Modelling Interactions Between Weather and Climate

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Fondation canadienne pour les sciences du climat et de l'atmosphère (FCSCA)

NSERC
CRSNG

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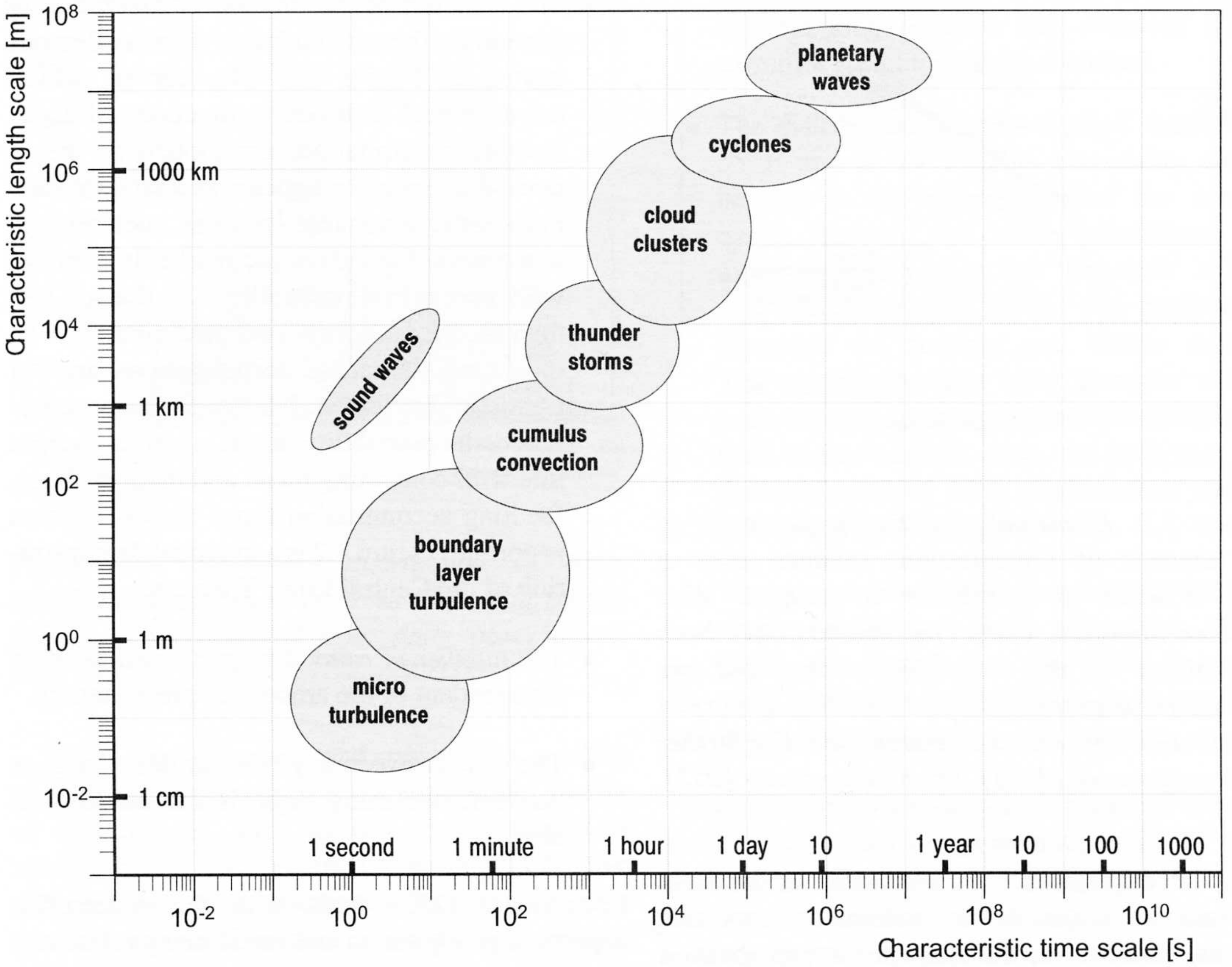

From von Storch and Zwiers, 1999

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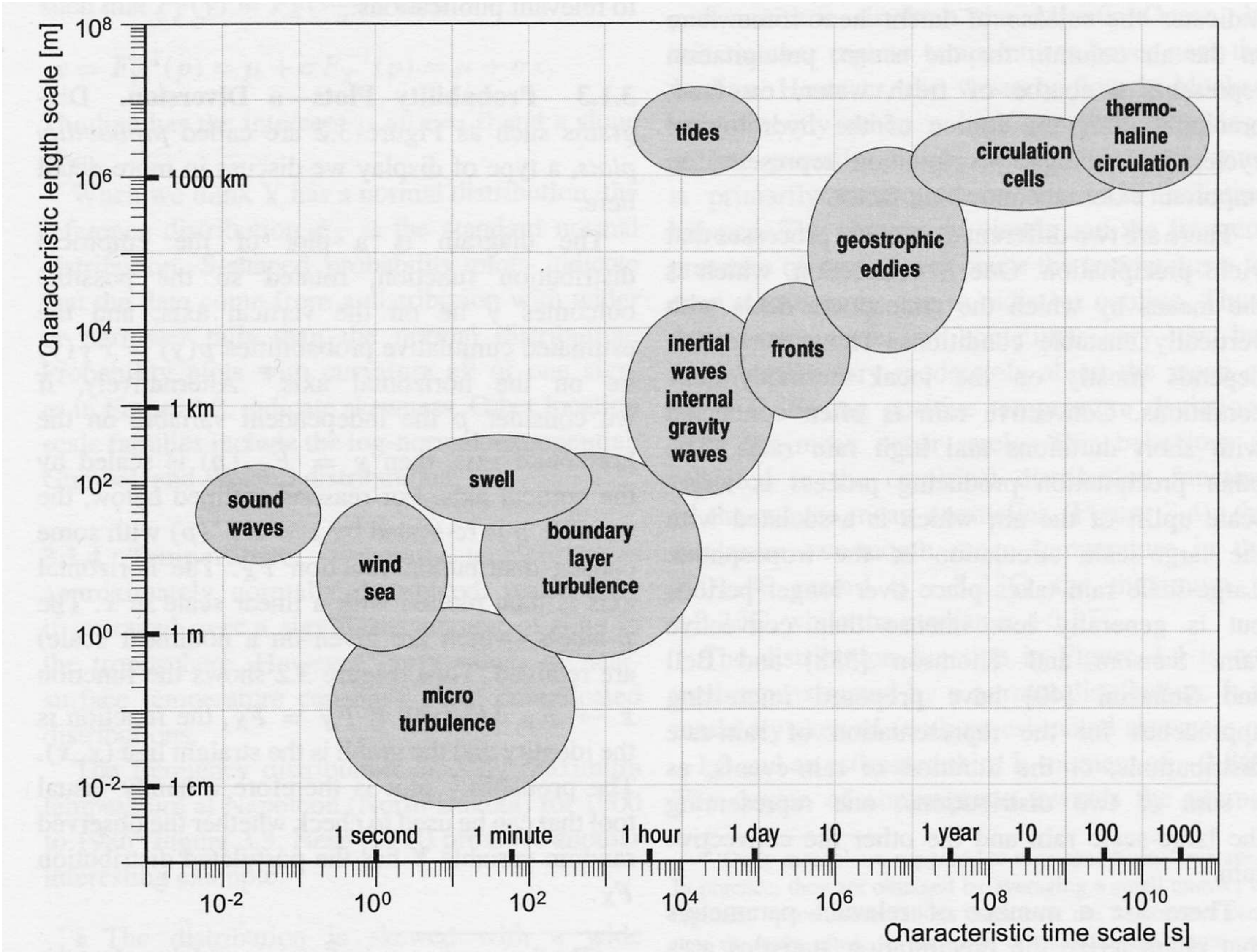

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  - **stochastic averaging** as a tool for obtaining stochastic climate models from coupled weather/climate system
  - two examples: coupled atmosphere/ocean boundary layers, extratropical atmospheric low-frequency variability

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A statistical mechanical perspective

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- Smaller (but still large) separation $\Rightarrow$ stochastic corrections needed

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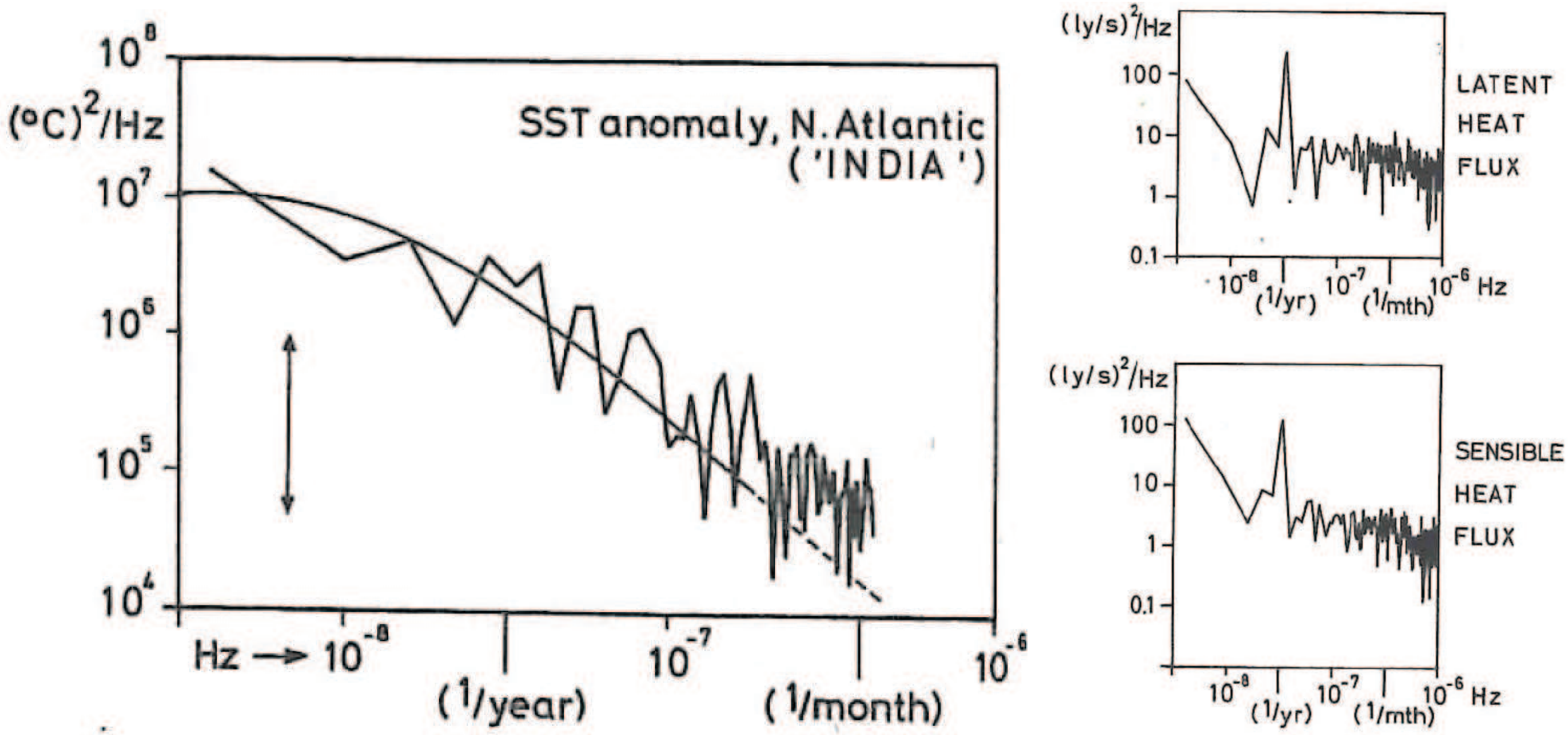
- Observations from North Atlantic weathership "INDIA"



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$$\frac{dT'}{dt} = -\frac{1}{\tau}T' + \gamma\dot{W}$$

with spectrum

$$\mathsf{E}\{T'(\omega)^2\} = \frac{\gamma^2\tau^2}{2\pi(1+\omega^2\tau^2)}$$

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- We'll be coming back to this again later on ....

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Coupled fast-slow systems

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$$\begin{aligned} \frac{d}{dt}\mathbf{x} &= f(\mathbf{x}, \mathbf{y}) \\ \frac{d}{dt}\mathbf{y} &= g(\mathbf{x}, \mathbf{y}) \end{aligned}$$

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where

$$\tau \sim \frac{f(\mathbf{x}, \mathbf{y})}{g(\mathbf{x}, \mathbf{y})}$$

is “small"

A simple example

- As a simple example of a coupled fast-slow system consider

$$
\begin{aligned}
\frac{d}{dt}x &= x - x^3 + (\Sigma + x)y \\
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- The "weather" process y is Gaussian with stationary autocovariance

$$\begin{aligned}C_{yy}(s) &= \mathsf{E}\left\{y(t)y(t+s)\right\} = \frac{|x|}{2}\exp\left(-\frac{|s|}{\tau|x|}\right) \\ &\overrightarrow{\text{as } \tau \to 0} \quad \tau x^2 \delta(s)\end{aligned}$$

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- Intuition suggests that as $\tau \to 0$

$$\frac{d}{dt}x \simeq x - x^3 + \sqrt{\tau}(\Sigma + x)|x| \circ \dot{W}$$

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  i.e. depending on $\overline{x}(0)$ system settles into bottom of one of two potential wells $\overline{x} = \pm 1$

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- Approximation (A) analogous to Reynolds averaging in fluid mechanics: split flow into "mean" and "turbulent" components

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- Note: averaging is projection operator only if scale separation between $\overline{u}$ and u'

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$$\sigma^2(\mathbf{x}) = \int_{-\infty}^{\infty} \mathsf{E}_{y|x}\left\{f'[\mathbf{x},\mathbf{y}(t+s)]f'[\mathbf{x},\mathbf{y}(t)]\right\}\,ds$$

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- Note that $\mathrm{std}(\zeta) \sim \sqrt{\tau}$ so corrections vanish in strict $\tau \rightarrow 0$ limit

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$\Rightarrow$ $x(t) \to 1 + \zeta$ such that

$$\frac{d}{dt}\zeta = -2\zeta + (1+\Sigma)^2 \dot{W}$$

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- Multiplicative noise introduced through:

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- **Crucially**: none of this assumes that the fast process is stochastic. Effective SDEs (L) and (N) arise in limit $\tau \to 0$ as approximation to fast *deterministic or stochastic* dynamics

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- Present analysis will not make MTV ansatz of increasing weather influence as τ decreases; rather than a “$\tau = 0$ theory", it is a “small τ theory"

Case study 1: Coupled atmosphere-ocean boundary layers

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\frac{d}{dt}u &= \langle \Pi_u \rangle - \frac{c_d(w)}{h(T_a,T_o)}wu - \frac{w_e}{h(T_a,T_o)}u + \sigma_u \dot{W}_1 \\
\frac{d}{dt}v &= -\frac{c_d(w)}{h(T_a,T_o)}wv - \frac{w_e}{h(T_a,T_o)}v + \sigma_u \dot{W}_2 \\
\frac{d}{dt}T_a &= \frac{\beta}{\gamma_a}(T_o - T_a)w - \frac{\beta}{\gamma_a}\theta(w-\mu_w) - \frac{\lambda_a}{\gamma_a}T_a + \Sigma_{11}\dot{W}_3 + \Sigma_{12}\dot{W}_4 \\
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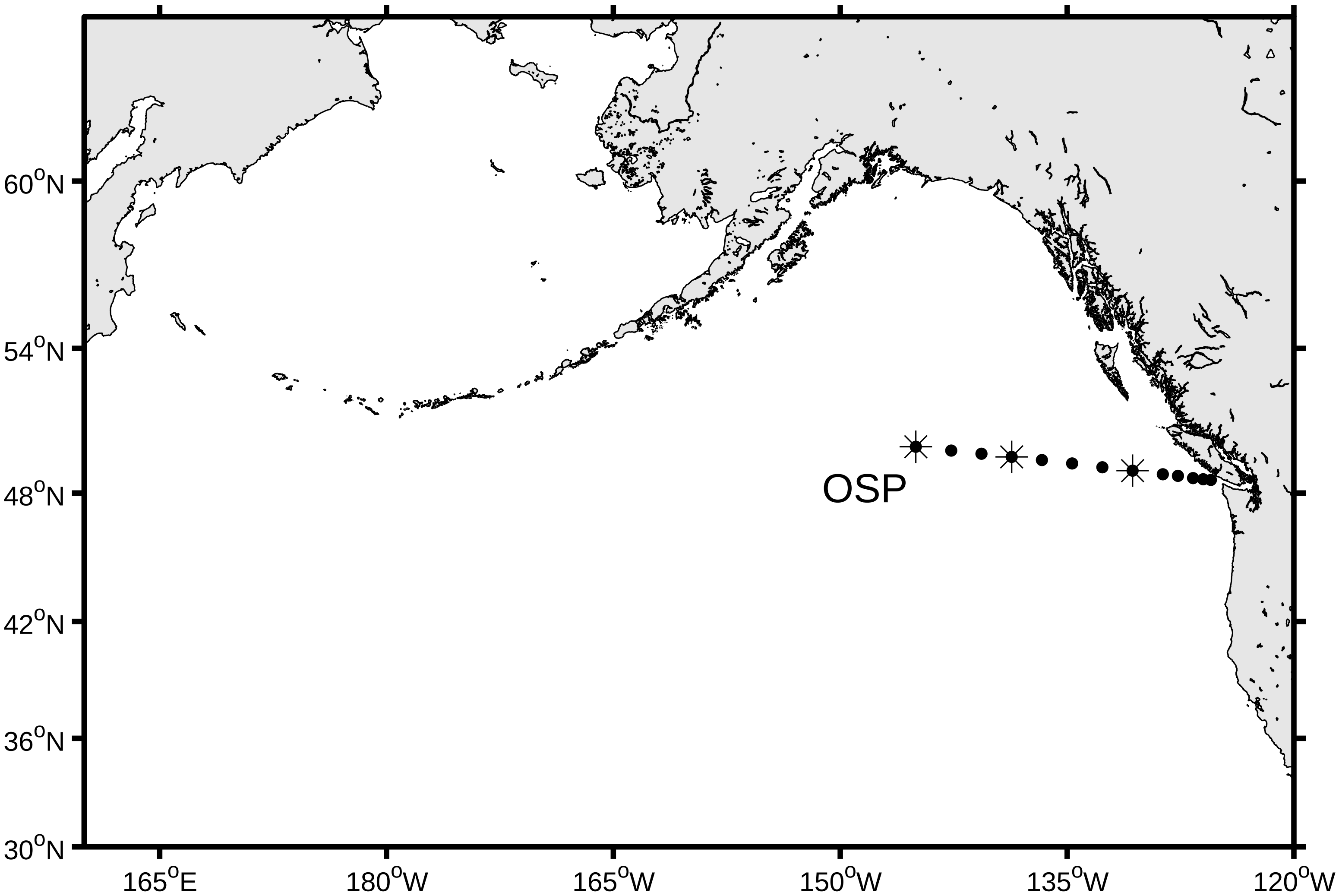
Case study 1: Coupled atmosphere-ocean boundary layers



Ocean station P

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- Atmospheric boundary layer thickness determined by surface stratification

$$h(T_a,T_o) = \max\left[h_{min}, \overline{h}(1-\alpha(T_a - T_o))\right]$$

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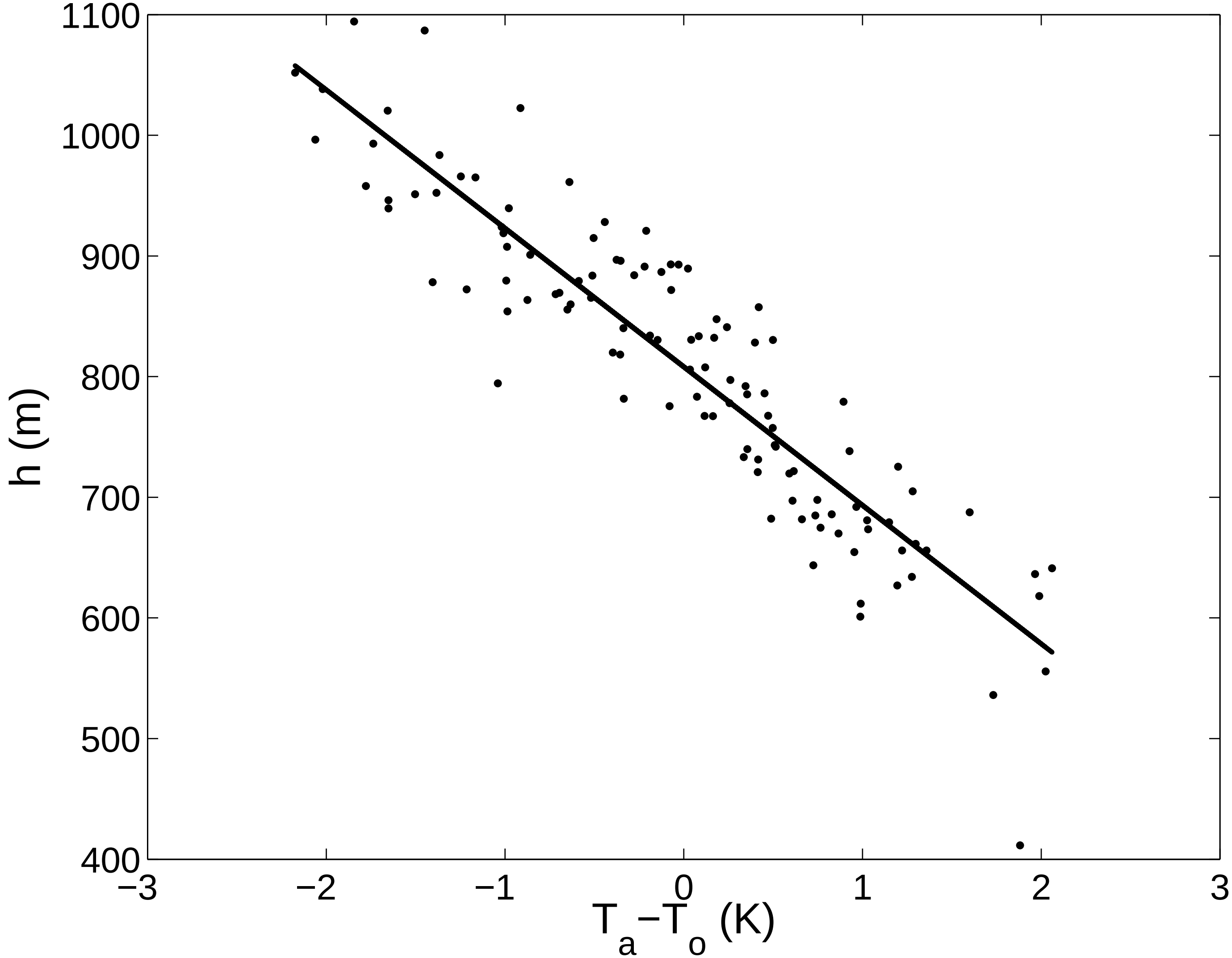

Boundary layer height and surface stability

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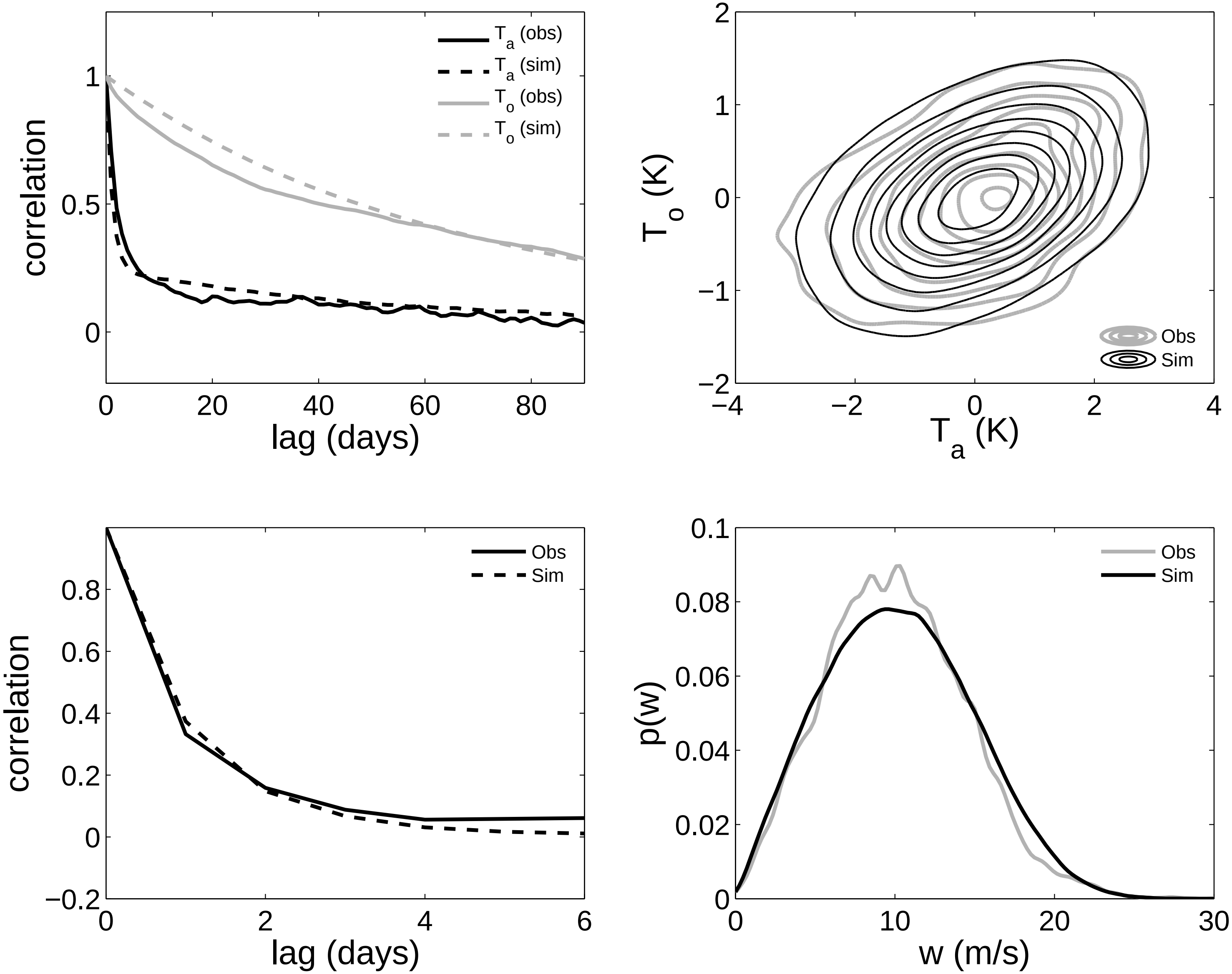

Observed and simulated statistics (Ocean Station P)

Case study 1: Coupled atmosphere-ocean boundary layers

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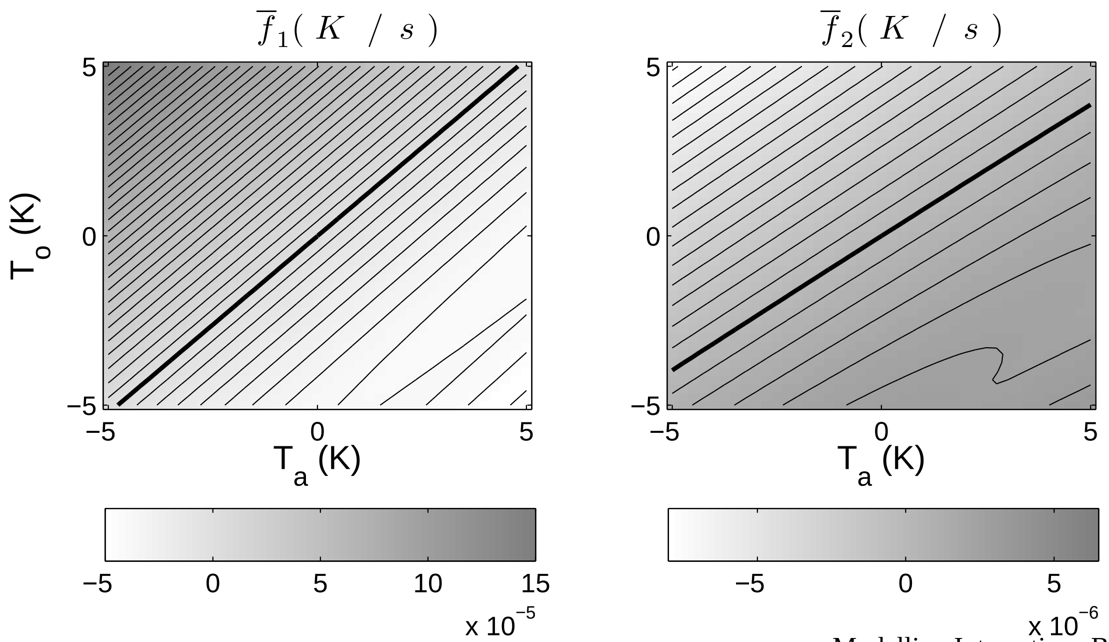

Case study 1: Coupled atmosphere-ocean boundary layers

- Analytic expression for $\sigma(T_a, T_o)$

$$\sigma(T_a, T_o) = \frac{\beta}{\sqrt{\gamma_a^2 + \gamma_o^2}} \begin{pmatrix} \frac{\gamma_o}{\gamma_a} & -1 \\ -1 & \frac{\gamma_a}{\gamma_o} \end{pmatrix} \psi(T_a - T_o)$$

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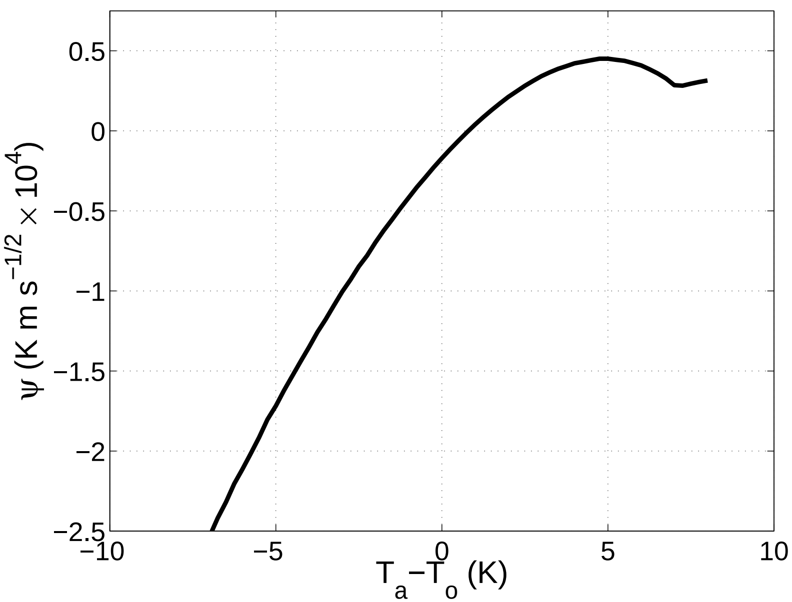

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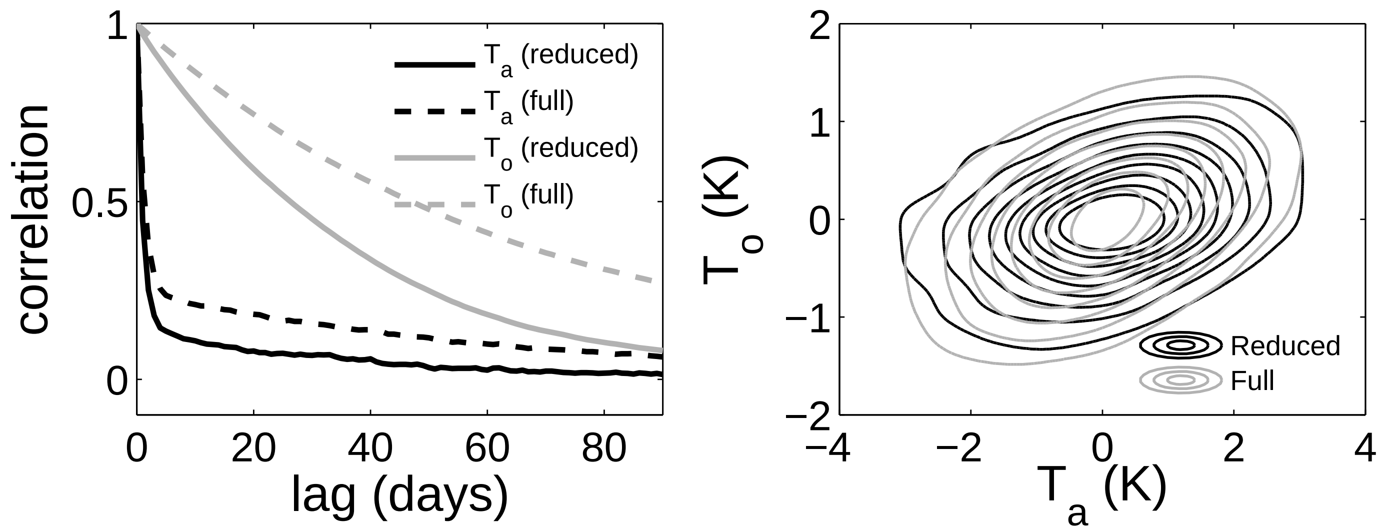
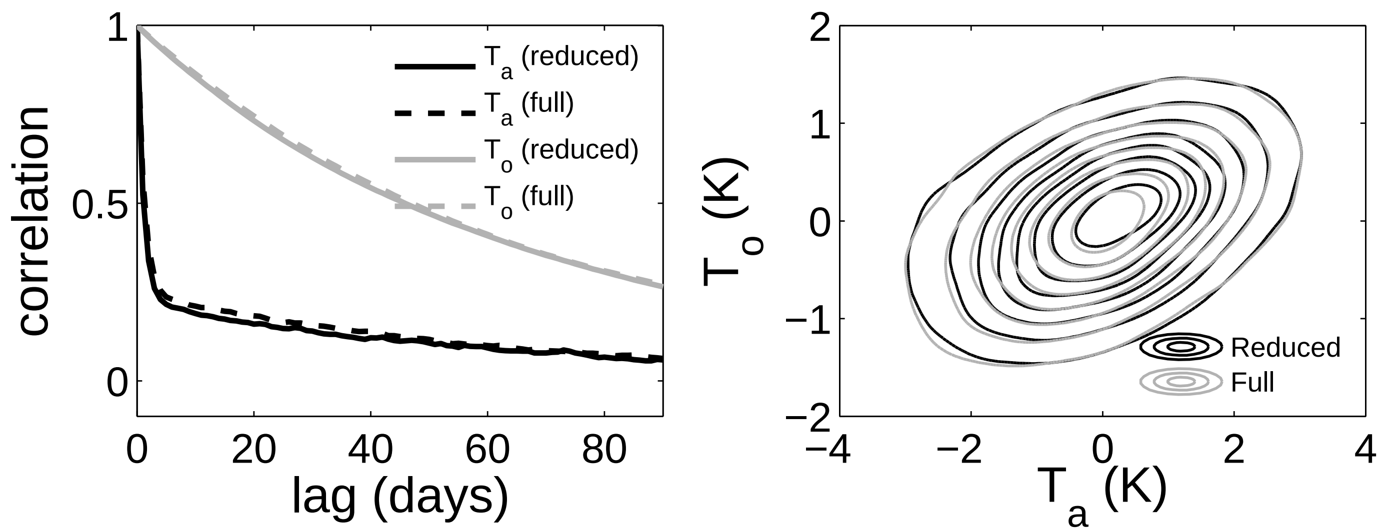


$\tau = 0.7$



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- Form of SDE did not account for feedback of stratification on h; physical origin of parameters in inverse model somewhat different than had been assumed

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Case study 2: extratropical atmospheric LFV

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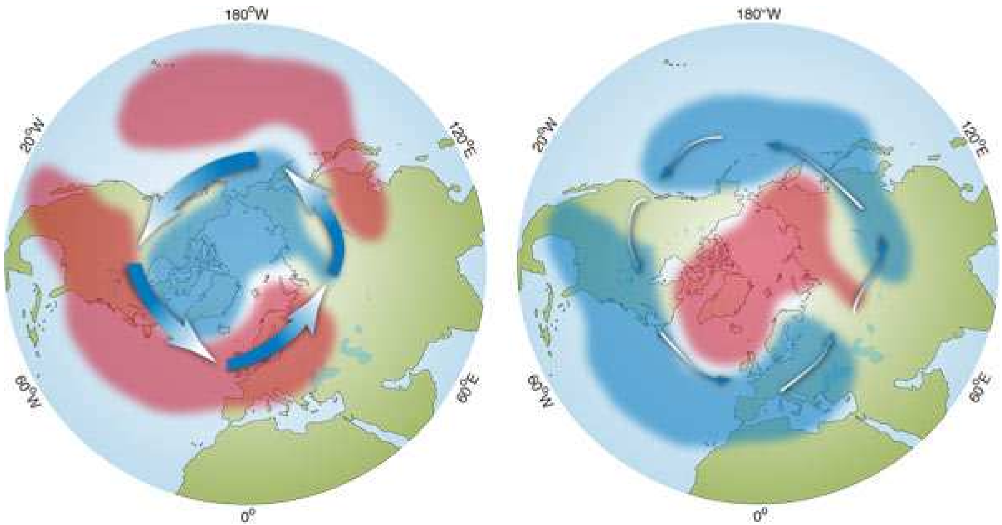

From `www.whoi.edu`

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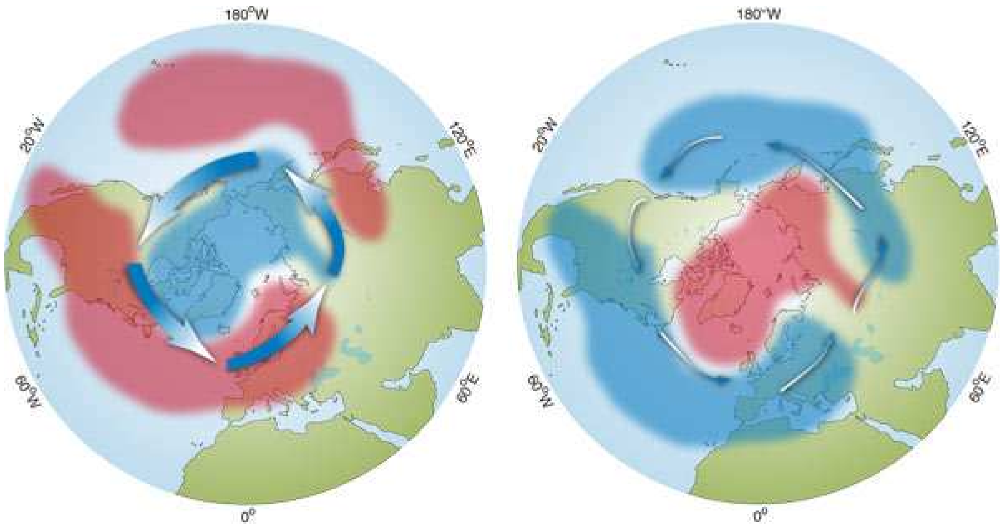

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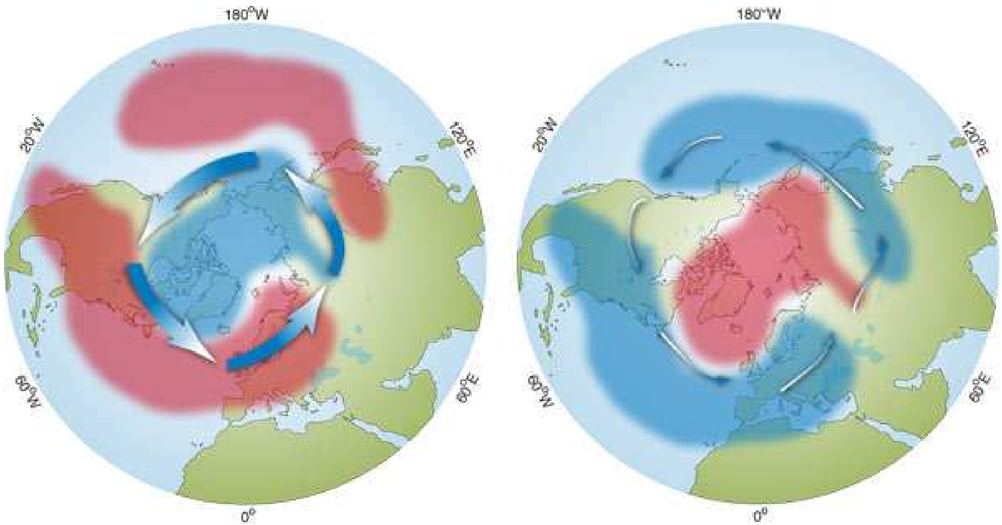

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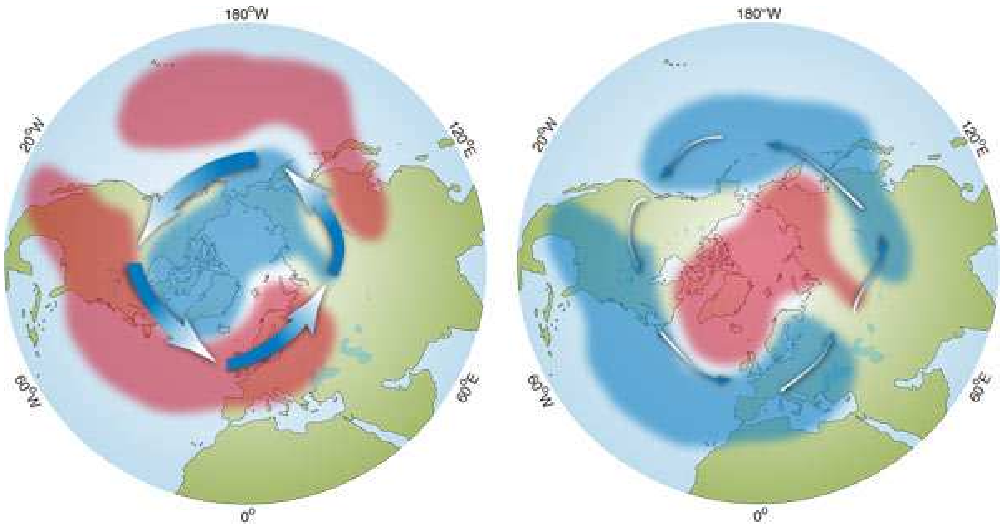

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- Stochastic reduction techniques $\Rightarrow$ natural tool for investigating dynamics

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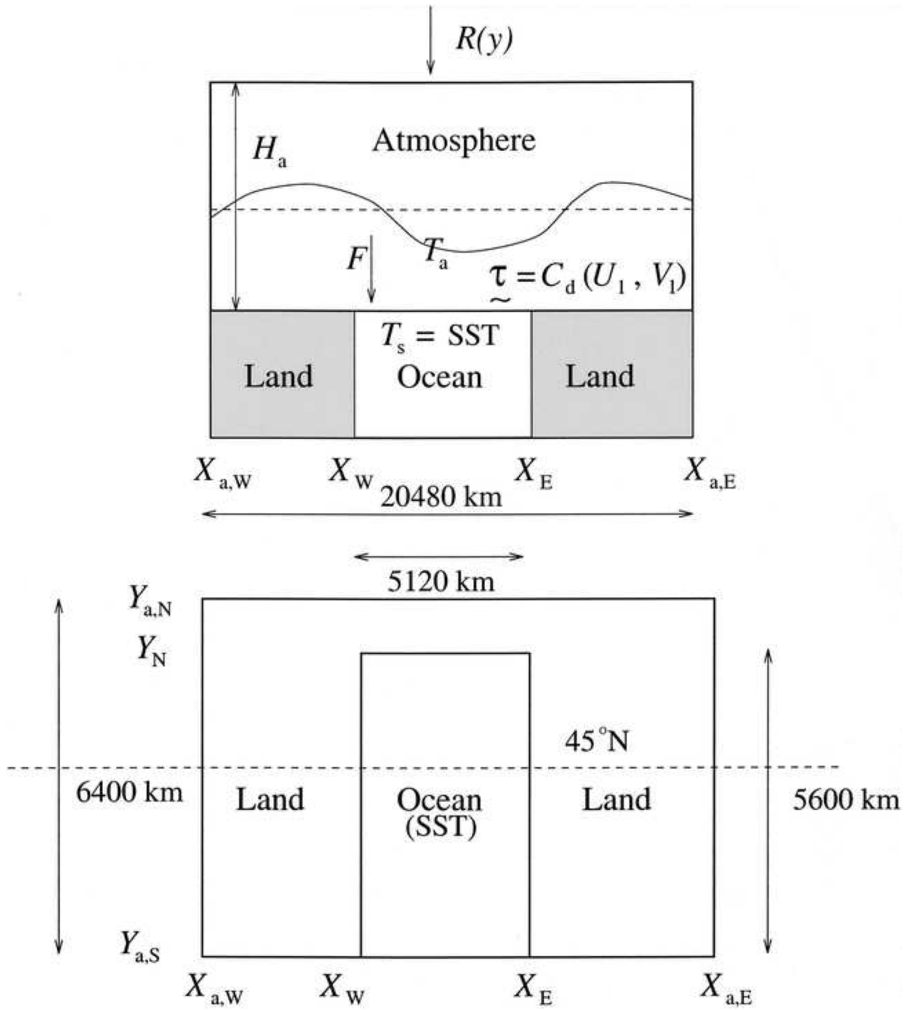

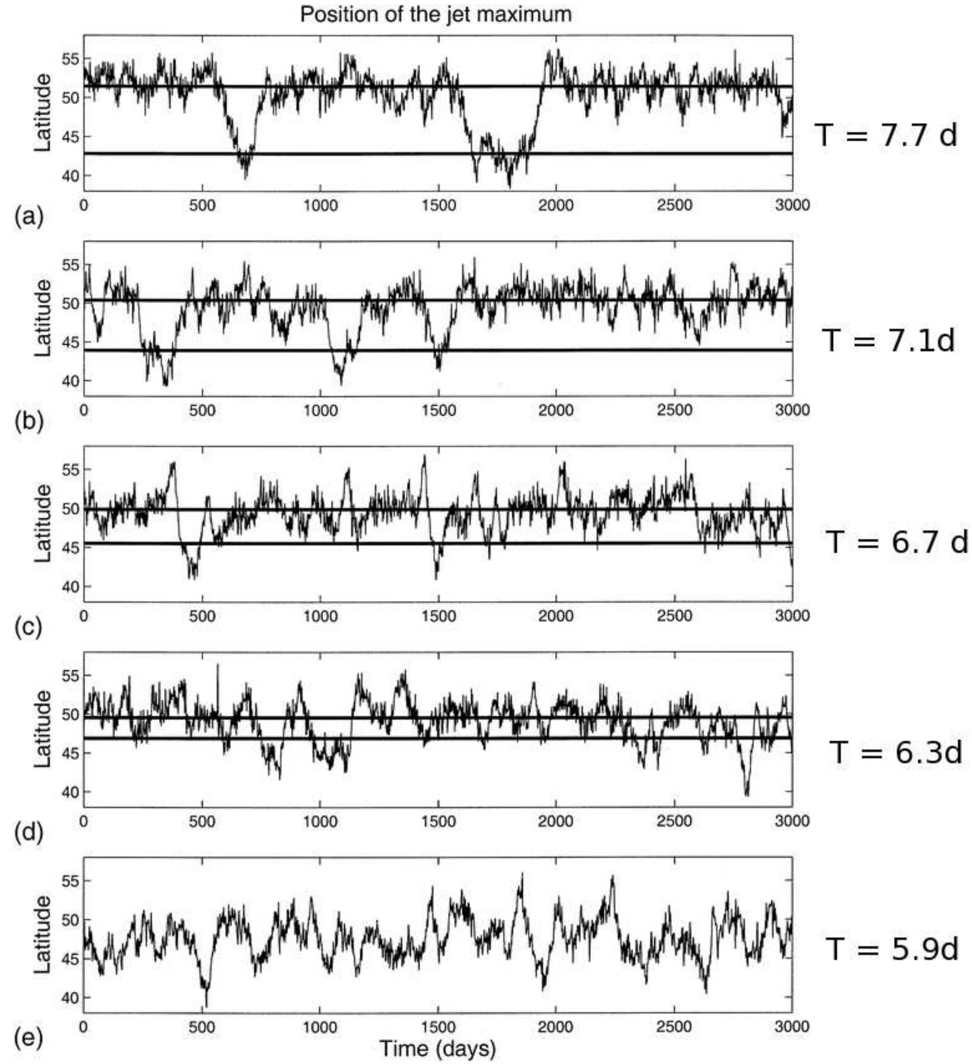

Kravtsov et al. JAS 2005

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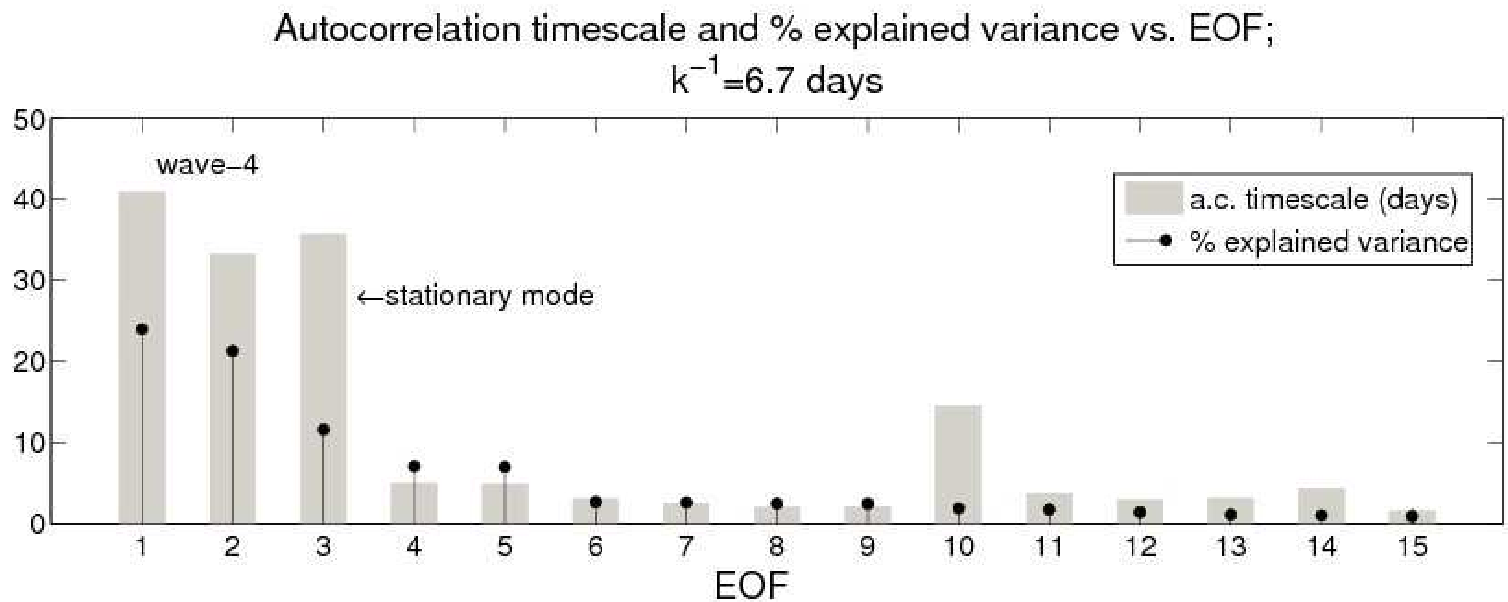

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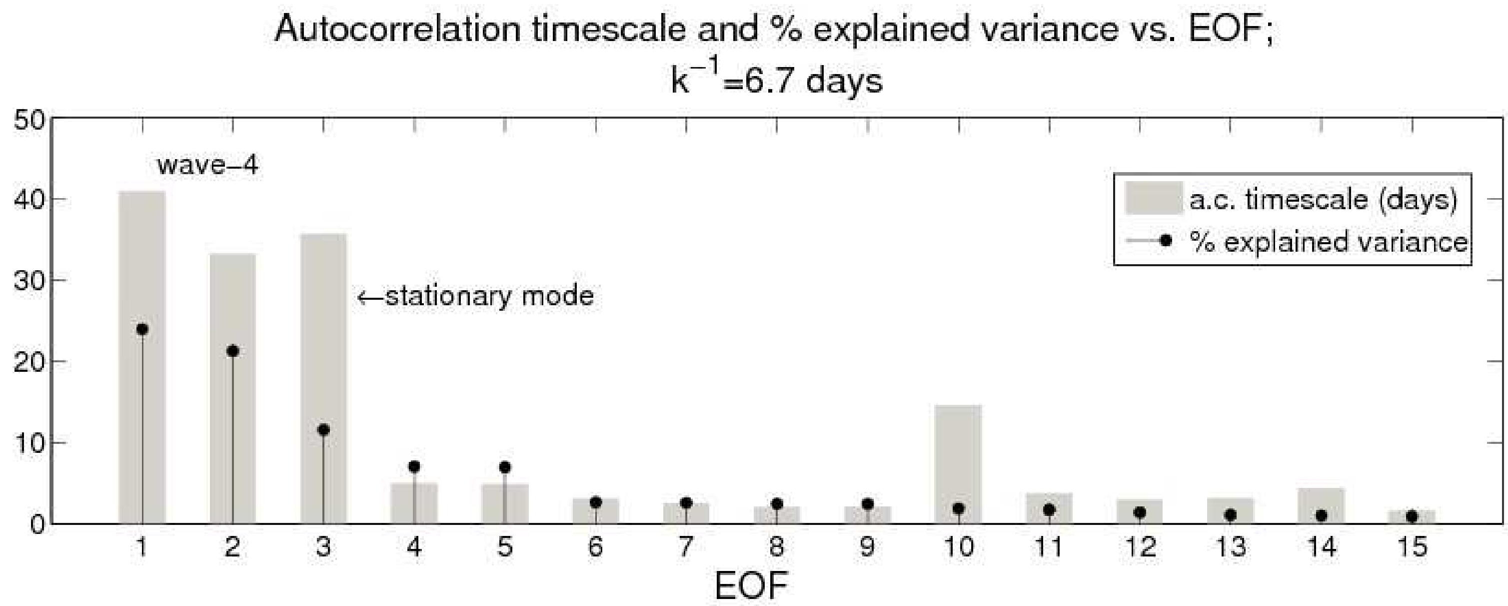

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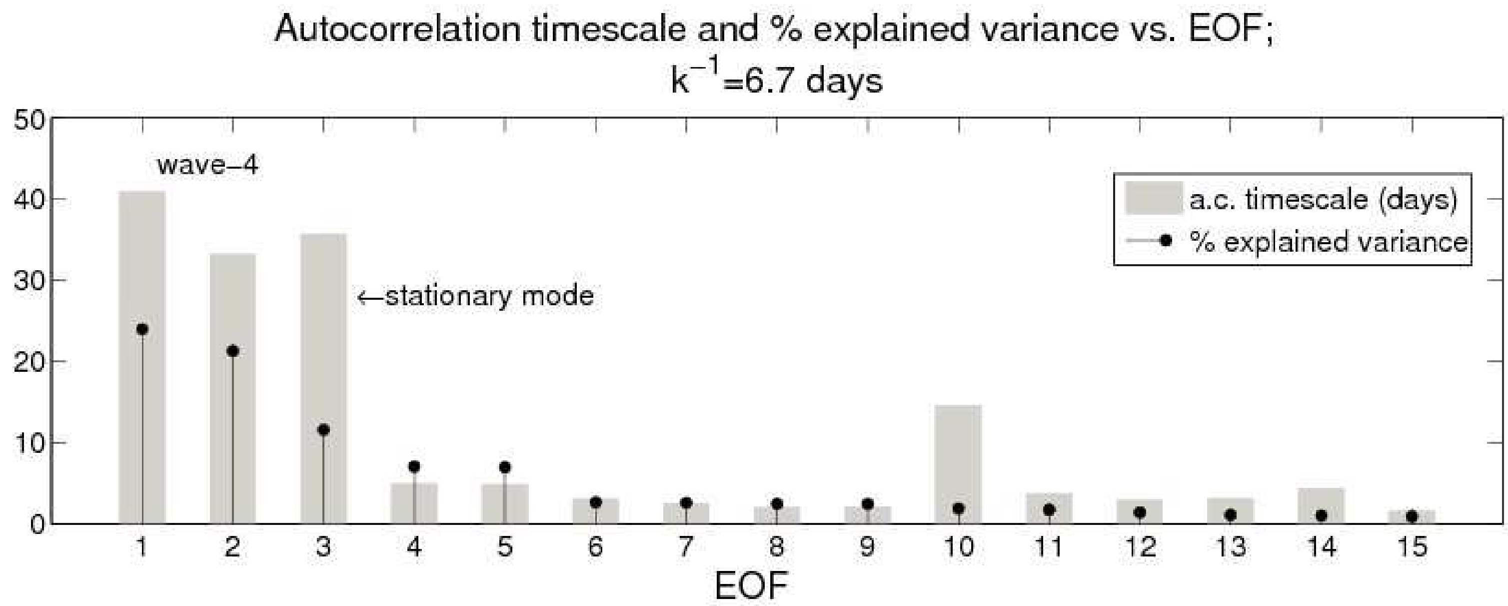

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- Consider reduction to 3-variable, 1-variable (stationary mode) models

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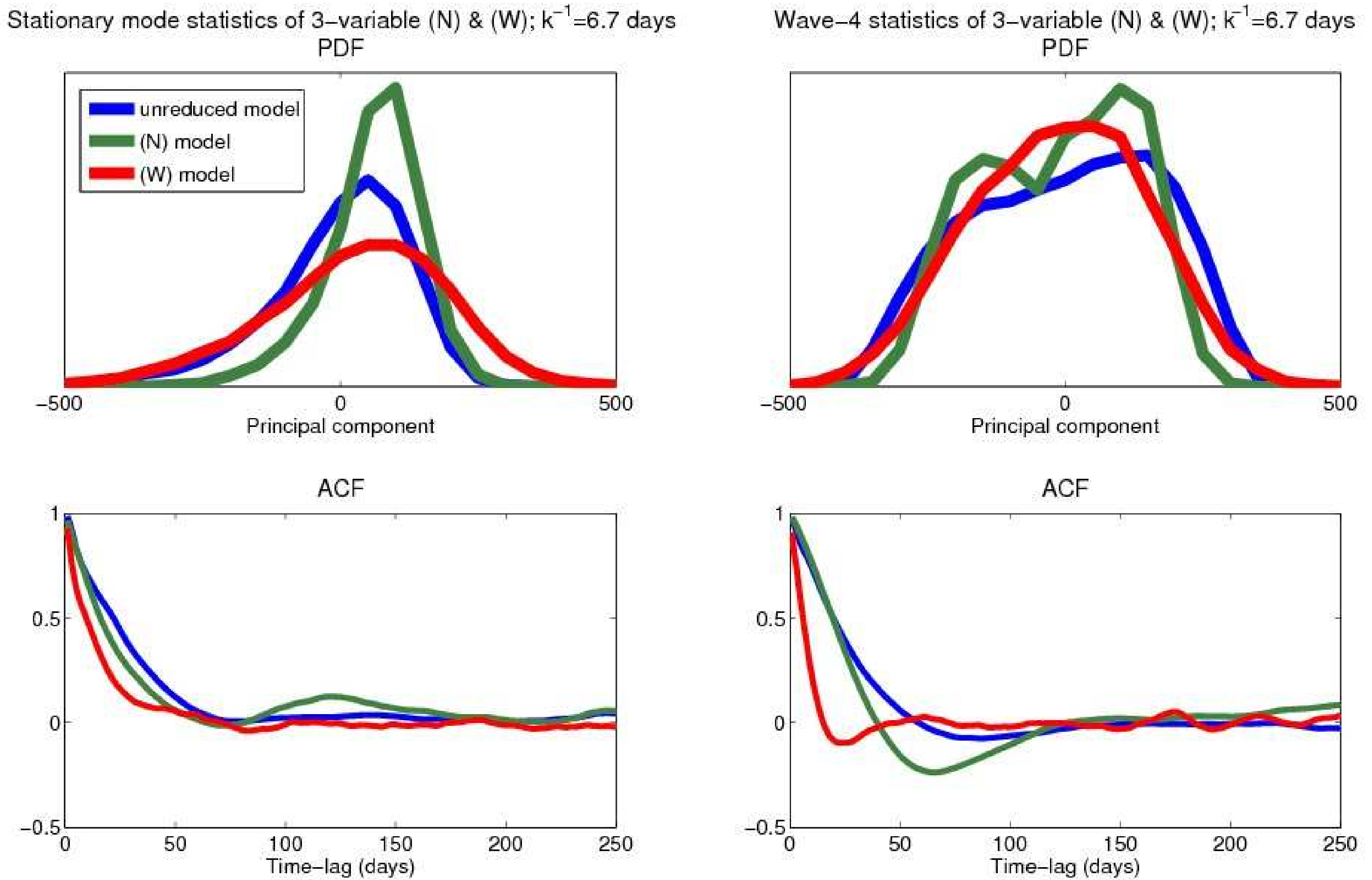

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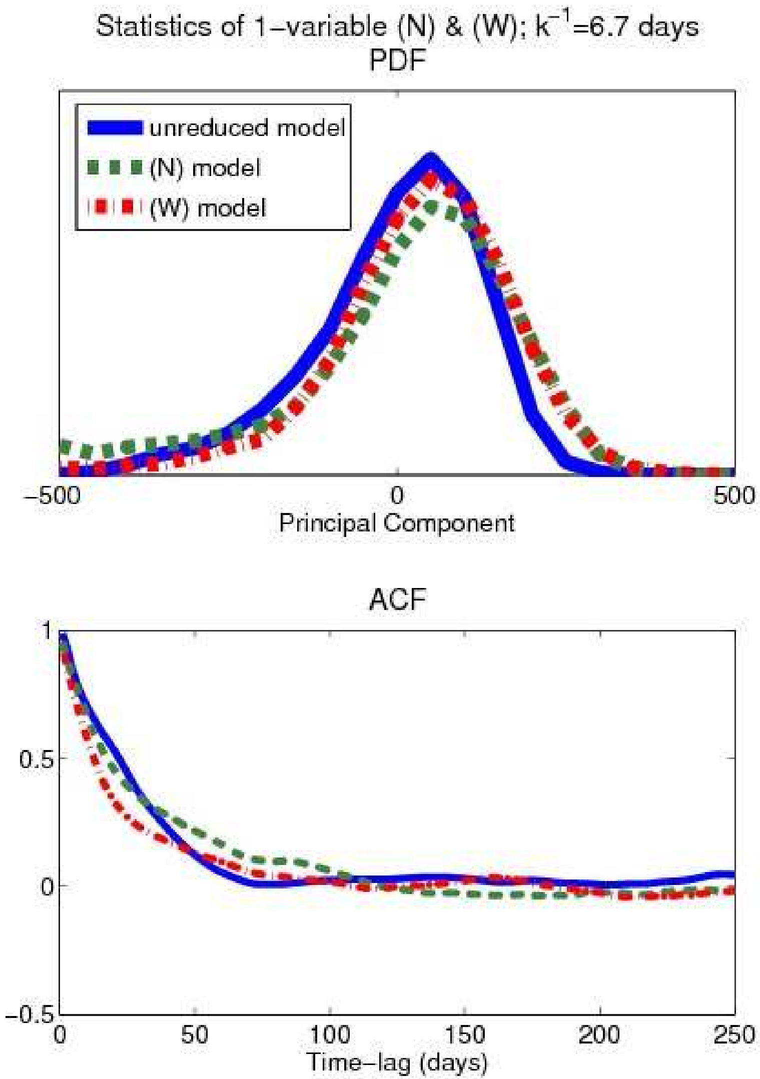

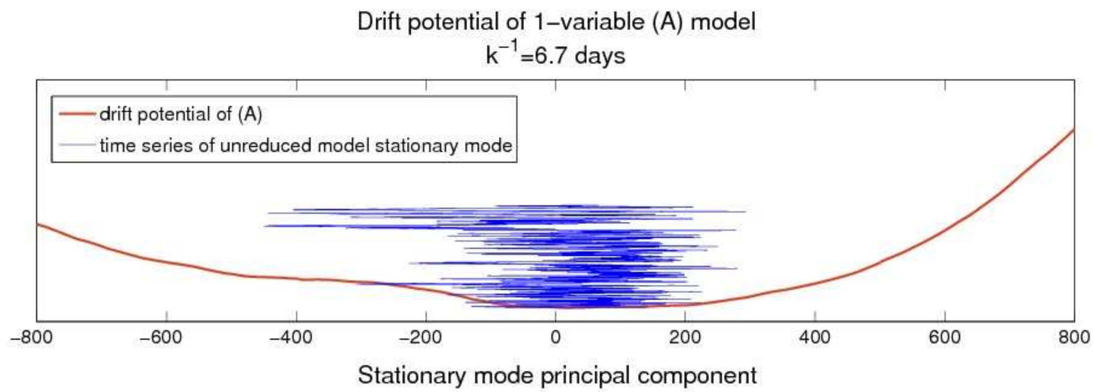

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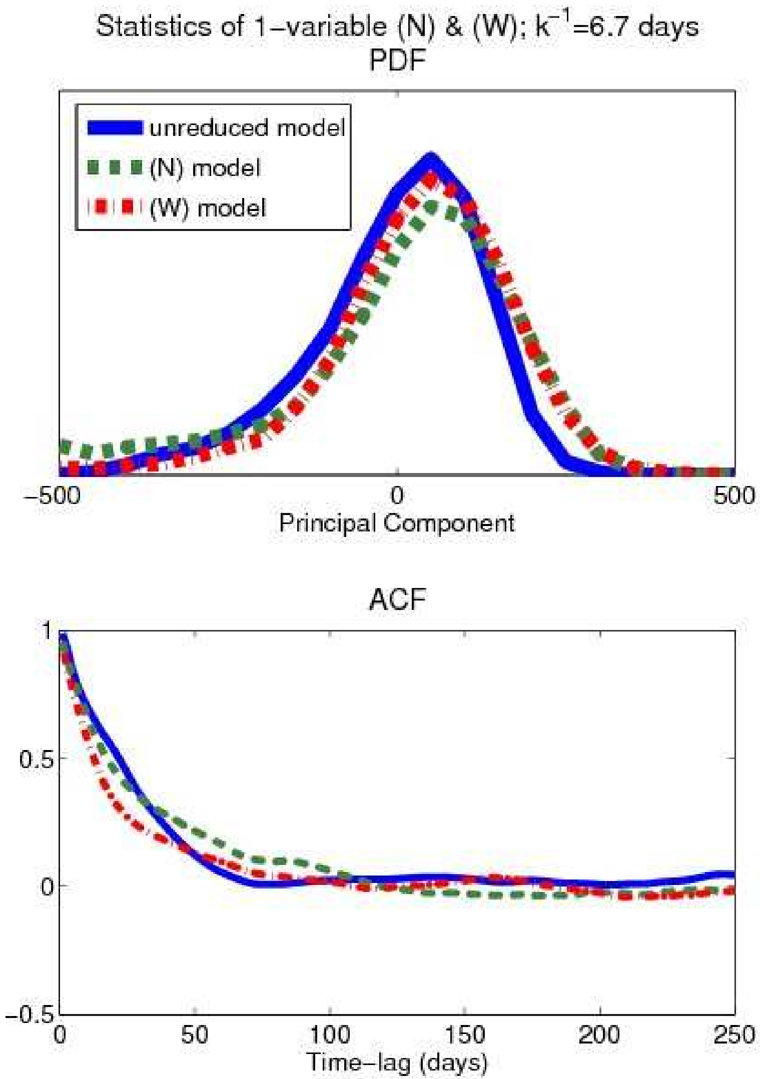

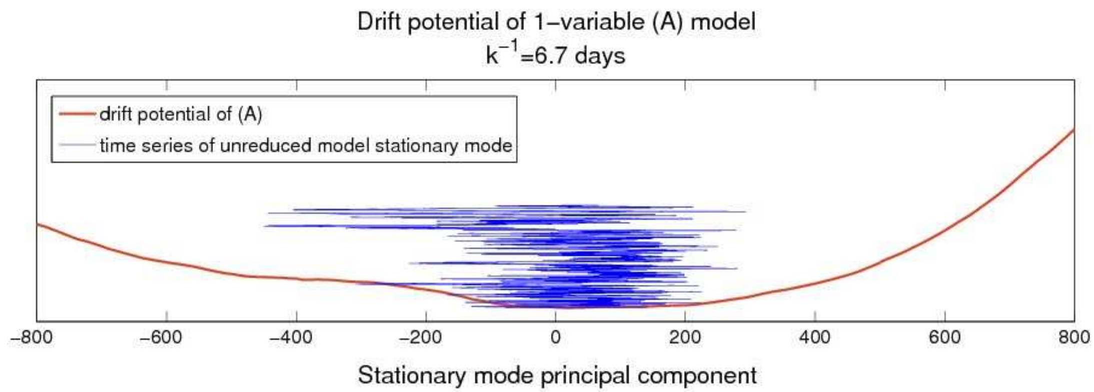

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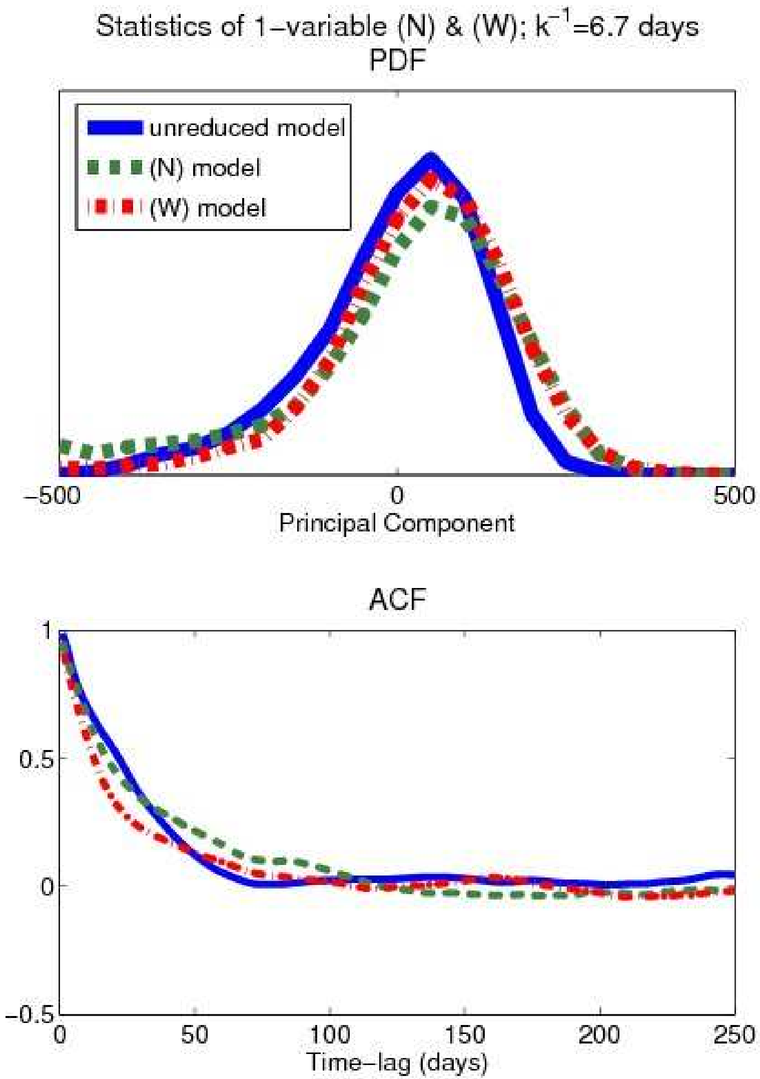

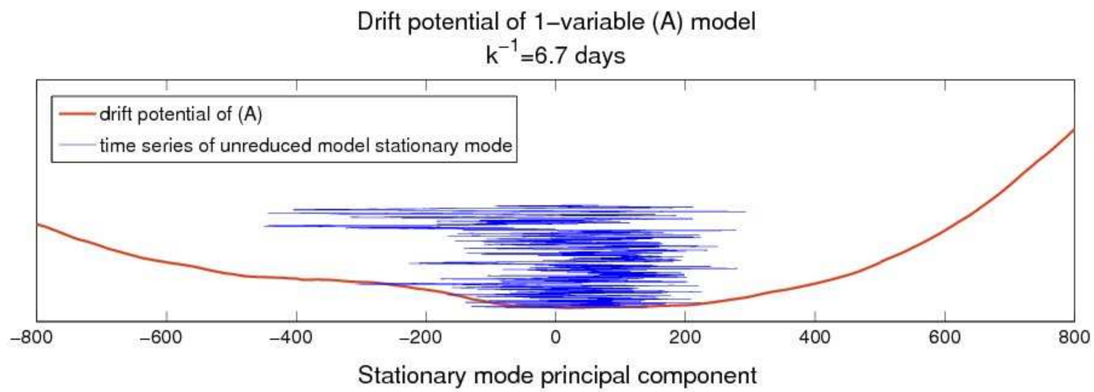

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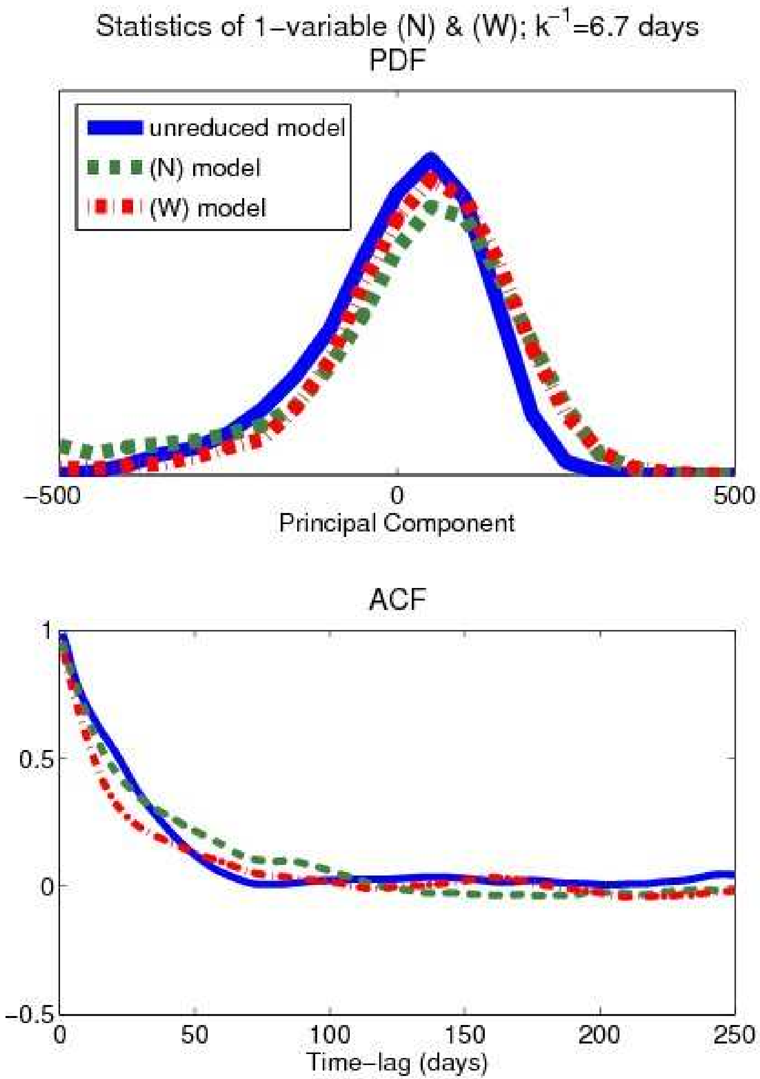

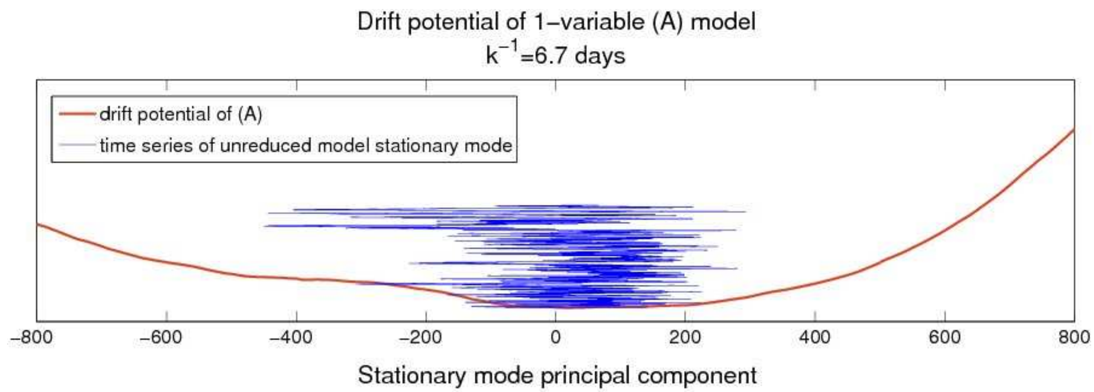

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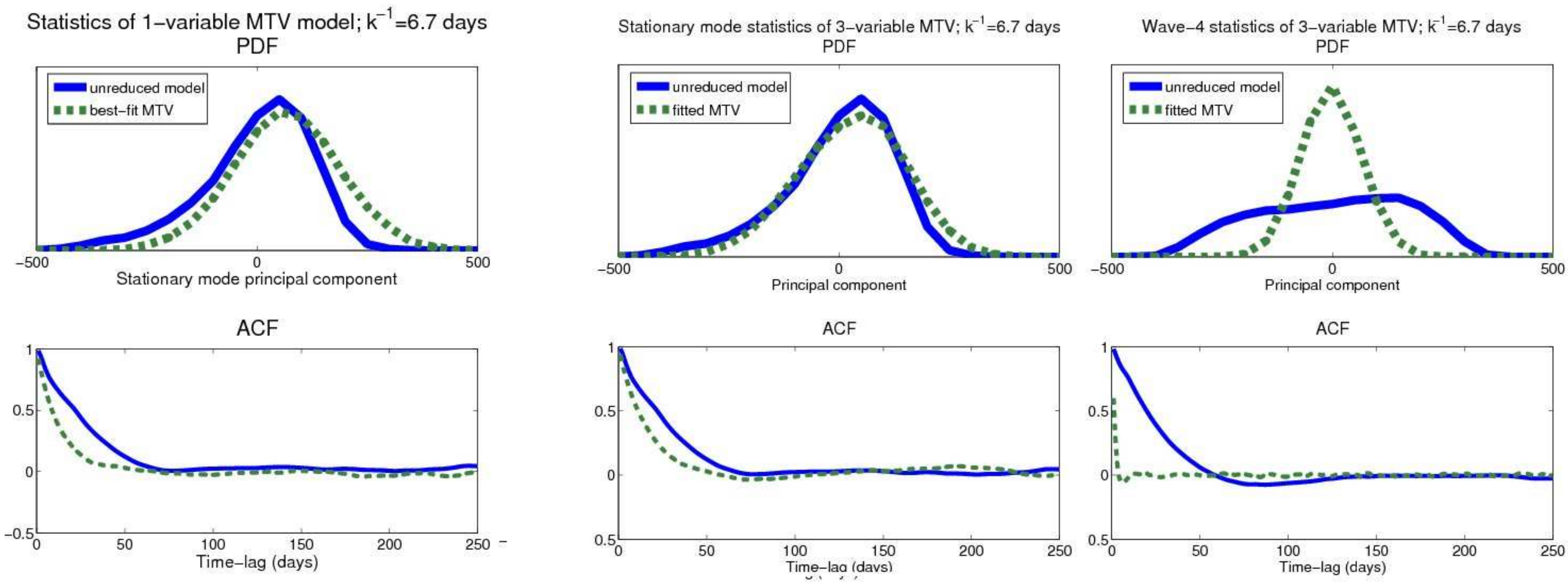

MTV approximations with “minimal regression” tuning

Conclusions

- Stochastic reduction tools allow systematic derivation of low-order climate models from coupled weather/climate systems

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- Real systems do not generally have a strong scale separation; an important outstanding problem is a more systematic approach to accounting for this

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