \\ \vdots \\ u_{g,18} \end{pmatrix} \]
Mortar Data Structures: Perform interpolations

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Adaptivity: IConnAMR

- Applies ‘forest of oct-trees’, weak data structures
- Uses a $2^d:1$ isotropic refinement decomposition
- Employs voxel database (VDB) to locate element neighbors and to set mortar properties
- Variety of a-posteriori refinement criteria; also user-defined
- 2-d and 3-d

Rosenberg, Fournier, Fischer, Pouquet,
Advection-diffusion: Adaptive 3-d linear advection
MHD Discretization

• Semi-discrete equations

\[
\frac{d\hat{Z}^\pm_j}{dt} = -MC^B j \hat{Z}^\pm_j + D^\pm_j \hat{p}^\pm - \nu_\perp j \hat{L}Z^\pm_j - \nu_\parallel j \hat{L}Z^\parallel_j
\]

\[
\hat{D}^j \hat{Z}^\pm_j = 0,
\]

• DNS==>one-step explicit time discretization:

\[
\hat{Z}^\pm_j \rightarrow \hat{Z}^\pm_{j+1} = \hat{Z}^\pm_j - \frac{1}{k} \Delta t \ M^{-1} (MC^B \hat{Z}^\pm_j - D^\pm_j \hat{p}^\pm + \nu_\perp j \hat{L}Z^\pm_j + \nu_\parallel j \hat{L}Z^\parallel_j).
\]

• Apply divergence constraint to discretized system:

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Pseudo-Poisson operator

\[ D_j^T M^{-1} D_j T \hat{\rho}^{\pm} = D_j^T g_j^{\pm} \]

Inhomogeneity:

\[ g_j^{\pm} = \frac{1}{k} \Delta t \ M^{-1} \left( M C^{\mp} Z_j^{\pm} + \nu \pm L Z_j^{\mp} \right) - Z_j^{\pm, n} \]

Communication is hidden in Stokes operators:

\[ D_j \rightarrow D_j \Phi A \]

This operator also appears in Navier-Stokes from a Schur decomposition of discretized equations accurate to second order in \( \Delta t \).
Orszag-Tang (OT) Problem: SEM AMR has excellent accuracy on challenging problem

- Current Density
- Vorticity


For OT, see: JFM 90, 129, 1979
OT SEM convergence: Truncation order matters!

Black: pseudo-spectral
Red: p=8
Blue: p=6
Green: p=4
Cyan: p=3
Iteration count scaling with Re in OT: Block Jacobi

Preconditioning clearly required!
Preconditioning strategy

Precondition the following pseudo-Laplacian operator:

\[ A \equiv D_j M^{-1} D_j^T \]

• Restricted Additive Schwarz (RAS), begin with conforming grids

• Apply results of St-Cyr, et al. (2007) to optimize Q1 blocks

• Use multilevel idea of Fischer (1997)
  Use equivalence between Q1 and SEM

RAS: Tiling extended grid

- Variable overlap
- Assemble 1d FE mass & stiffness matrices between nodes (Q1)
- Allow for communication of corner data

Fischer, JCP, 133:84 (1997)
RAS FEM Operator Assembly

- Use linear shape functions to build 1-d mass and stiffness operators;
- Do DSS.
- Construct 2- and 3-d Laplacian operators using tensor products.
RAS optimization

• Given a Schwarz method transform to optimized versions (RAS case)

\[ P_{RAS}^{-1} = \sum_{i=1}^{K} \tilde{R}_i^T A_i^{-1} R_i \quad ? \quad P_{ORAS}^{-1} = \sum_{i=1}^{K} \tilde{R}_i^T \tilde{A}_i^{-1} R_i \]

• St-Cyr et al. (2007) find under which conditions this is possible
• Conditions in the RAS case:

\[ B_{jk} R_k \tilde{R}_m = 0, \quad m \neq k. \]

In this case, transmission operator must have 2 points
Numerical Experiments

- Native stand-alone pseudo-Poisson solver
- Periodic boundary conditions
- Use BiCGStab
- Krylov vector initialized with random noise
- Corner communication
- Embedding: replace FE operator with SEM
- Extrapolation: allowed when not using corner communication
RAS vs overlap: Saturation at overlap of 2
ORAS corner communication: Corners necessary!
ORAS asymptotic scaling
Coarse grid correction

\[ P^{-1} = R_c^T \tilde{A}^{-1} R_c + \sum_{k=1}^{K} R_{E_k}^T \tilde{A}_k^{-1} R_{E_k} \]

• Coarse grid is fine grid skeleton but at low no. Fes (F=1, 2 or 3)
• \( R_c \) is simple interpolation from fine to coarse grid
• A-operator is FEM Laplacian; tiling same as in RAS w/o overlap
Coarse grid scaling: Conforming grids: Vary no. FEs: asymptotics look good!
Coarse grid scaling: Conforming grids: Vary degree: asymptotics again look good!
Considerations for nonconforming overlap

Applies to fine grid only; coarse correction already handled

Extended grid

Interp to child nodes from parent data
Conclusions & (near-)future work

• Demonstrated asymptotics of ORAS for pseudo-Laplacian operator on staggered grid,

• Demonstrated iteration plateauing/optimization with coarse grid correction

• Coarse solve using ‘factor once, use many’: e.g. AMG, SuperLU, XXT, MUMPS

• Complete 3D ORAS
Thank you!
Speedup
Adaptive SEM vs uniform FD on MICI problem
ORAS vs RAS and Block Jacobi
Embedding Results

![Graph showing iterations vs. time for different embedding methods.](COURTESY DUANE ROSENBERG)
Semi-discrete equations

Advection-Diffusion:

\[ M \frac{du_j}{dt} = -MCu_j - \nu Lu_j \]

Navier-Stokes:

\[ \begin{align*}
M \frac{du_j}{dt} &= -MCu_j + D^T_j p - \nu Lu_j \\
D^j_{uj} &= 0
\end{align*} \]
Continuity: Operators

Advection-Diffusion, semi-implicit form:

$$ H = \frac{\beta}{\Delta t} M + \nu L $$

$$ \Phi A A^T \Phi^T H \Phi A u = \Phi A A^T \Phi^T f $$

DSS operator

Navier-Stokes & MHD:

$$ D_j \rightarrow D_j \Phi A $$
Advection-diffusion: 2-d N-Wave
Navier Stokes: 3-vortex simulation

Vorticity (left) and energy spectra (right; Fournier, J. Comp. Phys., 215(1), (2006)) for Re=10^4. Note power law spectral behavior with filament formation.
Navier-Stokes: 3-vortex simulation

Fournier, Rosenberg, Pouquet, GAFD (2008)
MHD: Island coalescence instability

OT SEM convergence

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Evolution of iteration count for OT: Block Jacobi
MHD Turbulence

• Phenomenological & fundamental:

Trace

Mininni & Pouquet (2007)
Hydro Turbulence

Taylor-Green flow at $64^3$, $256^3$, $1024^3$, $2048^3$ (Mininni, Alexakis, Pouquet 2007)

Kelvin-Helmholtz rolls in match smoke