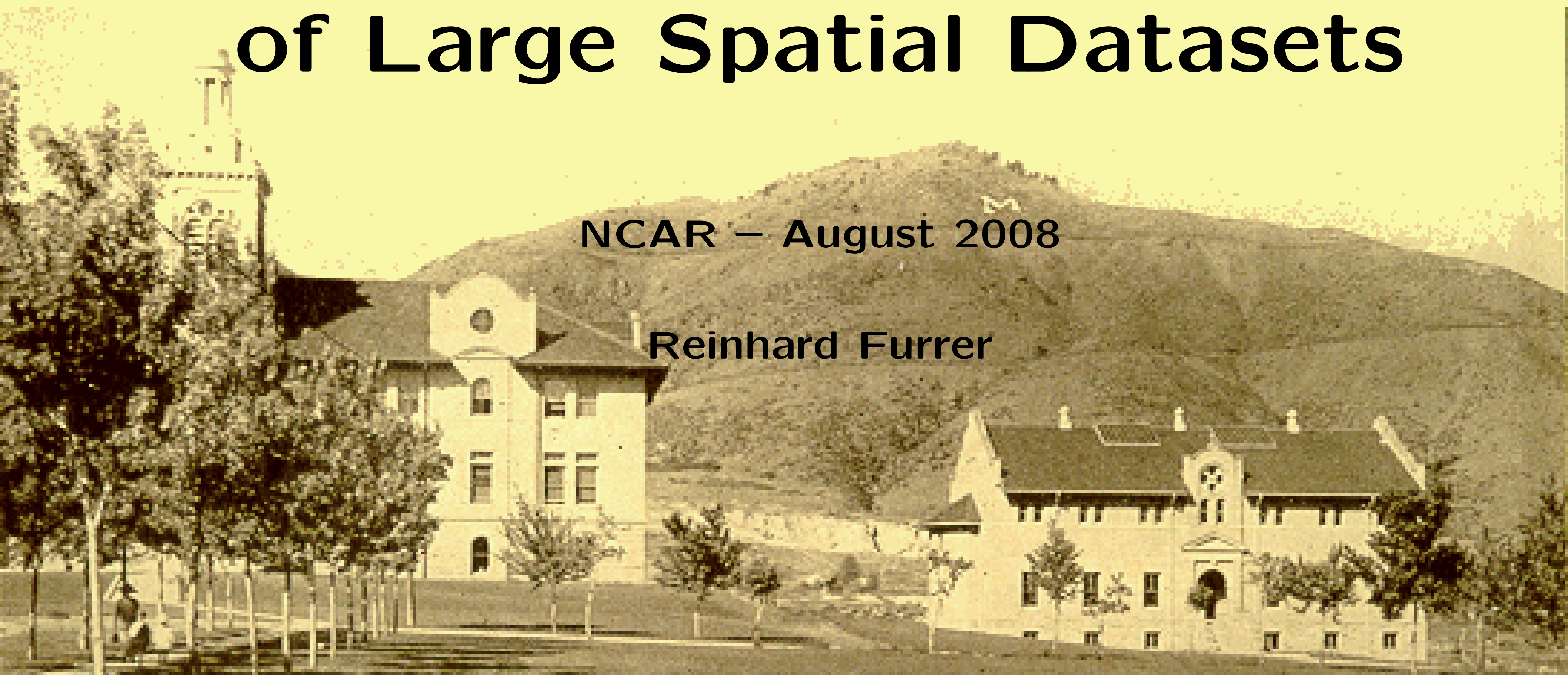
Inlet two:

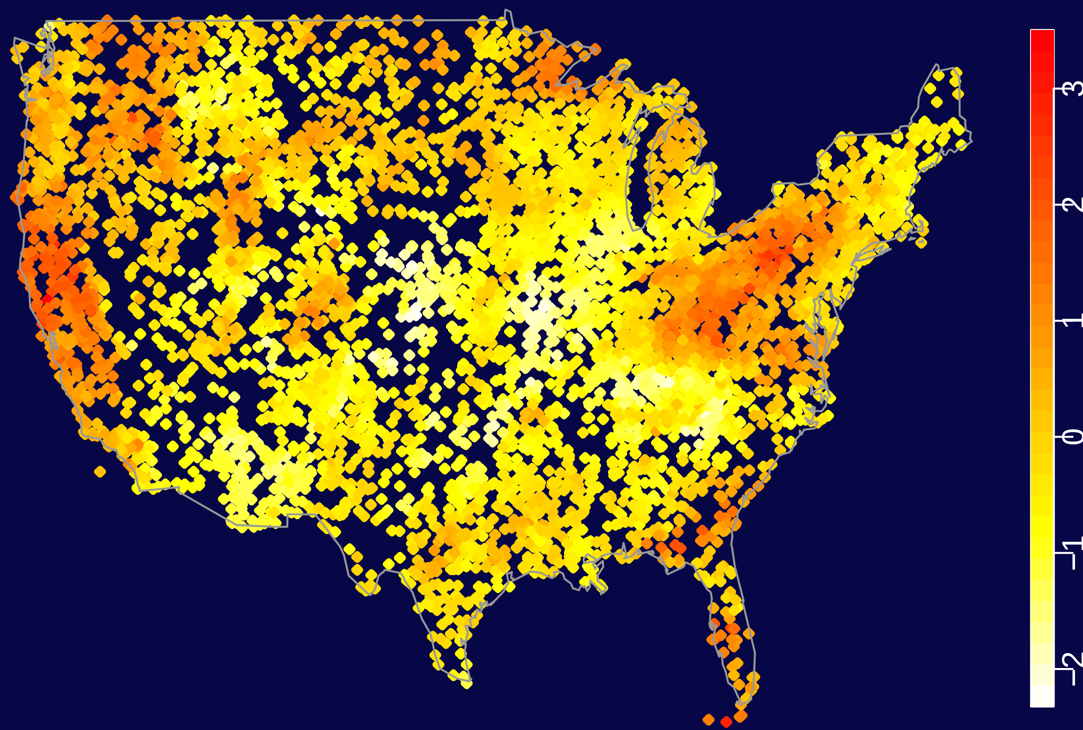
Covariance Tapering for Interpolation of Large Spatial Datasets



NCAR – August 2008

Reinhard Furrer

Motivation

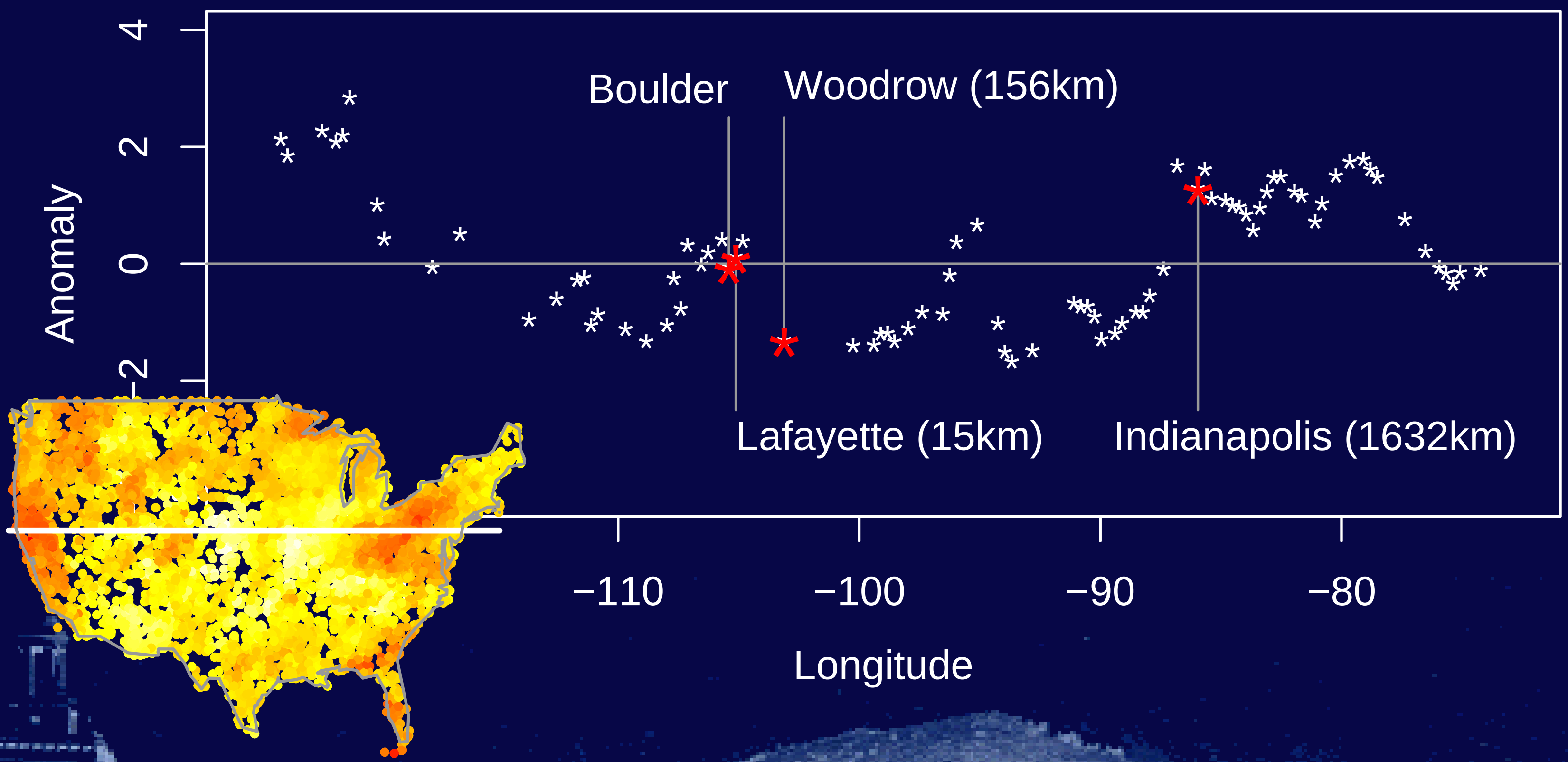


Precipitation anomaly in April 1948

Introduce a sparseness structure in the covariance via tapering to gain computational advantages in large kriging problems constraint to maintaining asymptotic optimality.

Motivation

Easy to think in one dimension: precipitation anomaly along 40° lat.



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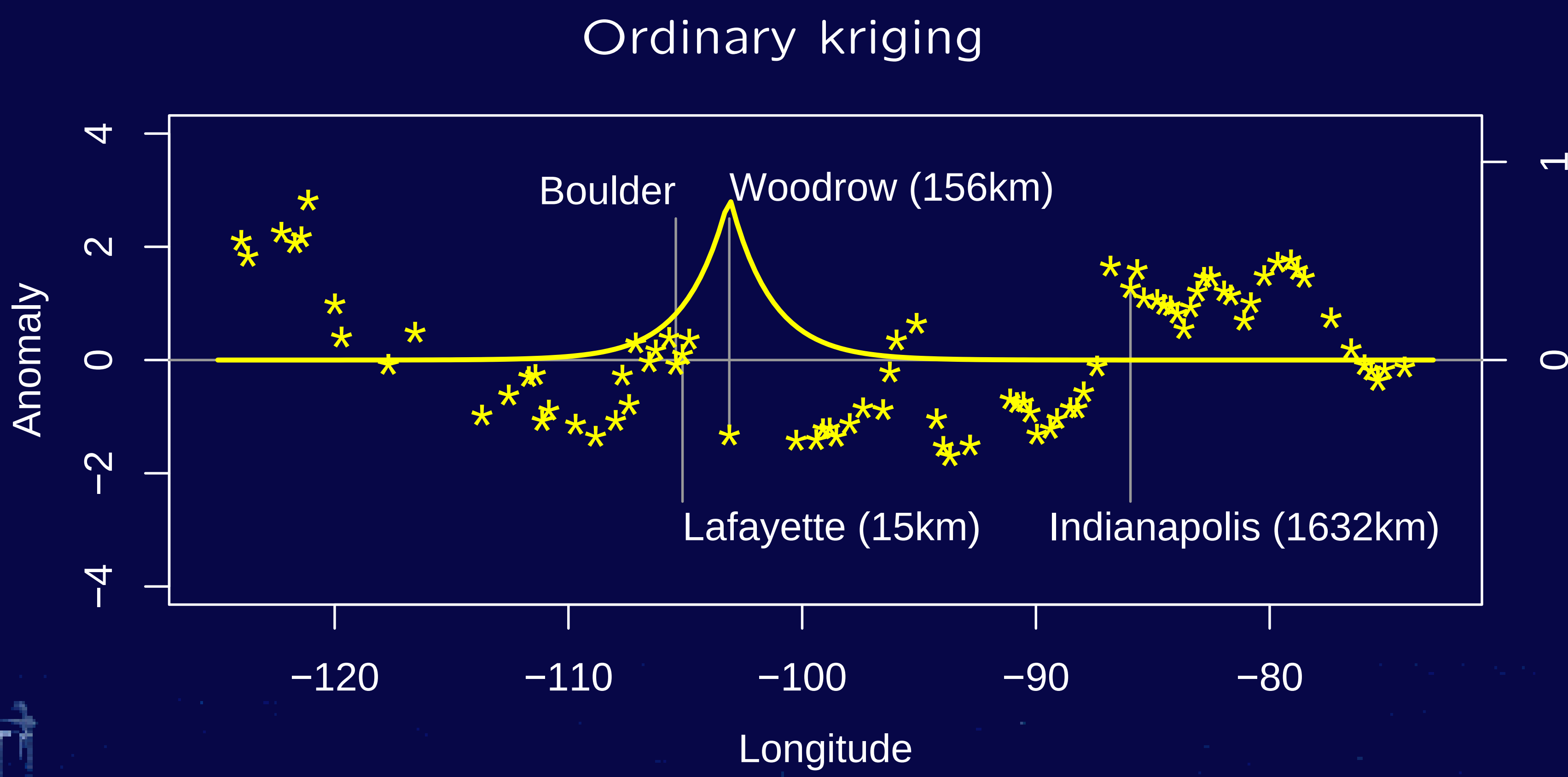

Motivation

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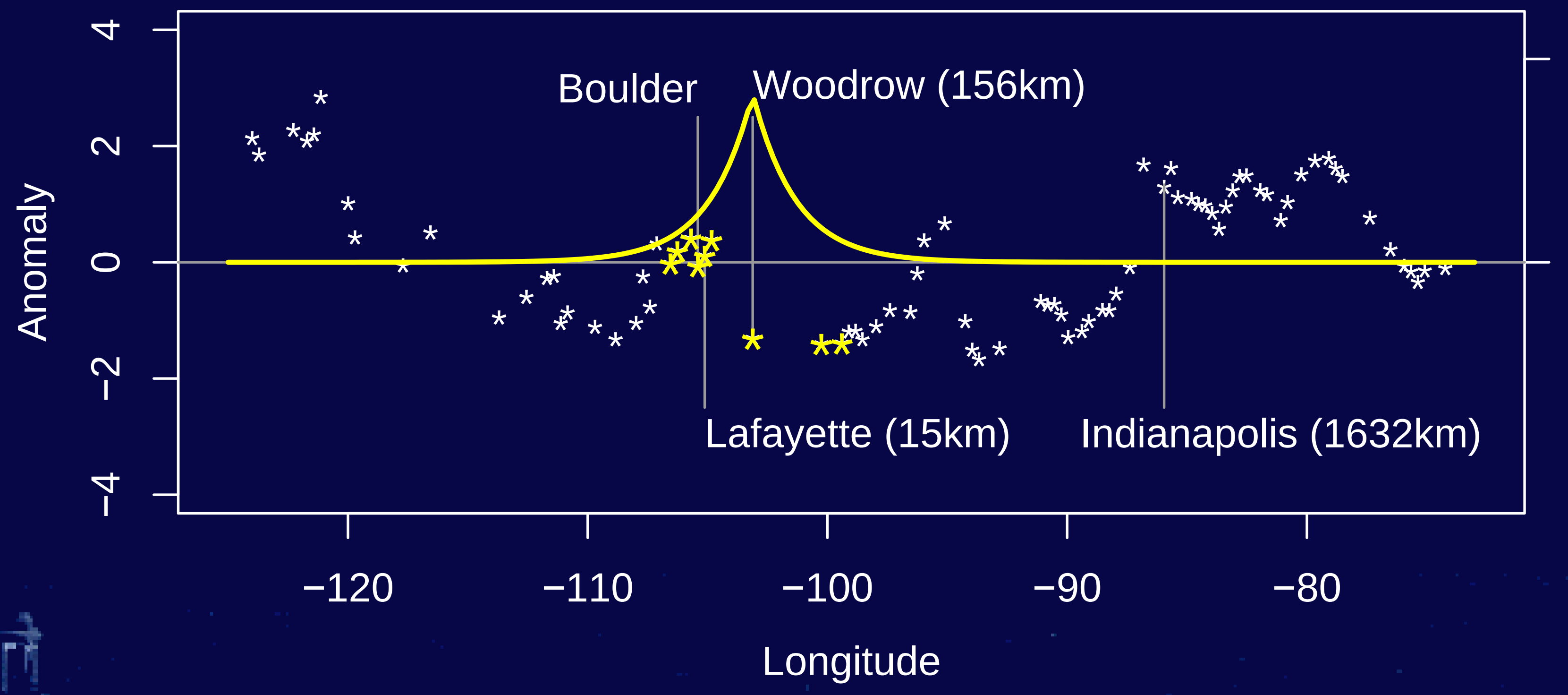
Nearest neighbor kriging with 8 observations



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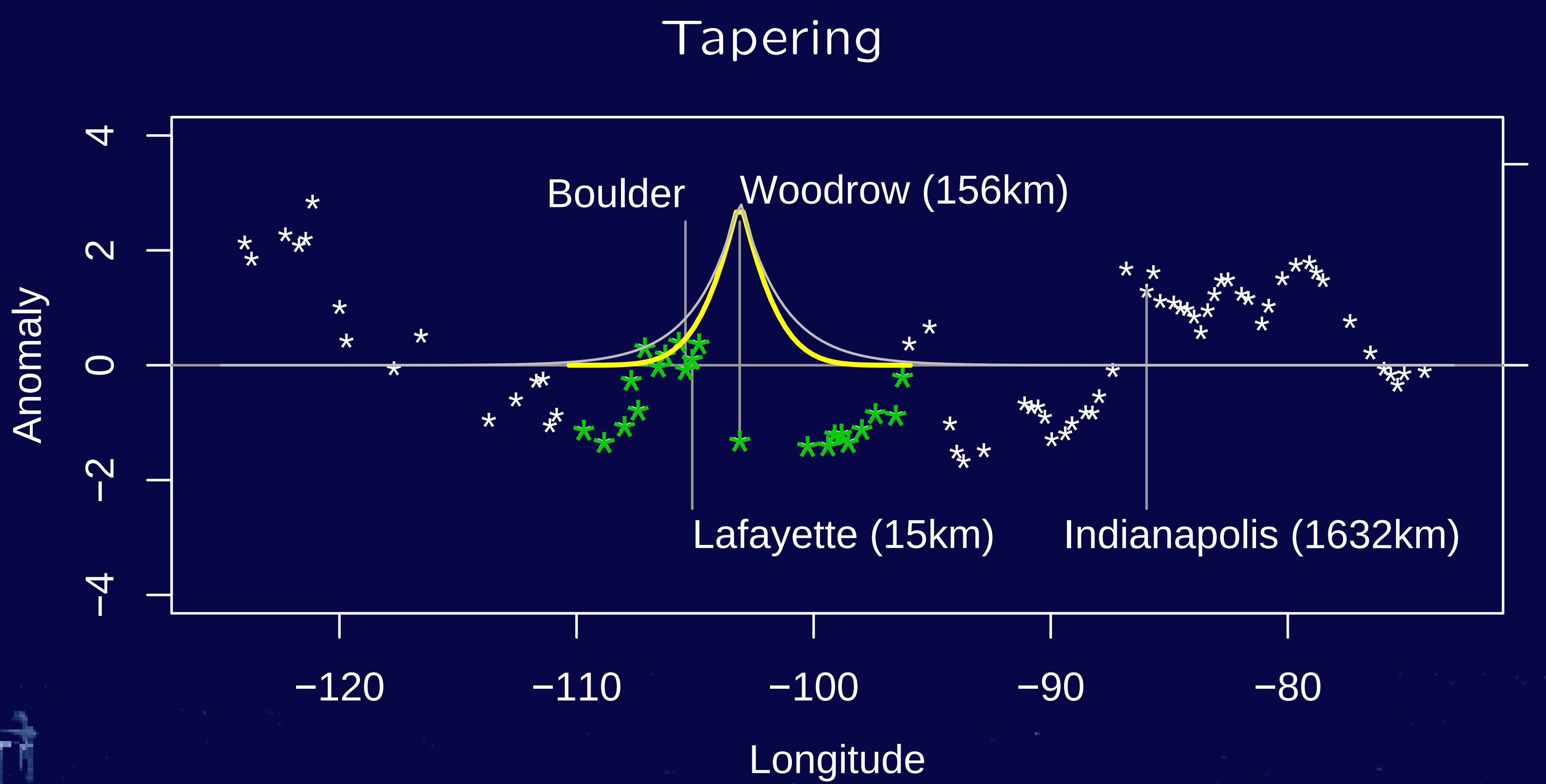
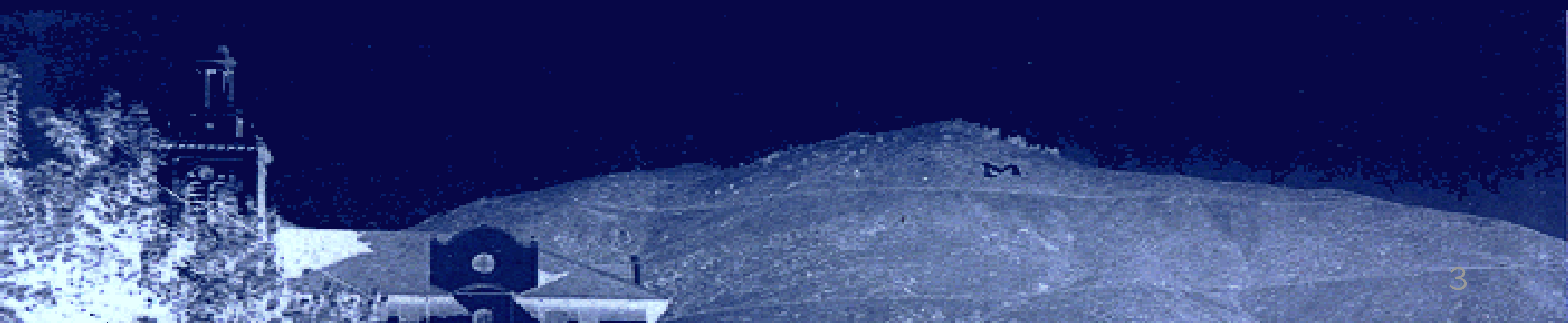


Objective

For an isotropic and stationary process
with covariance $C_0(h)$,
find a taper $C_\theta(h)$,
such that kriging with the product $C_0(h)C_\theta(h)$
is asymptotically optimal.



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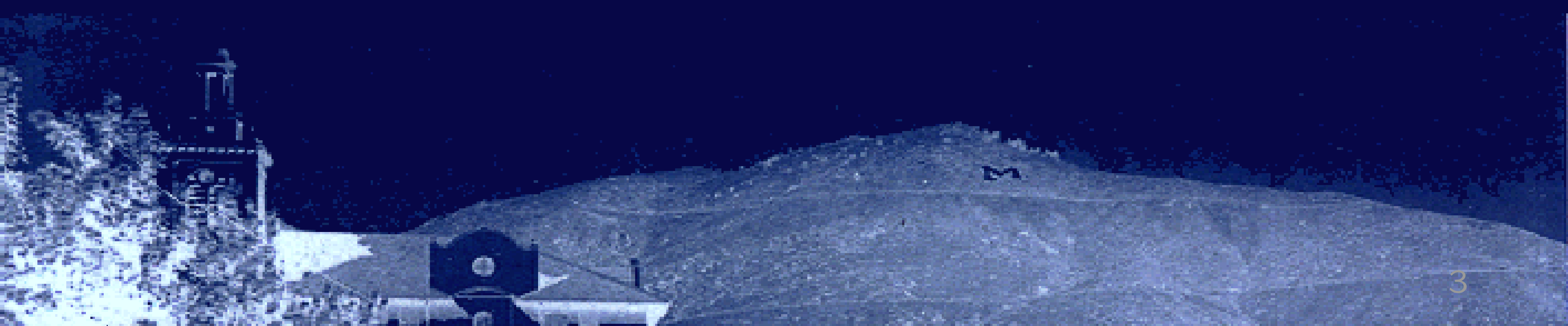
$$\frac{\text{MSE}(\mathbf{x}^*, C_0C_\theta)}{\text{MSE}(\mathbf{x}^*, C_0)} \to 1 \qquad \frac{\varrho(\mathbf{x}^*, C_0C_\theta)}{\text{MSE}(\mathbf{x}^*, C_0)} \to \gamma$$

$$\varrho(\mathbf{x}^*, C) = C(0) - \mathbf{c}^{*\mathsf{T}}\mathbf{C}^{-1}\mathbf{c}^*$$



Matérn Covariance

We need a broad, flexible class of covariances to describe spatial processes.

Matérn class covariance

$$C_{\alpha,\nu}(h) \propto \left(\alpha h\right)^{\nu} \mathcal{K}_{\nu}\left(\alpha h\right) \qquad h = \|\mathbf{h}\|$$

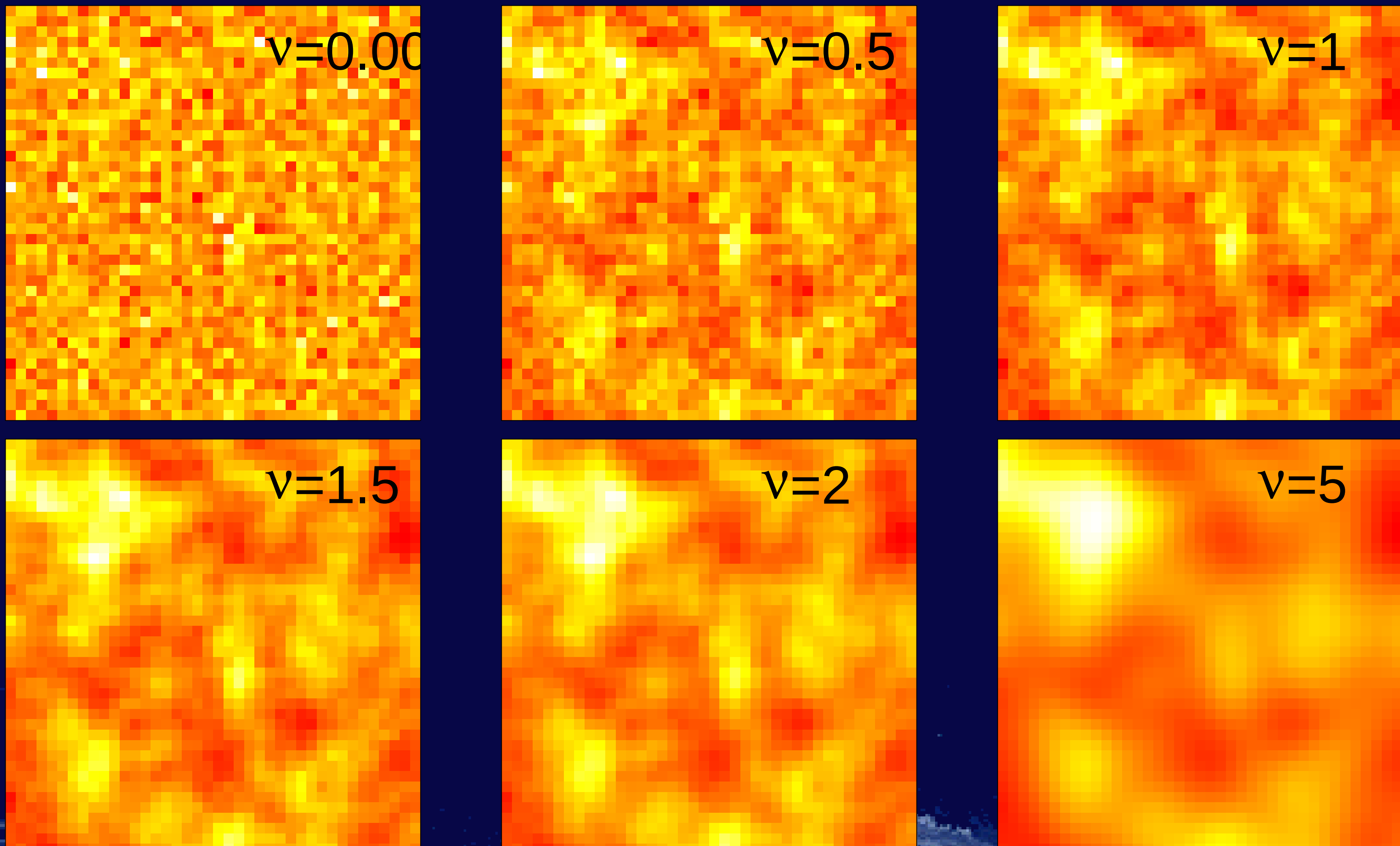
and spectral density

$$f_{\alpha,\nu}(\rho) \propto \frac{1}{\left(\alpha^2 + \rho^2\right)^{\nu + d/2}} \qquad \rho = \|\boldsymbol{\omega}\|$$

Differentiability at the origin of the covariance is related to the tail behavior of the spectrum, i.e. the smoothness of the process.

The process is m times mean squared differentiable iff $m < \nu$.

Matérn Covariance



Taper Functions

We impose on the taper C_θ the conditions

- C_θ is a positive definite function in $\mathbb{R}^d$
- $C_\theta(h) = 0$ for $h > \theta$

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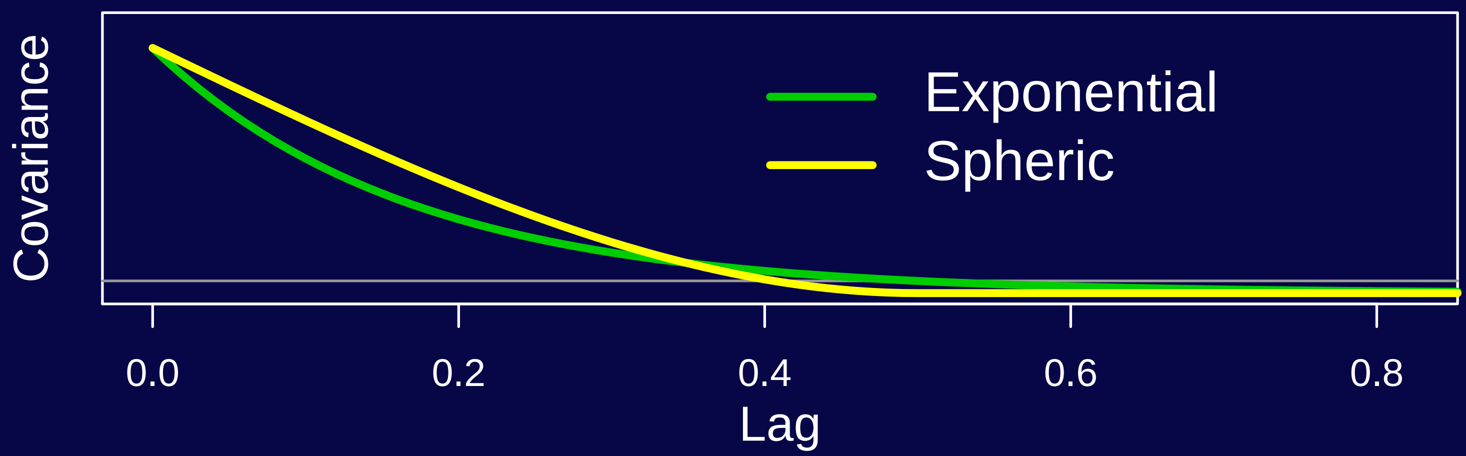
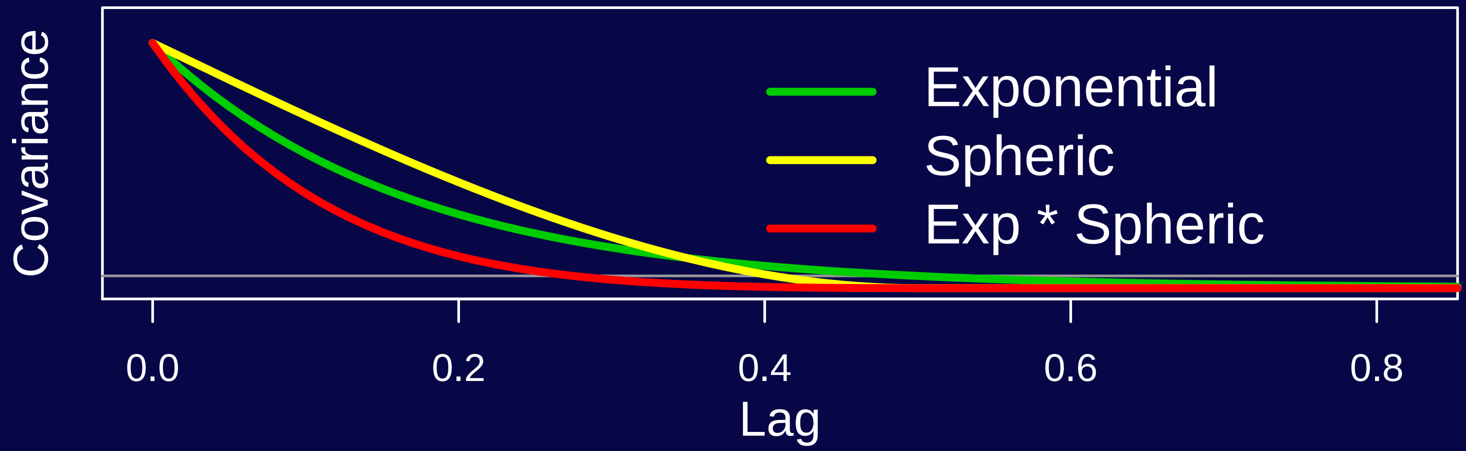
For example:

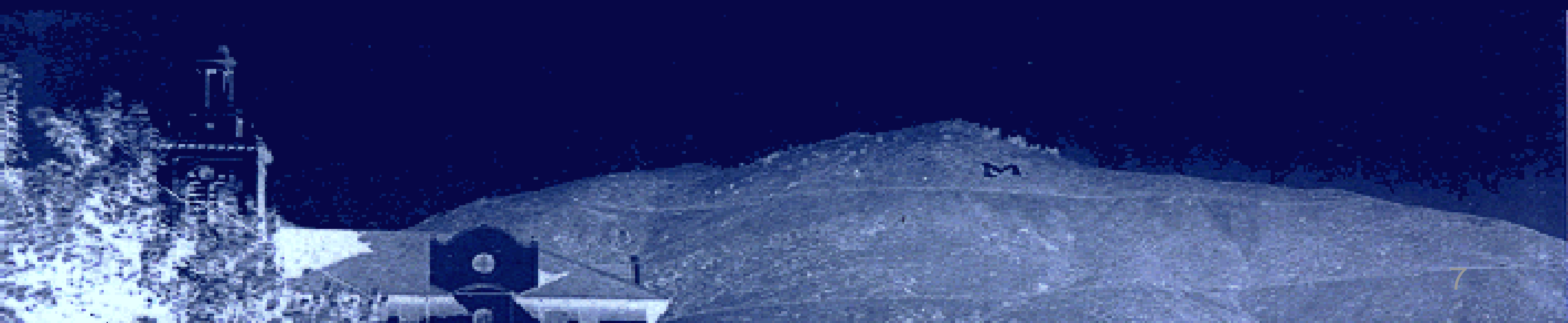
- triangular: $C_\theta(h) = \left(1 - \frac{|h|}{\theta}\right)_+$ $\qquad (x)_+ = \max(0, x)$
- spherical: $C_\theta(h) = \left(1 - \frac{|h|}{\theta}\right)_+^2 \left(1 + \frac{|h|}{\theta}\right)$
- Wendland-type: $C_\theta(h) = \left(1 - \frac{|h|}{\theta}\right)_+^{\ell+k} \cdot$ polynom in $\frac{|h|}{\theta}$ of deg k

Examples

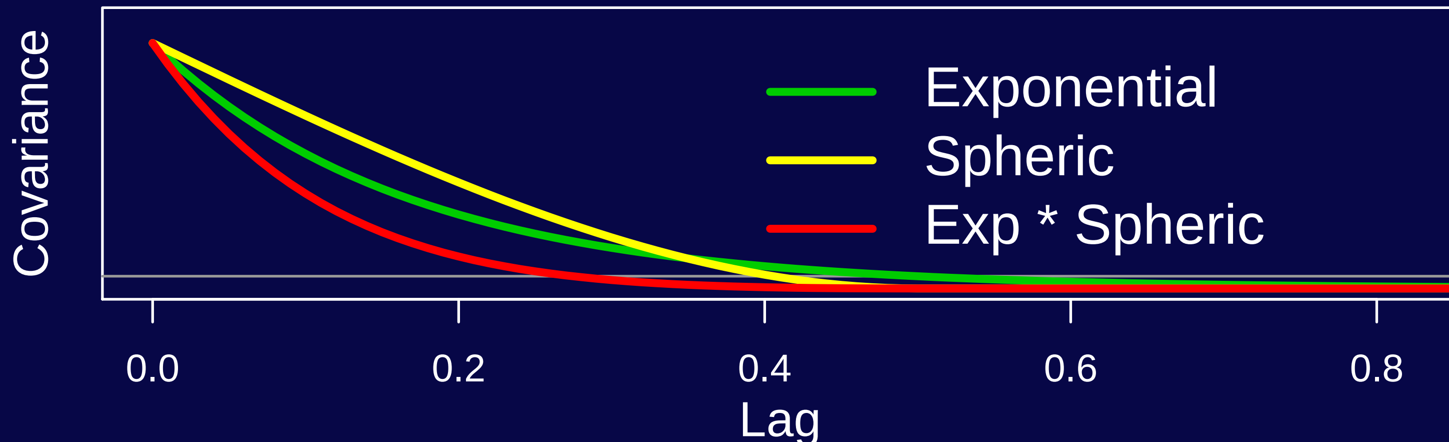


Examples


Covariance
Exponential
Spheric
Exp * Spheric
0.0
0.2
0.4
0.6
0.8
Lag



Examples



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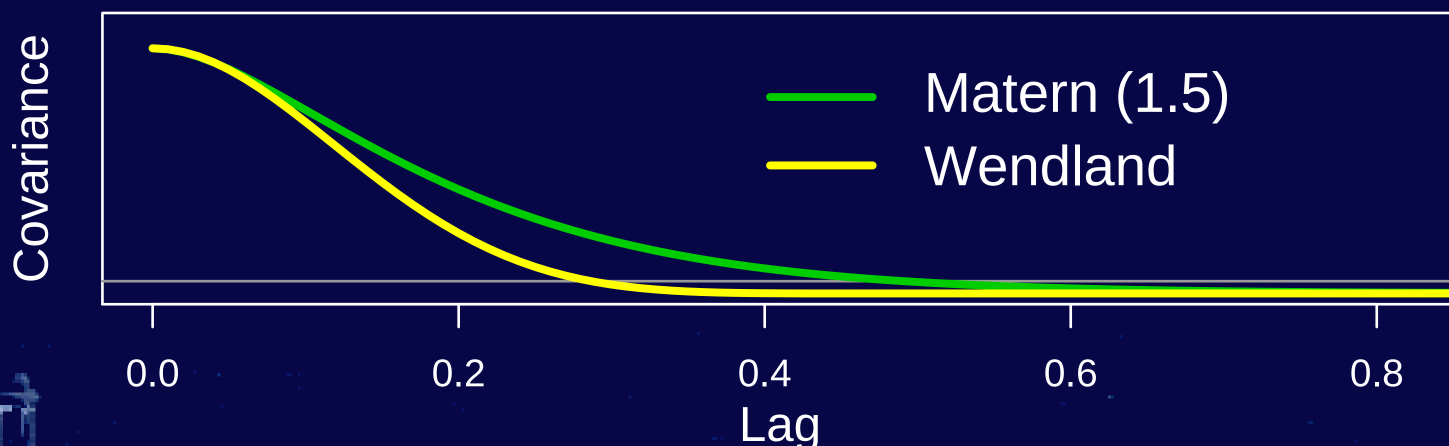

Covariance
Matern (1.5)
Wendland
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0.2
0.4
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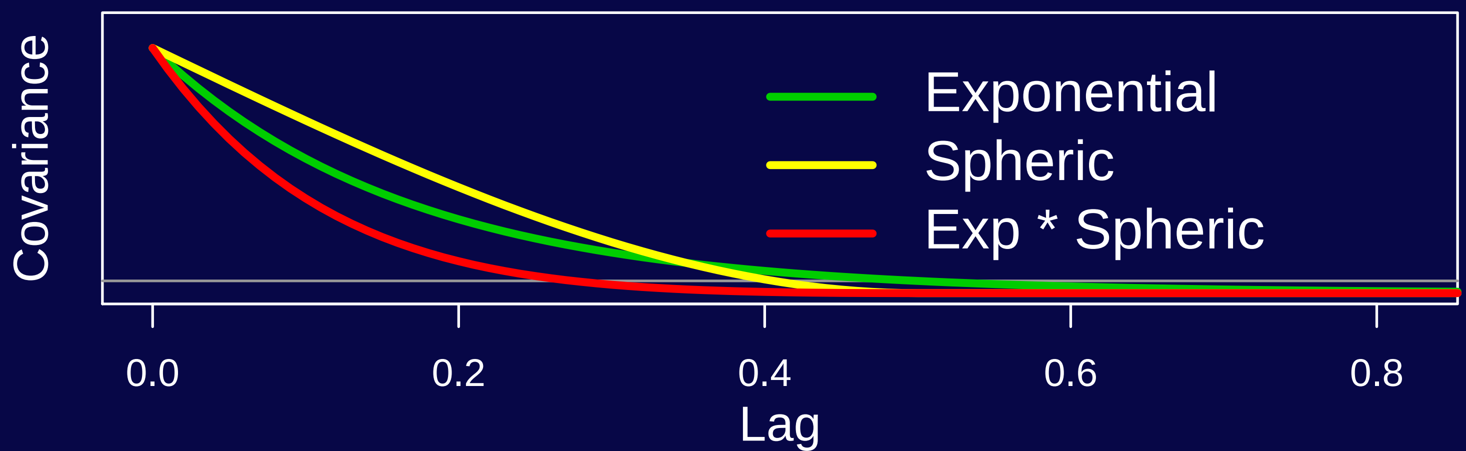
0.8
Lag

Examples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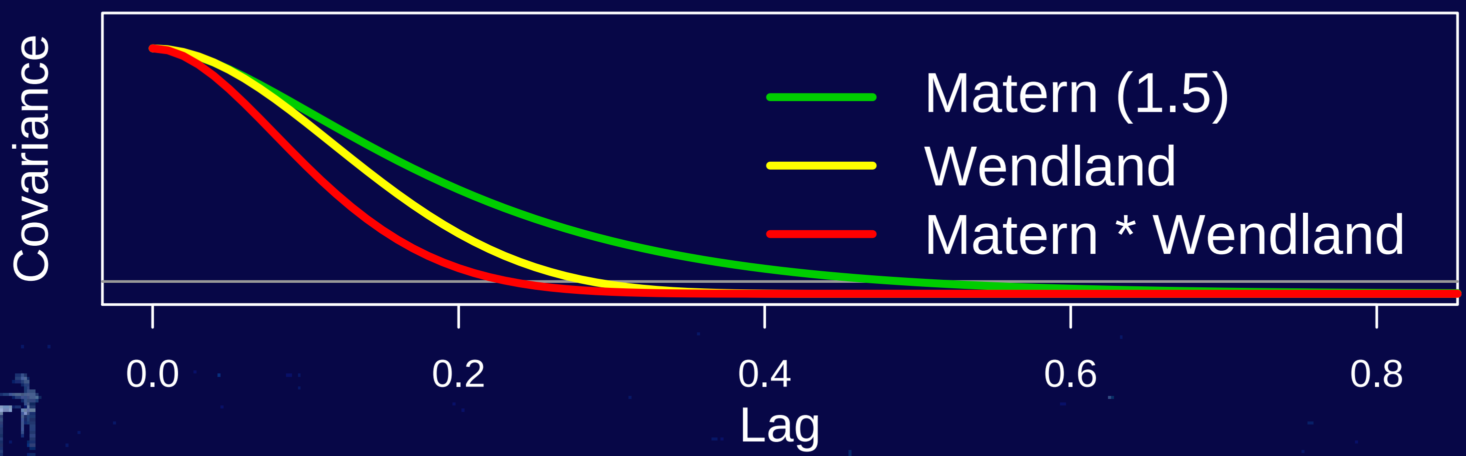
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Misspecified Covariances

In a series of (Annals) papers, Stein gives asymptotic results for misspecified covariances.

Suppose the true covariance C_0 and spectrum f_0. If we krig with the misspecified covariance C_1 characterized by f_1 then under the **Tail Condition**

$$\frac{f_1(\boldsymbol{\omega})}{f_0(\boldsymbol{\omega})} = \gamma \quad 0 < \gamma < \infty \qquad \text{as } \|\boldsymbol{\omega}\| \to \infty \qquad \text{and } \ldots$$

we have asymptotic optimality

$$\frac{\text{MSE}(\mathbf{x}^*, C_1)}{\text{MSE}(\mathbf{x}^*, C_0)} \to 1 \qquad\qquad \frac{\varrho(\mathbf{x}^*, C_1)}{\text{MSE}(\mathbf{x}^*, C_0)} \to \gamma$$

Tapered Covariances

Tapering is a form of misspecification if

$$\frac{\mathcal{F}\Big(C_{\alpha,\nu}(h)C_\theta(h)\Big)}{\mathcal{F}\Big(C_{\alpha,\nu}(h)\Big)} \to \gamma \text{ as } \|\boldsymbol{\omega}\| \to \infty$$

Which taper satisfies this condition?

The taper has to be

- as differentiable at the origin as the original covariance
- more differentiable throughout the domain than at the origin

Taper Theorem

Infill Condition: *Let* $\mathbf{x}^* \in \mathcal{D}$ *and* $\mathbf{x}_1, \mathbf{x}_2, \ldots$ *be a dense sequence in* $\mathcal{D}$.

Taper Condition: *Let* f_θ *be the spectral density of the taper covariance,* C_θ, *and for some* $\epsilon > 0$ *and* $M < \infty$

$$f_\theta(\rho) < \frac{M}{(1+\rho^2)^{\nu+d/2+\epsilon}}$$

Taper Theorem: *Assume that* $C_{\alpha,\nu}$ *is a Matérn covariance with smoothness parameter* ν *and the Infill and Taper Conditions hold. Then*

$$\lim_{n\to\infty} \frac{\mathrm{MSE}(\mathbf{x}^*, C_{\alpha,\nu}C_\theta)}{\mathrm{MSE}(\mathbf{x}^*, C_{\alpha,\nu})} = 1 \qquad \lim_{n\to\infty} \frac{\varrho(\mathbf{x}^*, C_{\alpha,\nu}C_\theta)}{\mathrm{MSE}(\mathbf{x}^*, C_{\alpha,\nu})} = 1 (= \gamma)$$

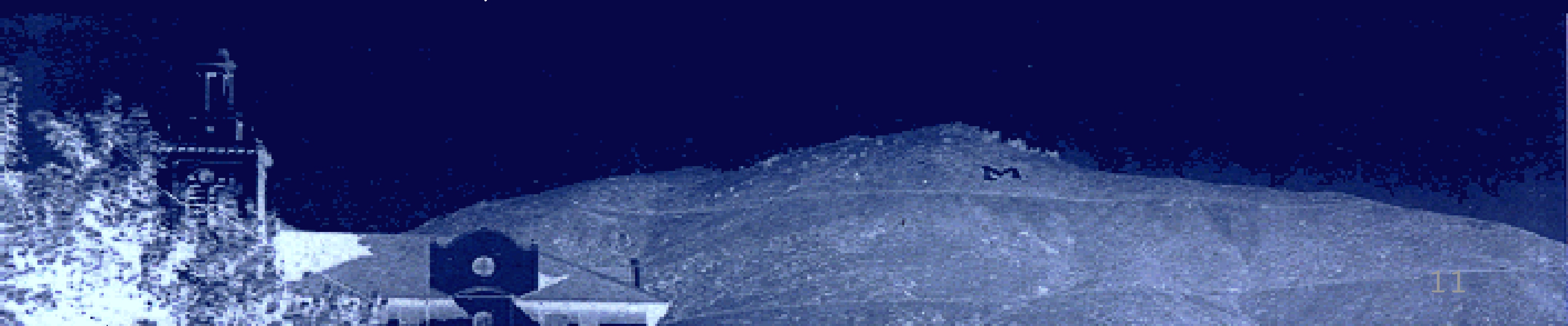
Conditions in Terms of Covariances

The principal irregular term (PIT) relates the tail behavior of the spectrum and the behavior at the origin of the covariance.

Formally, the PIT of C is the first term as a function of h in this series expansion about zero that is not raised to an even power.

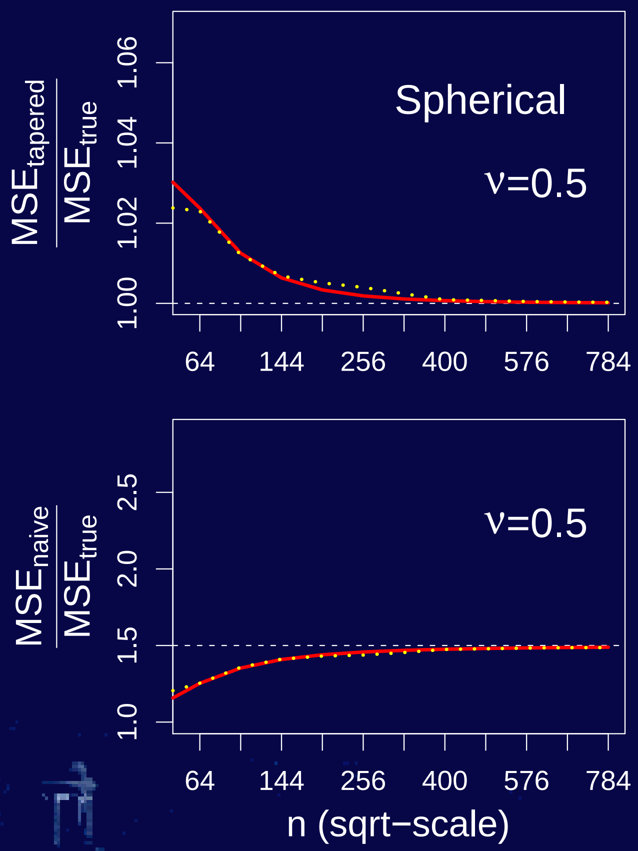
Conjecture: *Assume a polynomial isotropic covariance function* C_θ *in* $\mathbb{R}^d$ *that is integrable with* PIT Bh^μ *and . . . . Then the* PIT *and the tail behaviour are related by*

$$\lim_{\rho\to\infty} \rho^{\mu+d} f_\theta(\rho) = \left| B \cdot T(\mu, d)/\theta^{d-1} \right|$$



Simulation Study

When does infill asymptotics kick in?



Simulation setup:

– n equispaced or random (100 samples) observations in $[0,1]^2$

– $C_{\alpha,\nu}$: Matérn covariance, eff. range 0.4

– C_θ: spherical and Wendland-type, range $\theta = 0.4$

– prediction on $(0.5, 0.5)$.

Simulation Study

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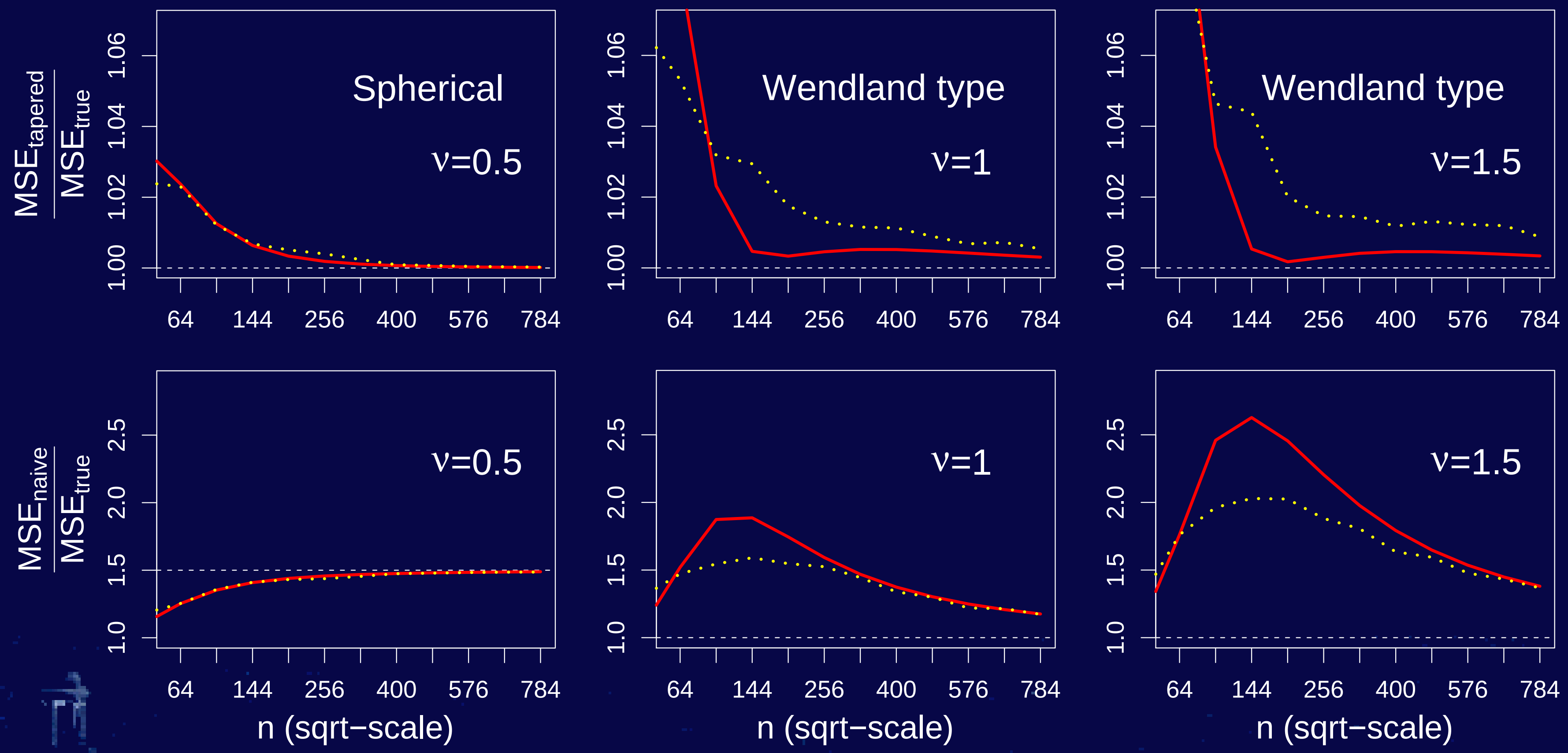
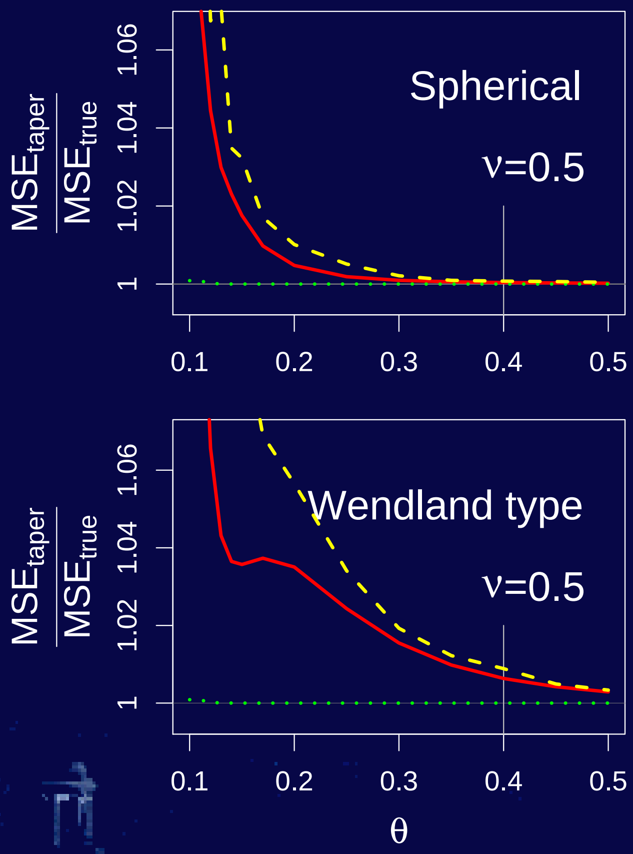


Simulation Study

What is an efficient taper?



Simulation setup:

– $n = 400$ equispaced or random observations in $[0,1]^2$

– $C_{\alpha,\nu}$: Matérn covariance, eff. range 0.4

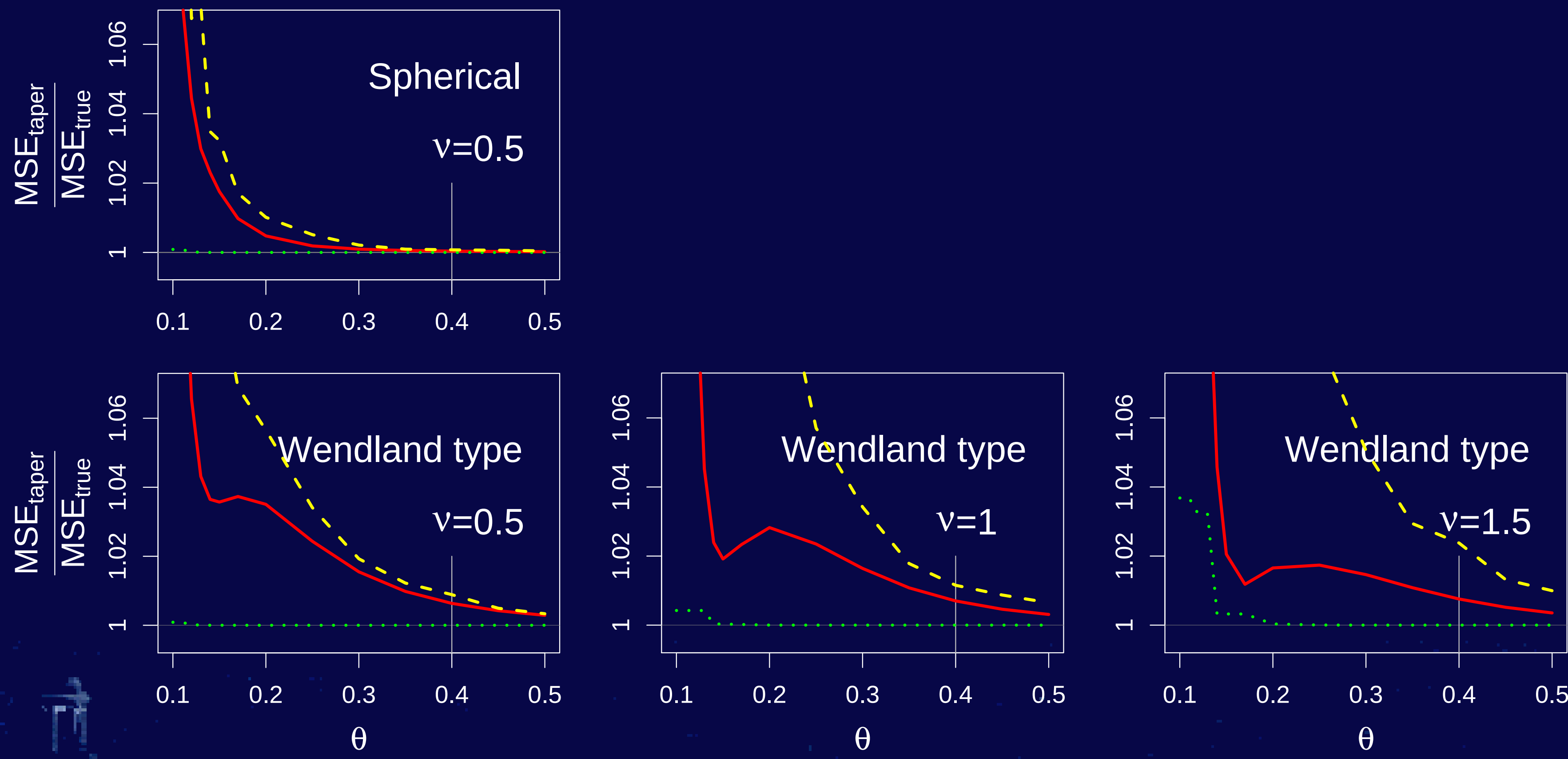
– C_θ: spherical, Wendland-type, range θ

– NN-kriging with neighborhood θ

– prediction on $(0.5, 0.5)$.

Simulation Study

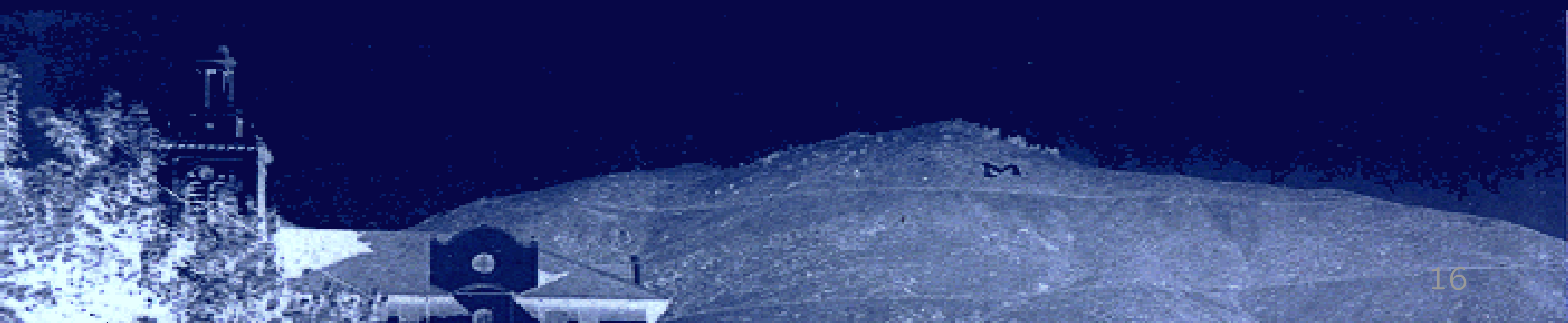
What is an efficient taper?



Conclusion

Tapering is an (asymptotically and computationally) efficient technique to create sparse covariance matrices.

Taper range can be justified by computing resources. However, 20–30 locations within the taper range is often sufficient.



References

For example:

Furrer, R., Genton, M. G. and Nychka, D. (2006). Covariance Tapering for Interpolation of Large Spatial Datasets. *Journal of Computational and Graphical Statistics*, 15(3), 502–523.

Furrer, R. and Sain, S. R. (2008). Spatial Model Fitting for Large Datasets with Applications to Climate and Microarray Problems. *Statistics and Computing*, doi:10.1007/s11222-008-9075-x.