Some Applications of HOMs

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Overview:

- Preconditioning: Optimized Schwarz for climate modeling
- Adaptive mesh refinements (on the sphere)
- Efficient time-stepping for AMR and SEM
- Discontinuous Galerkin for non-hydrostatic modeling

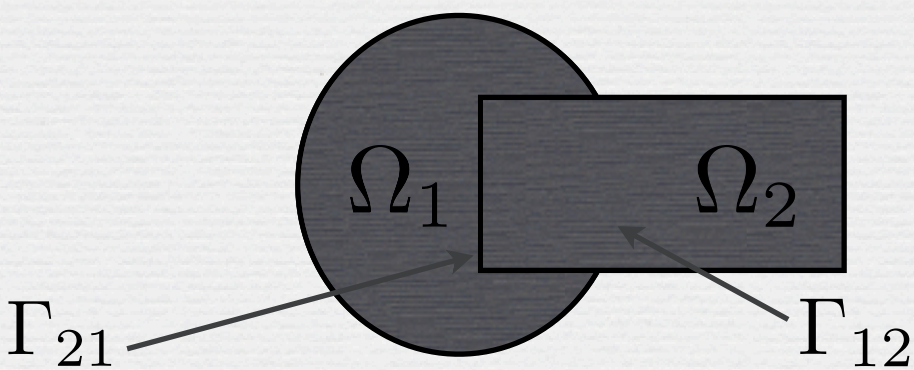
Classical Schwarz

Suppose we need to solve:

$$\mathcal{L}u = f \quad \text{in } \Omega, \qquad \mathcal{B}u = g \quad \text{on } \partial\Omega$$

Partition the original domain into 2 domains:

$$\begin{aligned} \mathcal{L}u_1^{n+1} &= f && \text{in } \Omega_1, & \mathcal{L}u_2^{n+1} &= f && \text{in } \Omega_2, \\ \mathcal{B}(u_1^{n+1}) &= g && \text{on } \partial\Omega_1, & \mathcal{B}(u_2^{n+1}) &= g && \text{on } \partial\Omega_2, \\ u_1^{n+1} &= u_2^n && \text{on } \Gamma_{12}, & u_2^{n+1} &= u_1^n && \text{on } \Gamma_{21}. \end{aligned}$$



The Robin method

- Lions (1990)
- Used to accelerate convergence of Schwarz
- Free positive parameter: how to find its correct value?
- Convergence rate not demonstrated theoretically

$$\mathcal{L}u_j^{k+1} = u_j^{k+1} - \Delta u_j^{k+1} = f_j$$
$$pu_j^{k+1} + \frac{\partial u_j^{k+1}}{\partial \mathbf{n}_{jl}} = pu_l^k + \frac{\partial u_l^k}{\partial \mathbf{n}_{jl}} \text{ on } \partial\Omega_j \cap \partial\Omega_l \text{ for } l \in \mathcal{N}(\Omega_j)$$
$$u_j^{k+1} = u_0 \text{ on } \partial\Omega_j \cap \partial\Omega$$

Optimized approach

- Inspired by the Robin problem:

$$\begin{aligned} (\eta-\Delta)u_1^{n+1} &= 0 && \text{in } \Omega_1, & (\eta-\Delta)u_2^{n+1} &= 0 && \text{in } \Omega_2, \\ (\partial_x+S_1)u_1^{n+1} &= (\partial_x+S_1)u_2^n && \text{on } \Gamma_{12}, & (\partial_x+S_2)u_2^{n+1} &= (\partial_x+S_2)u_1^n && \text{on } \Gamma_{21}. \end{aligned}$$

We are looking for the best possible forms of in Fourier space

Proceeding as before leads to the solutions: $(\sigma_r(k)=\mathcal{F}(S_r))$

$$\hat{u}_1^n(x,k) = \frac{\sigma_1(k)-\sqrt{k^2+\eta}}{\sigma_1(k)+\sqrt{k^2+\eta}}e^{-\sqrt{k^2+\eta}(x-L)}\hat{u}_2^{n-1}(L,k), \qquad \hat{u}_2^n(x,k) = \frac{\sigma_2(k)+\sqrt{k^2+\eta}}{\sigma_2(k)-\sqrt{k^2+\eta}}e^{-\sqrt{k^2+\eta}x}\hat{u}_1^{n-1}(0,k)$$

New convergence rate:

$$\rho_{opt}=\rho_{opt}(k,\eta,L)=\frac{\sigma_1(k)-\sqrt{k^2+\eta}}{\sigma_1(k)+\sqrt{k^2+\eta}}\frac{\sigma_2(k)+\sqrt{k^2+\eta}}{\sigma_2(k)-\sqrt{k^2+\eta}}e^{-2\sqrt{k^2+\eta}L}$$

Optimized Schwarz: algebraic results

- SGT 2006: show how to modify existing Schwarz algorithm to yield optimized versions
- The augmented or "enhanced" system is rediscovered
- Spectral elements are natural candidates:
  - Overlapping grids are cumbersome to construct
  - Block preconditioning costly: FDM when possible
  - Optimal preconditioner is known (SD Kim 2006)
  - Q1-GLL based problem costly to invert does not scale: use MG or other solver to invert

Creating the augmented system from a weak form

Consider:

$$w_j^{k+1} - \Delta w_j^{k+1} = G_j(x,y)$$
$$pw_j^{k+1} + \frac{\partial w_j^{k+1}}{\partial \mathbf{n}_{jl}} = pw_l^k + \frac{\partial w_l^k}{\partial \mathbf{n}_{jl}} \text{ on } \partial\Omega_j \cap \partial\Omega_l \text{ for } l \in \mathcal{N}(\Omega_j)$$
$$w_j^{k+1} = w_0 \text{ on } \partial\Omega_j \cap \partial\Omega$$

To be solved for all k on any Ω_j : it converges (Lions 1990).

Weak form:

$$\int_{\Omega_j} \phi_j w_j^{k+1} + \int_{\Omega_j} \nabla\phi_j \cdot \nabla w_j^{k+1} - \int_{\partial\Omega_j} \phi_j \left(\frac{\partial w_j}{\partial \mathbf{n}}\right)^{k+1} = \int_{\Omega_j} \phi_j G_j.$$

Where test functions are in:

$$H^1(\mathcal{Q}_h) = \{v \in L^2(\Omega) | v|_Q \in H^1(Q)\ \forall Q \in \mathcal{Q}_h\}$$

Decomposition: $\mathcal{Q}_h = \cup_j \Omega_j$

Creating the augmented system from a weak form

Define:

$$a_j(w_j^{k+1}, \phi_j) \equiv \int_{\Omega_j} \phi_j w_j^{k+1} + \int_{\Omega_j} \nabla \phi_j \cdot \nabla w_j^{k+1}$$

$$f_j(\phi_j) \equiv \int_{\Omega_j} \phi_j G_j$$

$$T_j(w_j^{k+1}, \phi_j) \equiv \sum_{l \in \mathcal{N}(\Omega_j)} \int_{\Gamma_{jl}} \phi_j \left(\frac{\partial w_j}{\partial \mathbf{n}_{jl}} \right)^{k+1}$$

Leads to:

$$a_j(w_j^{k+1}, \phi_j) - T_j(w_j^{k+1}, \phi_j) = f_j(\phi_j)$$

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Leads to:

$$a_j(w_j^{k+1}, \phi_j) - T_j(w_j^{k+1}, \phi_j) = f_j(\phi_j)$$

Remains to introduce the artificial transmission condition...

Creating the augmented system from a weak form

- The normal derivative can be written in terms of the original bilinear operator (Toselli, Widlund 2005)
- Avoids the difficult duality pairing for functions on the edges of the subdomains

$$\begin{aligned} T_j(w_j^{k+1}, \phi_j) &= \int_{\Omega_j} \nabla \phi_j \cdot \nabla w_j^{k+1} + \int_{\Omega_j} \phi_j \Delta w_j^{k+1} \\ &= \int_{\Omega_j} \nabla \phi_j \cdot \nabla w_j^{k+1} + \int_{\Omega_j} \phi_j w_j^{k+1} - f_j(\phi_j) \\ &= a_j(w_j^{k+1}, \phi_j) - f_j(\phi_j) \end{aligned}$$

Where we pick $\phi_j \in H^1(\partial\Omega_j)$

Creating the augmented system from a weak form

Boundary condition is:

$$\begin{aligned} T_j(w_j^{k+1}, \phi_j) &= \sum_{l \in \mathcal{N}(\Omega_j)} T_j(w_j^{k+1}, \phi_j|_{\Gamma_{jl}}) \\ &= \sum_{l \in \mathcal{N}(\Omega_j)} \{ \int_{\Gamma_{jl}} p\phi_j(w_l^k - w_l^{k+1}) - T_l(w_l^k, \phi_j|_{\Gamma_{jl}}) \} \\ &= -\int_{\Omega_j} p\phi_j w_j^{k+1} + \sum_{l \in \mathcal{N}(\Omega_j)} \{ \int_{\Gamma_{jl}} p\phi_j w_l^k - T_l(w_l^k, \phi_j|_{\Gamma_{jl}}) \} \end{aligned}$$

where a sum on neighbors appears.

Creating the augmented system from a weak form

Boundary condition is:

$$
\begin{aligned}
T_j(w_j^{k+1}, \phi_j) &= \sum_{l \in \mathcal{N}(\Omega_j)} T_j(w_j^{k+1}, \phi_j|_{\Gamma_{jl}}) \\
&= \sum_{l \in \mathcal{N}(\Omega_j)} \{ \int_{\Gamma_{jl}} p\phi_j (w_l^k - w_l^{k+1}) - T_l(w_l^k, \phi_j|_{\Gamma_{jl}}) \} \\
&= -\int_{\Omega_j} p\phi_j w_j^{k+1} + \sum_{l \in \mathcal{N}(\Omega_j)} \{ \int_{\Gamma_{jl}} p\phi_j w_l^k - T_l(w_l^k, \phi_j|_{\Gamma_{jl}}) \}
\end{aligned}
$$

where a sum on neighbors appears.

Leads to the form required by the algorithm

$$
\begin{aligned}
a_j(w_j^{k+1}, \phi_j) + \int_{\Omega_j} p\phi_j w_j^{k+1} &= f_j(\phi_j) + \sum_{l \in \mathcal{N}(\Omega_j)} f_l(\phi_l|_{\Gamma_{jl}}) \\
&\quad - \sum_{l \in \mathcal{N}(\Omega_j)} a_l(w_l^k, \phi_l|_{\Gamma_{jl}}) + \sum_{l \in \mathcal{N}(\Omega_j)} \int_{\Gamma_{jl}} p\phi_l w_l^k
\end{aligned}
$$

Creating the augmented system from a weak form

$$a_j(w_j^{k+1}, \phi_j) + \int_{\Omega_j} p\phi_j w_j^{k+1} = f_j(\phi_j) + \sum_{l \in \mathcal{N}(\Omega_j)} f_l(\phi_l|_{\Gamma_{jl}}) - \sum_{l \in \mathcal{N}(\Omega_j)} a_l(w_l^k, \phi_l|_{\Gamma_{jl}}) + \sum_{l \in \mathcal{N}(\Omega_j)} \int_{\Gamma_{jl}} p\phi_l w_l^k$$

After (any)

$$\tilde{A}_j \boldsymbol{u}_j^{n+1} = \boldsymbol{f}_j + \sum_{k=1}^{J} \tilde{B}_{jk} \boldsymbol{u}_k^n, \quad j = 1, .., J$$

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Creating the augmented system from a weak form

$$a_j(w_j^{k+1},\phi_j)+\int_{\Omega_j} p\phi_j w_j^{k+1} = f_j(\phi_j) + \sum_{l\in\mathcal{N}(\Omega_j)} f_l(\phi_l|_{\Gamma_{jl}})$$

$$\boxed{-\sum_{l\in\mathcal{N}(\Omega_j)} a_l(w_l^k,\phi_l|_{\Gamma_{jl}}) + \sum_{l\in\mathcal{N}(\Omega_j)} \int_{\Gamma_{jl}} p\phi_l w_l^k}$$

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Possible to create augmented system!

Creating the augmented system from a weak form

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Possible to create augmented system!

Not mentioned: difficulties at corners

Creating the augmented system from a weak form

$$a_j(w_j^{k+1}, \phi_j) + \int_{\Omega_j} p\phi_j w_j^{k+1} = f_j(\phi_j) + \sum_{l \in \mathcal{N}(\Omega_j)} f_l(\phi_l|_{\Gamma_{jl}})$$
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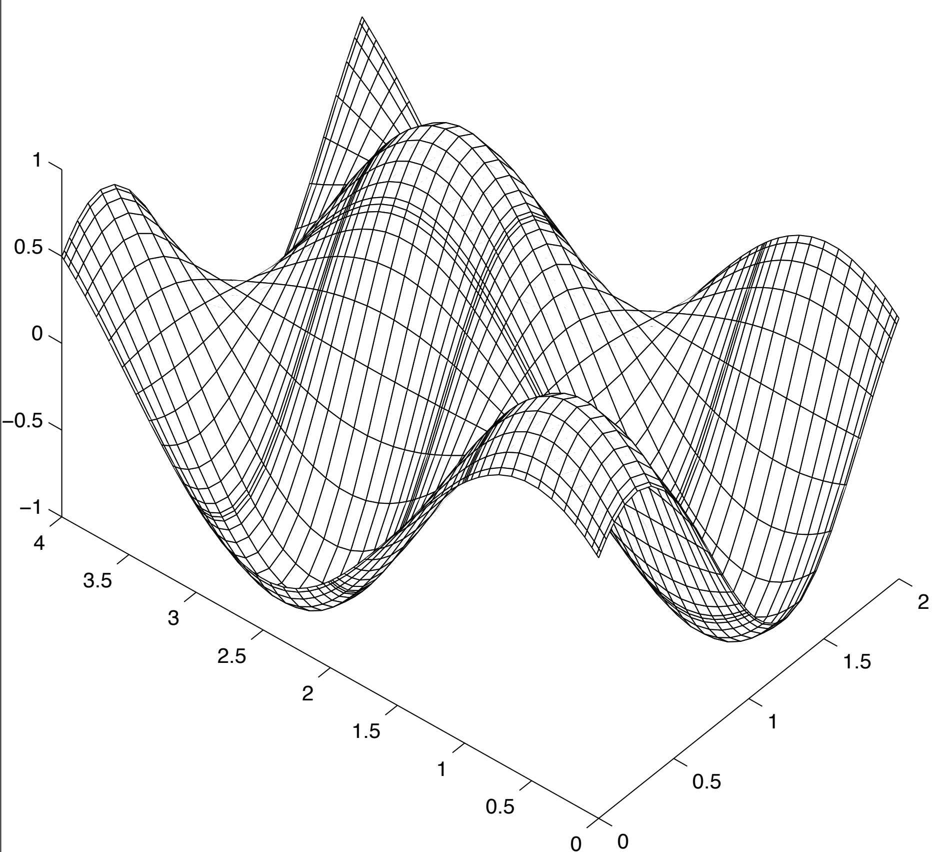
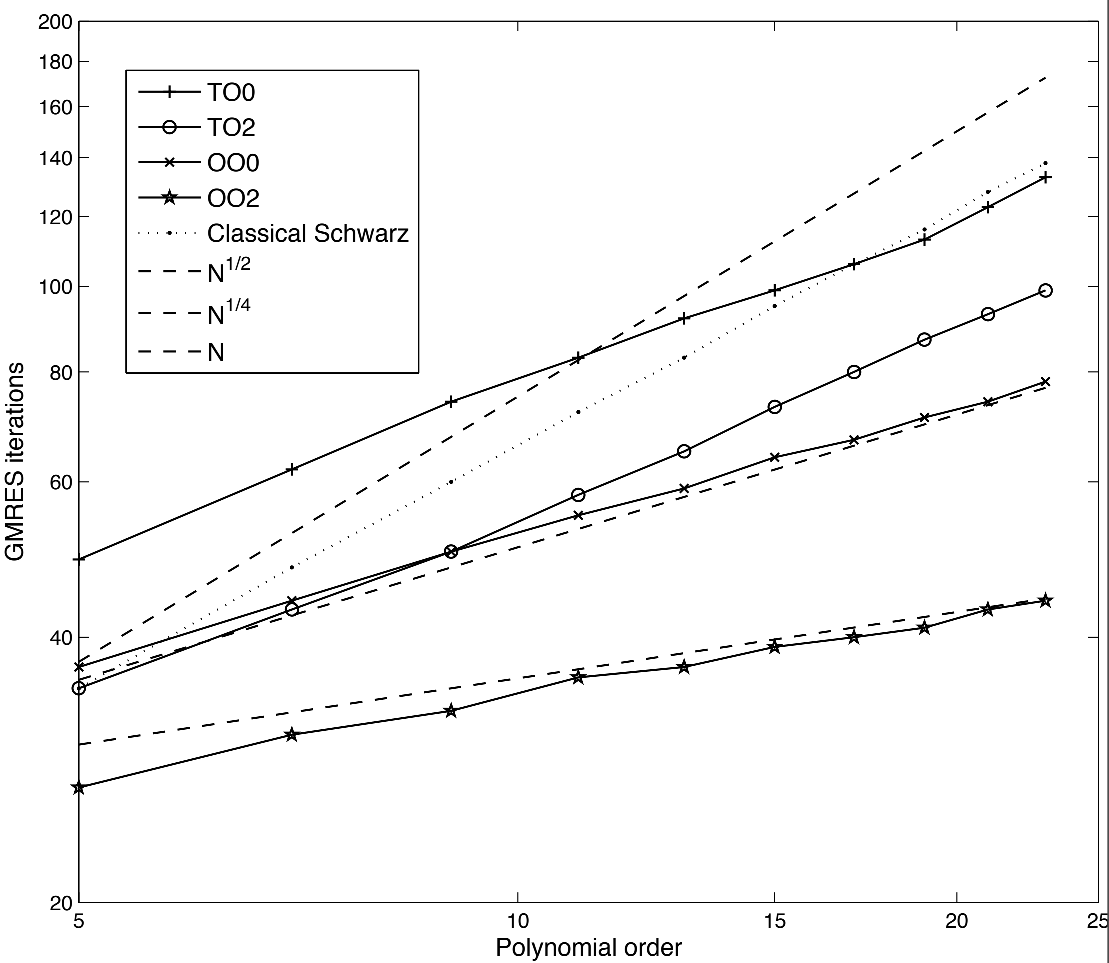
Not mentioned: “under” integration for SEM

SEM simple problem

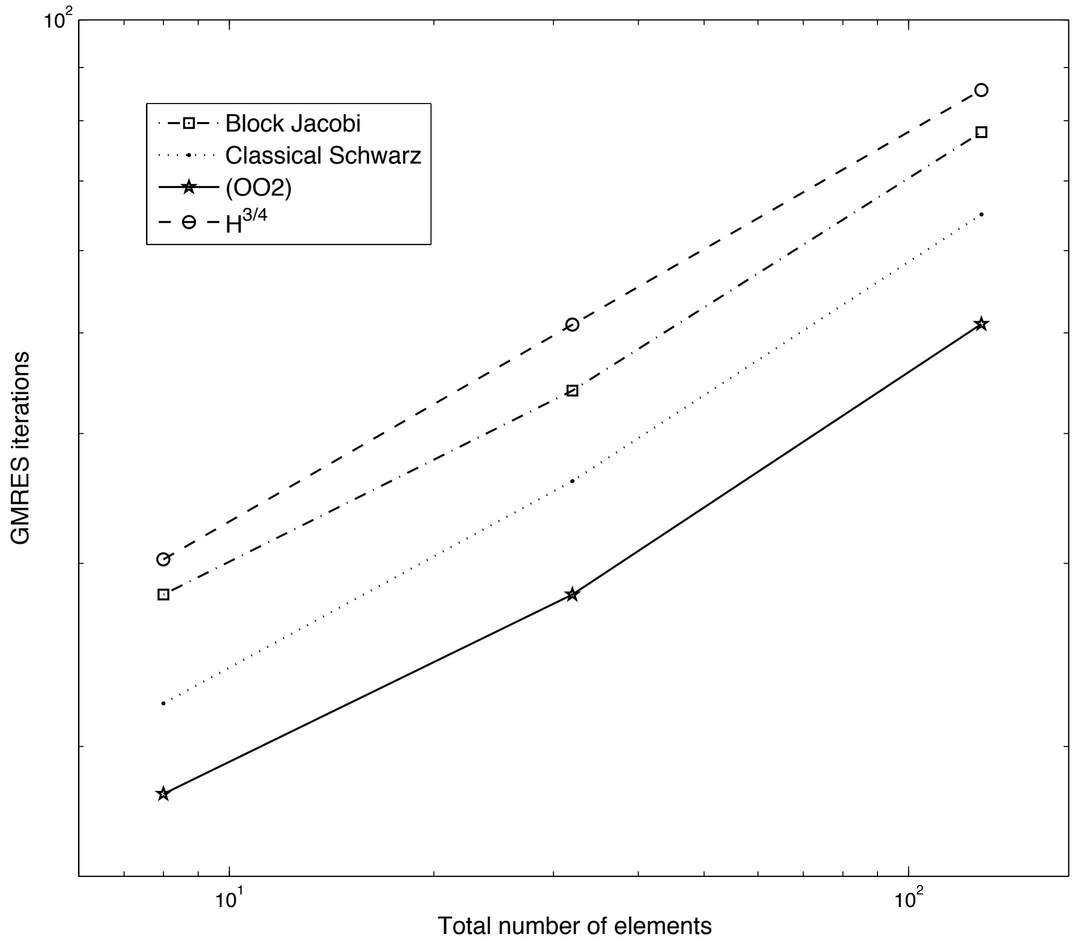
$\mathcal{L}u = (\eta - \Delta)u = f, \quad \text{in } \Omega,$

Gander 2006

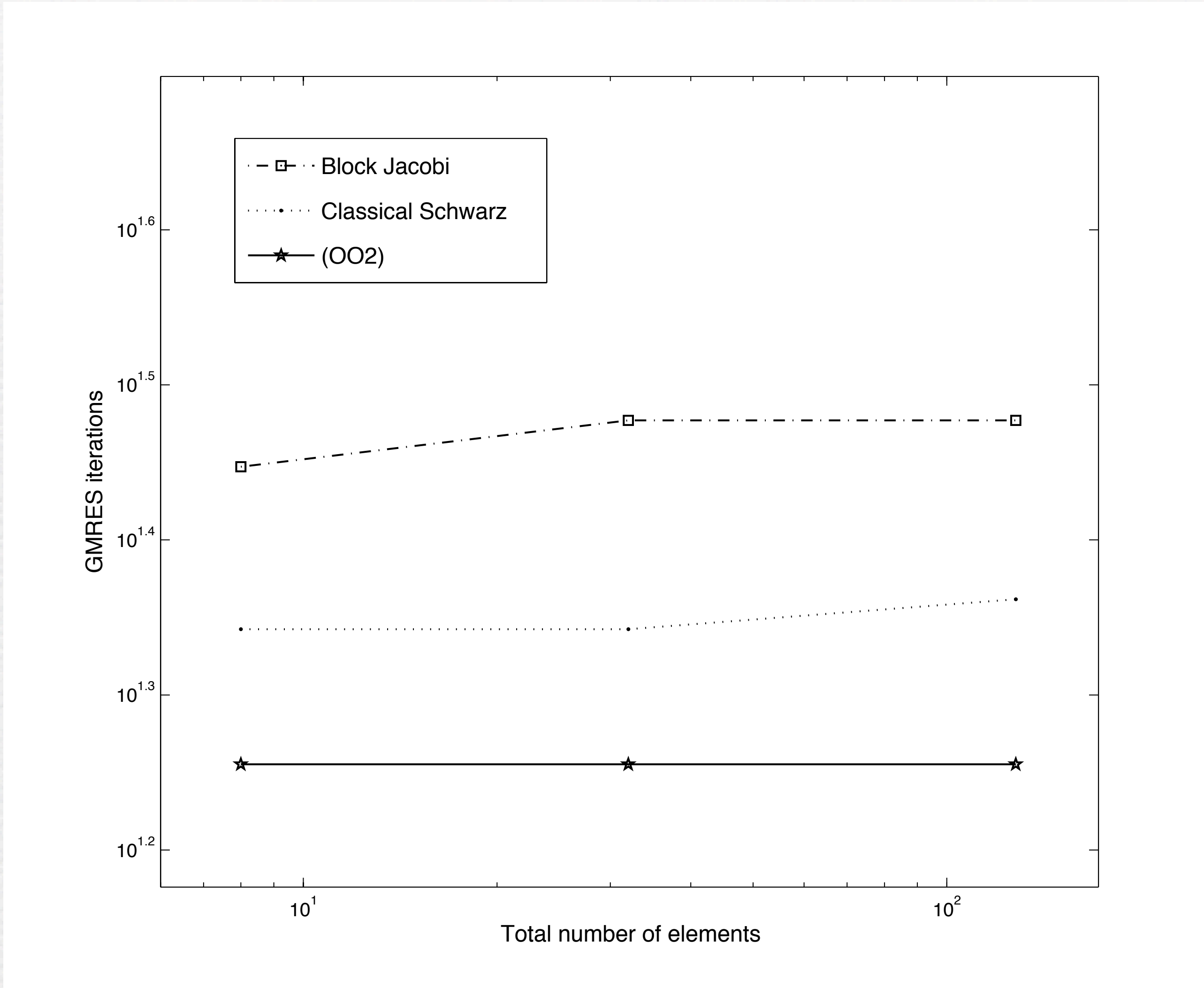




SEM simple problem



SEM simple problem



Primitive equations

Momentum: $$\frac{d\mathbf{v}}{dt} + f\,\mathbf{k} \times \mathbf{v} + \nabla\Phi + R\,T\,\nabla \ln p = 0$$

Thermodynamic: $$\frac{d\,T}{d\,t} - \frac{\kappa\,T\,\omega}{p} = 0$$

Continuity: $$\frac{\partial}{\partial t}\left(\frac{\partial p}{\partial \eta}\right) + \nabla \cdot \left(\mathbf{v}\,\frac{\partial p}{\partial \eta}\right) + \frac{\partial}{\partial \eta}\left(\dot{\eta}\,\frac{\partial p}{\partial \eta}\right) = 0$$

HOMME: high order multiscale modeling environment

Primitive equations: SI

Hydrostatic assumption: $\frac{\partial \Phi}{\partial \eta} = -\frac{R\,T}{p}\frac{\partial p}{\partial \eta}.$

Linearization (barotropic state): $T^r = 300K, \;\; p_s^r = 1000hPa$

Semi-Implicit:

$$\frac{dX}{dt} = \mathcal{M}(X)$$

Add zero: $\frac{dX}{dt} = \mathcal{M}(X) + \mathcal{L}X - \mathcal{L}X = \mathcal{N}(X) - \mathcal{L}X$

$$\frac{X^{n+1} - X^{n-1}}{2\Delta t} = \mathcal{N}(X^n) - \frac{1}{2}\mathcal{L}(X^{n+1} + X^{n-1}) = \mathcal{M}(X^n) + \mathcal{L}X^n - \frac{1}{2}\mathcal{L}(X^{n+1} + X^{n-1})$$

$$\frac{X^{n+1} - X^{n-1}}{2\Delta t} = \mathcal{M}(X^n) - \frac{1}{2}\Delta_{tt}\mathcal{L}X$$

"Time diffusion"

PE: vertical structure matrix

Results of hydrostatic assumption and vertical coordinate choice:

$$p(\eta, p_s) = A(\eta)p_0 + B(\eta)p_s$$

$$\mathbf{A} = R\mathbf{H}^r\mathbf{T} + RT^r P,$$

$$G^r - \Delta t^2 \mathbf{A}\nabla^2 G^r = B - \Delta t \mathbf{A}\nabla \cdot \mathcal{V}$$

Solve for each k:

$$\left(\nabla^2 - \frac{1}{\Delta t^2 \lambda_k}\right)\Gamma_k^r = C_k$$

Series of 2D Helmholtz

Barotropic eigenmodes of atmosphere

Backsub:

$$\begin{aligned} D &= \Delta t^{-1}\,\mathbf{A}^{-1}\,(B - G^r) \\ \ln p_s &= \mathcal{P} - \Delta t\, P \cdot D \\ T &= \mathcal{T} - \Delta t\, \mathbf{T} D \\ \mathbf{v} &= \mathcal{V} - \Delta t\, \nabla G^r \end{aligned}$$

PE: vertical structure matrix

Results of hydrostatic assumption
and vertical coordinate choice: $p(\eta, p_s) = A(\eta)p_0 + B(\eta)p_s$

$$\mathbf{A} = R\mathbf{H}^r\mathbf{T} + RT^r P, \longleftarrow \text{Diagonalize}$$

$$G^r - \Delta t^2 \mathbf{A}\nabla^2 G^r = B - \Delta t \mathbf{A}\nabla \cdot \mathcal{V}$$

Solve for each k:

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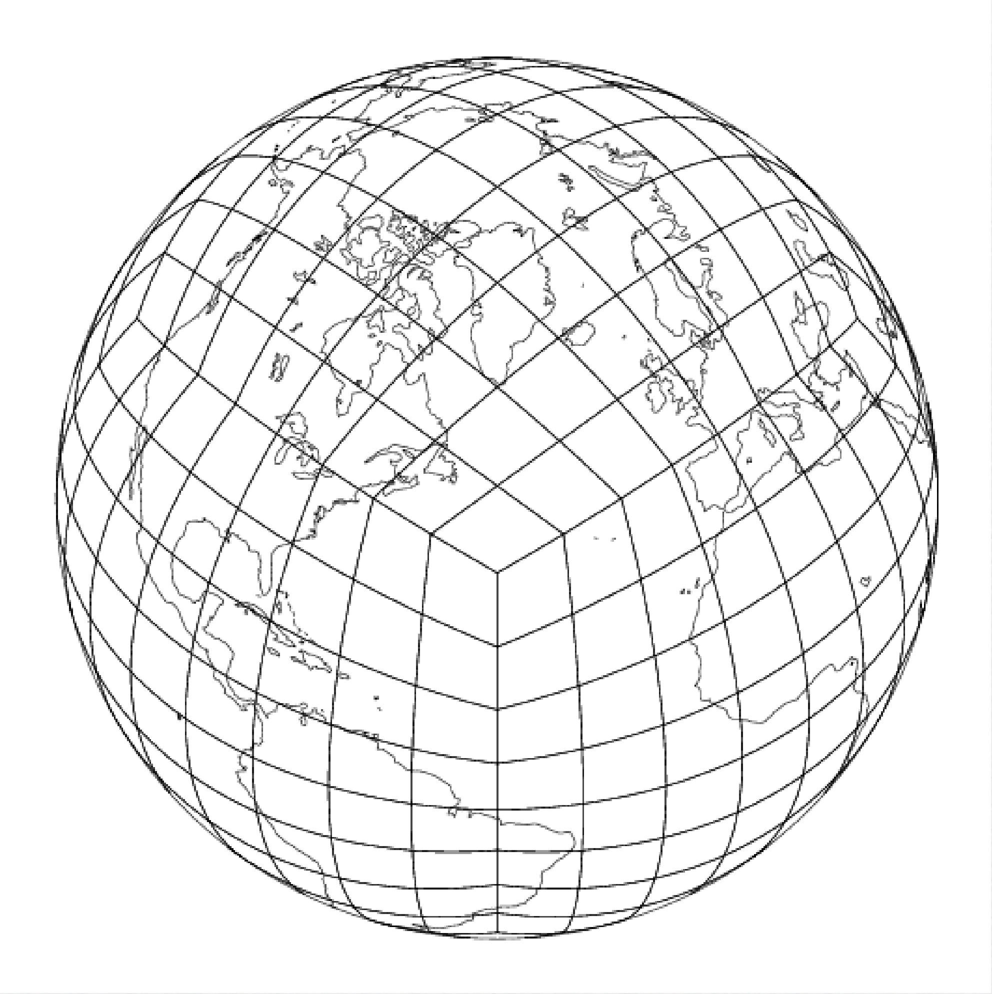
Cubed sphere

- Equiangular projection
- Sadourny (72), Rancic (96), Ronchi (96)
- Most models moving towards this approach
- SFC: Dennis 2003



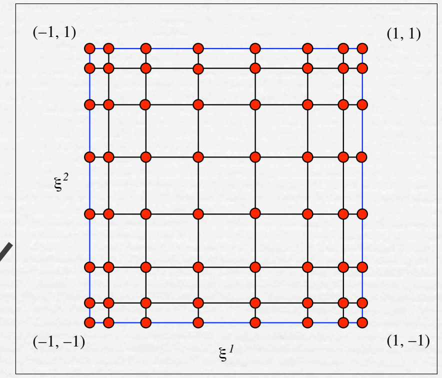
Metric tensor

$$g_{ij} = \frac{1}{r^4 \cos^2 x_1 \cos^2 x_2} \left[\begin{array}{cc} 1 + \tan^2 x_1 & -\tan x_1 \tan x_2 \\ -\tan x_1 \tan x_2 & 1 + \tan^2 x_2 \end{array} \right].$$

Rewrite div and vorticity

$$g\,\nabla \cdot \mathbf{v} = \frac{\partial}{\partial x^j}\,(\,g\,u^j\,), \quad g\,\zeta = \epsilon_{ij}\,\frac{\partial u_j}{\partial x^i}.$$

Cubed sphere



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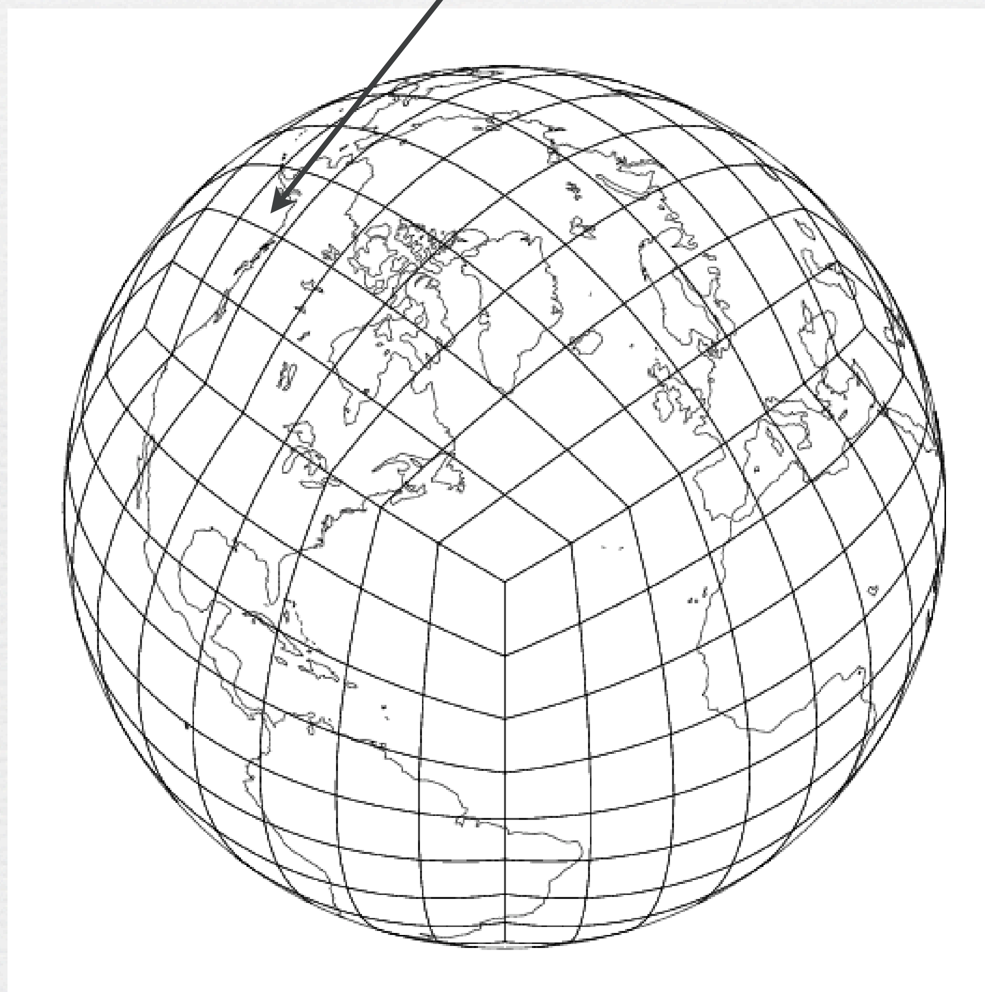

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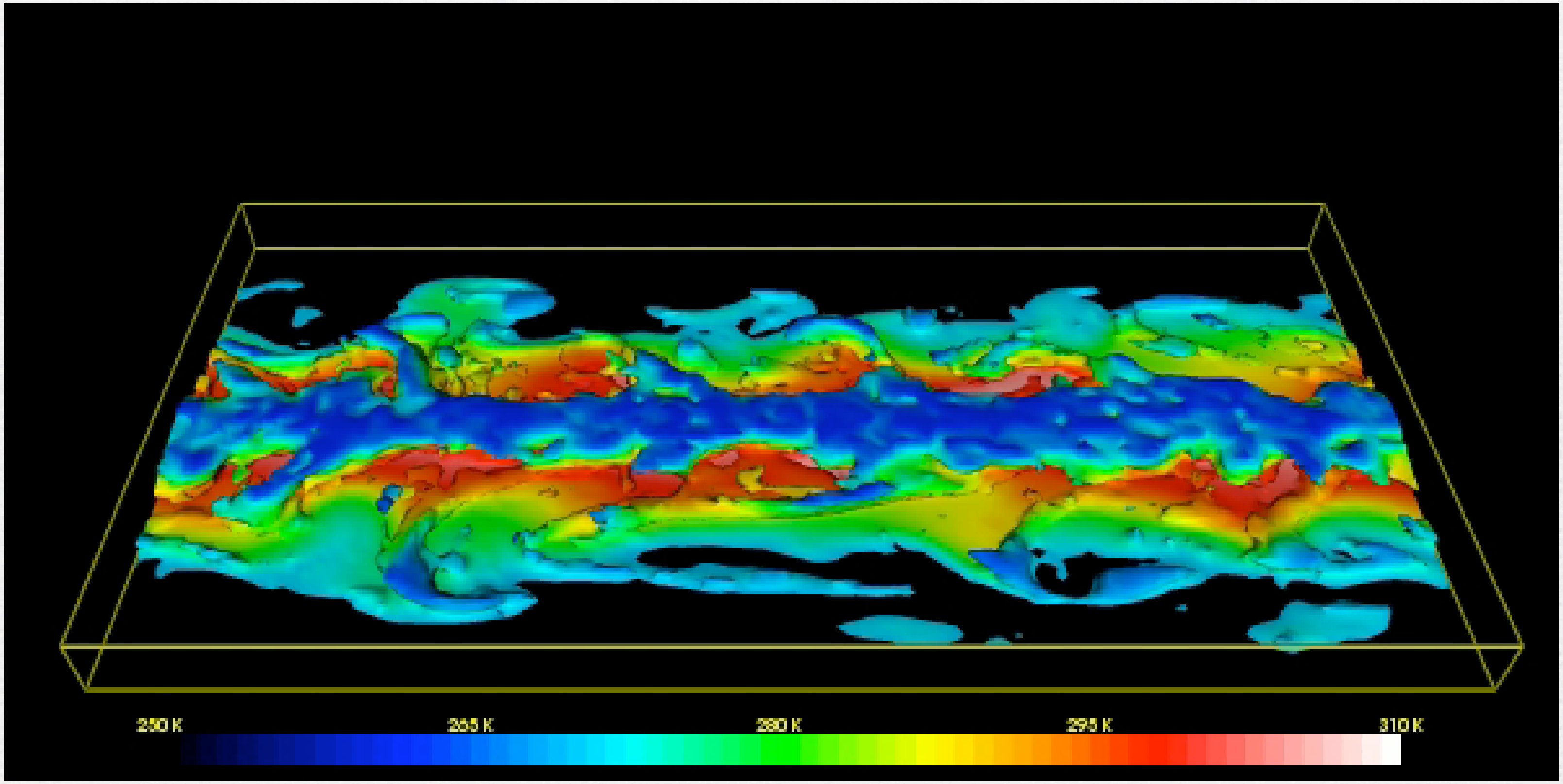
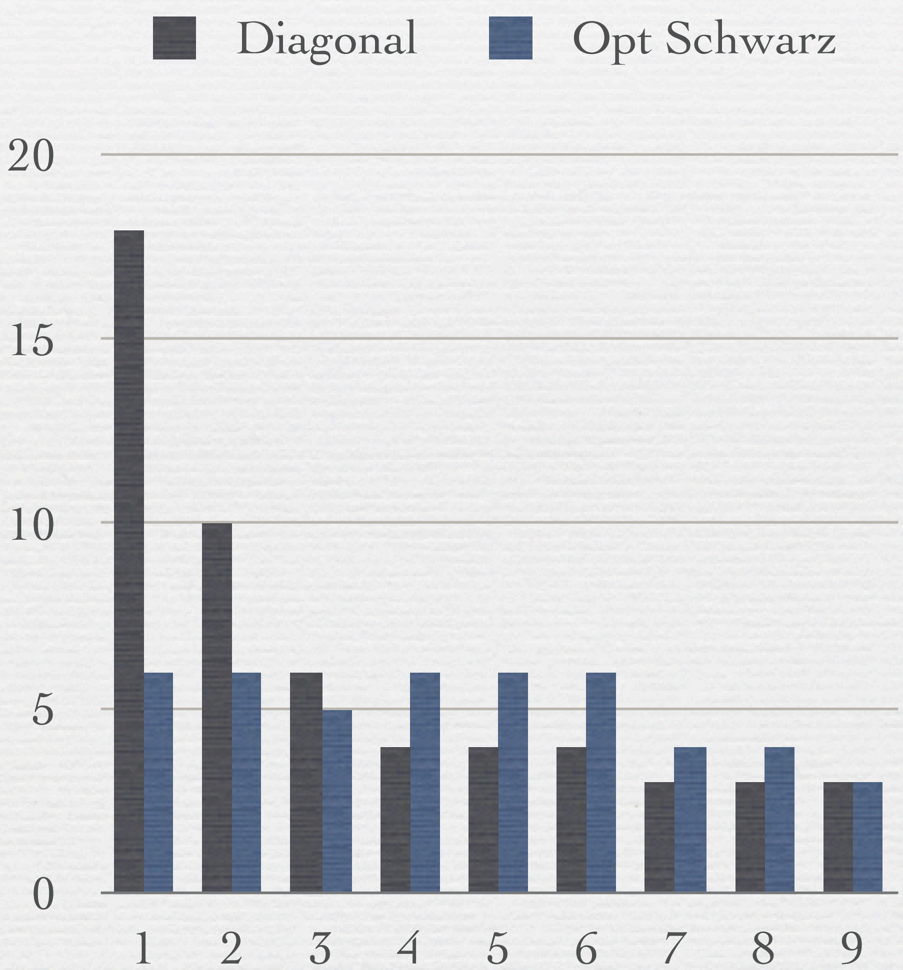
Held-Suarez numerical experiment: with moisture



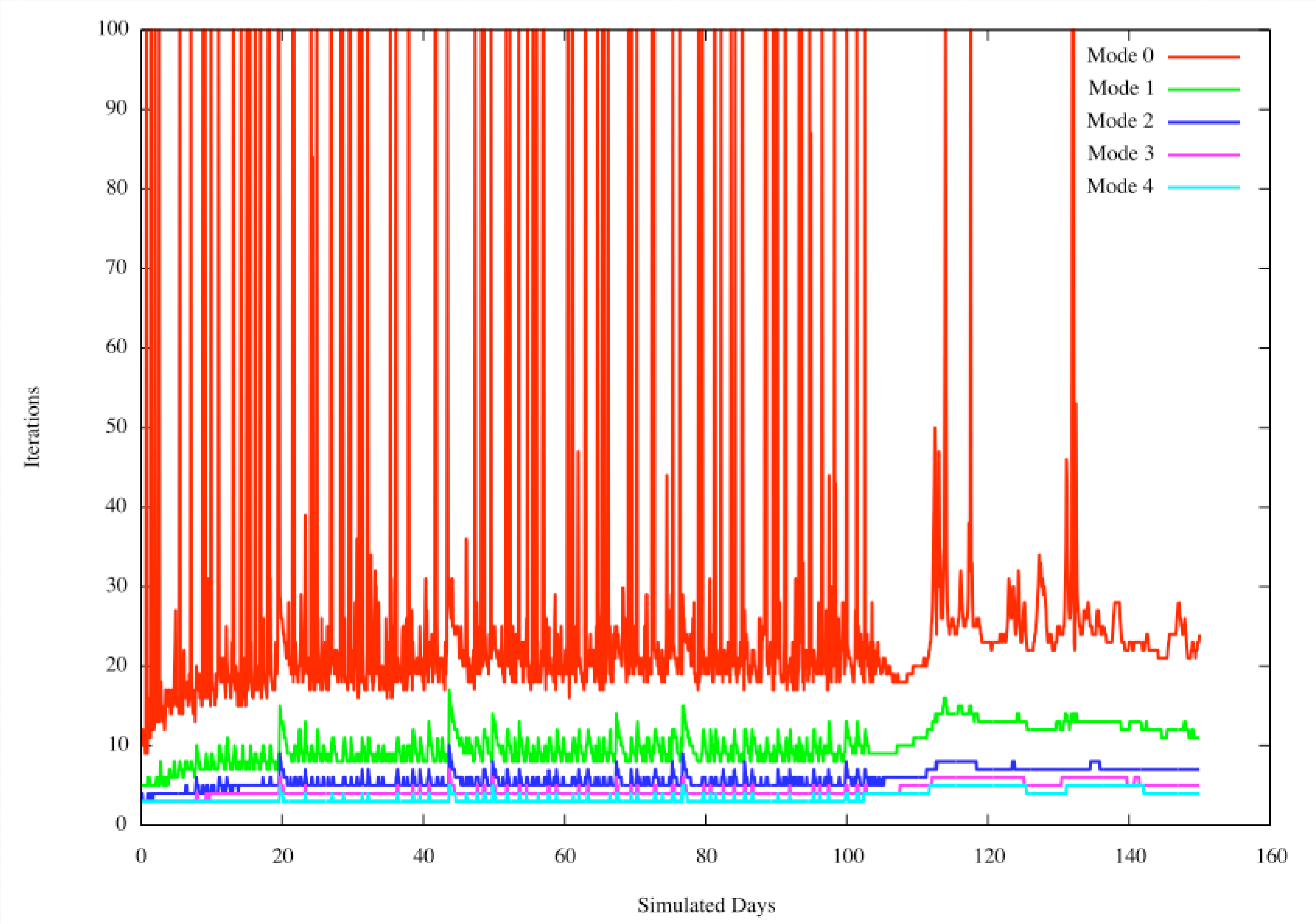
Galewski, Sobel and Held 2004

Convergence per mode

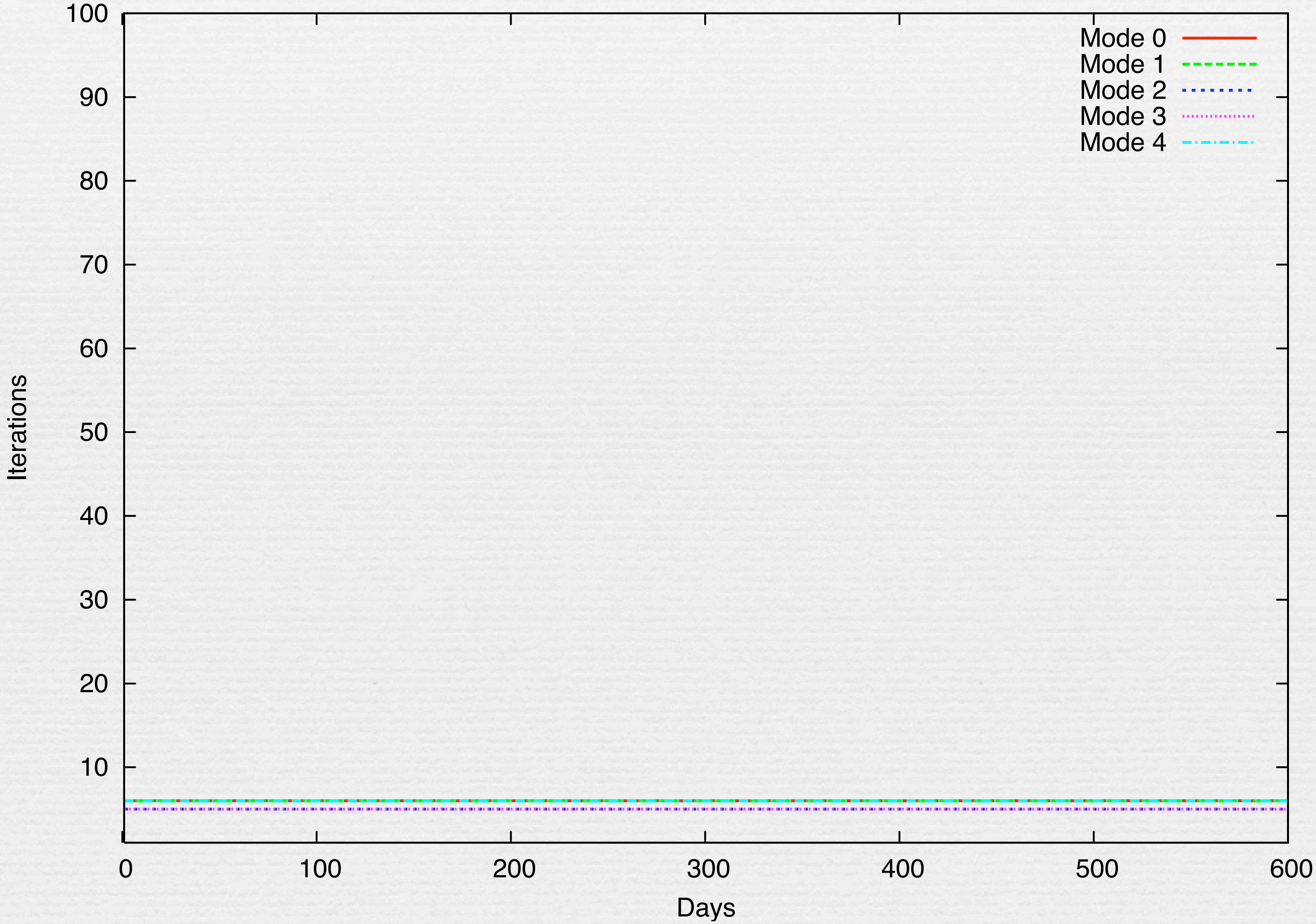


- Optimized algorithm: no maxing out
- Communication cost <u>identical</u>
- Twice the cost of CG per iteration
- Diagonal O(N) while OS is $O(N^3)$
- Best strategy: use OS on first few barotropic modes and diagonal elsewhere
- No coarse solver needed: because of time dependance

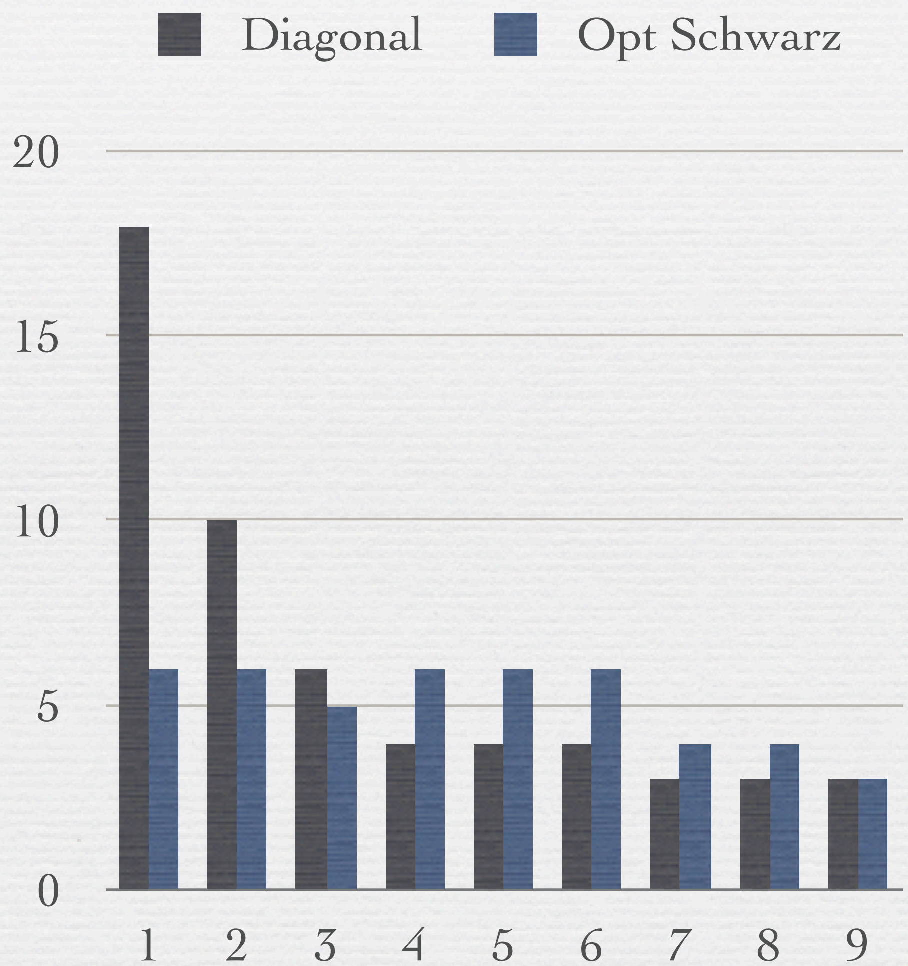
Diagonal preconditioning



Optimized Schwarz



New approach



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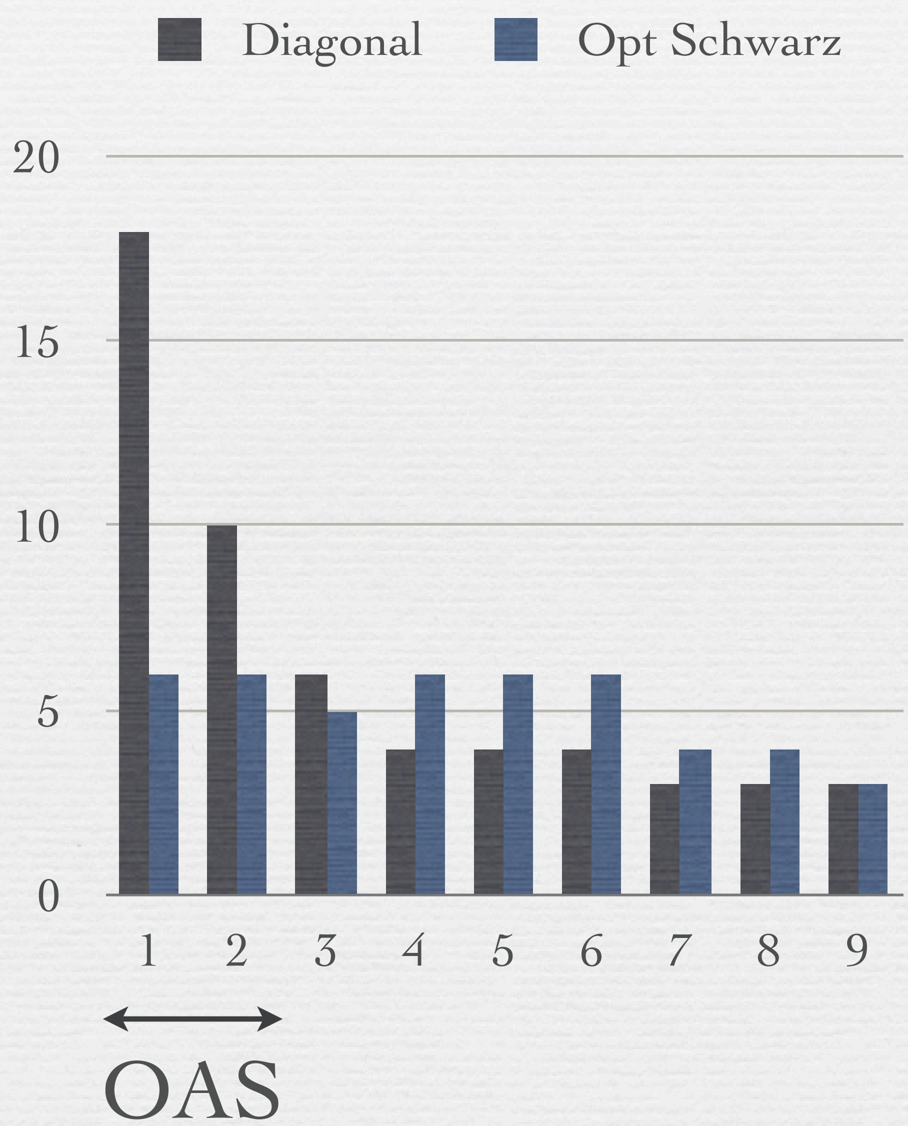

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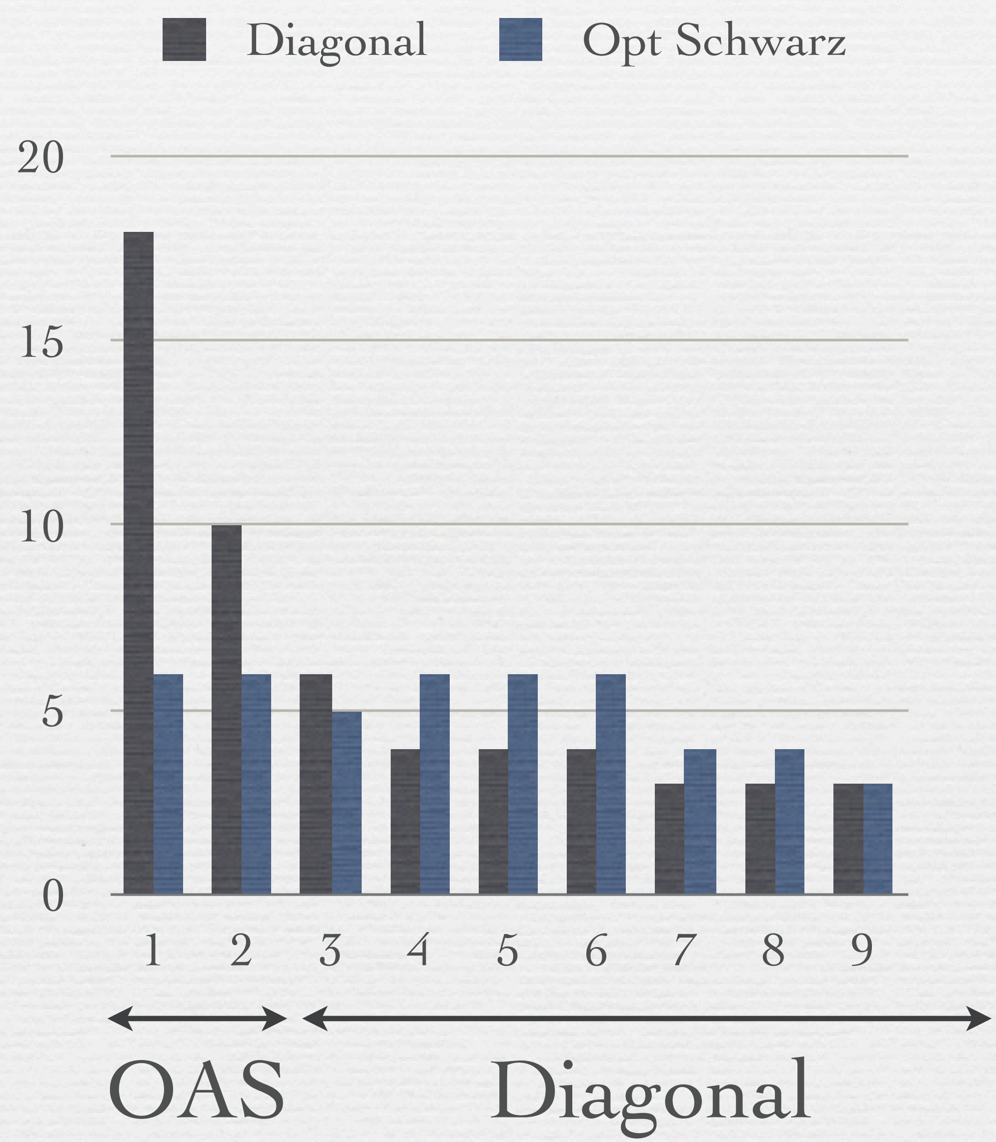
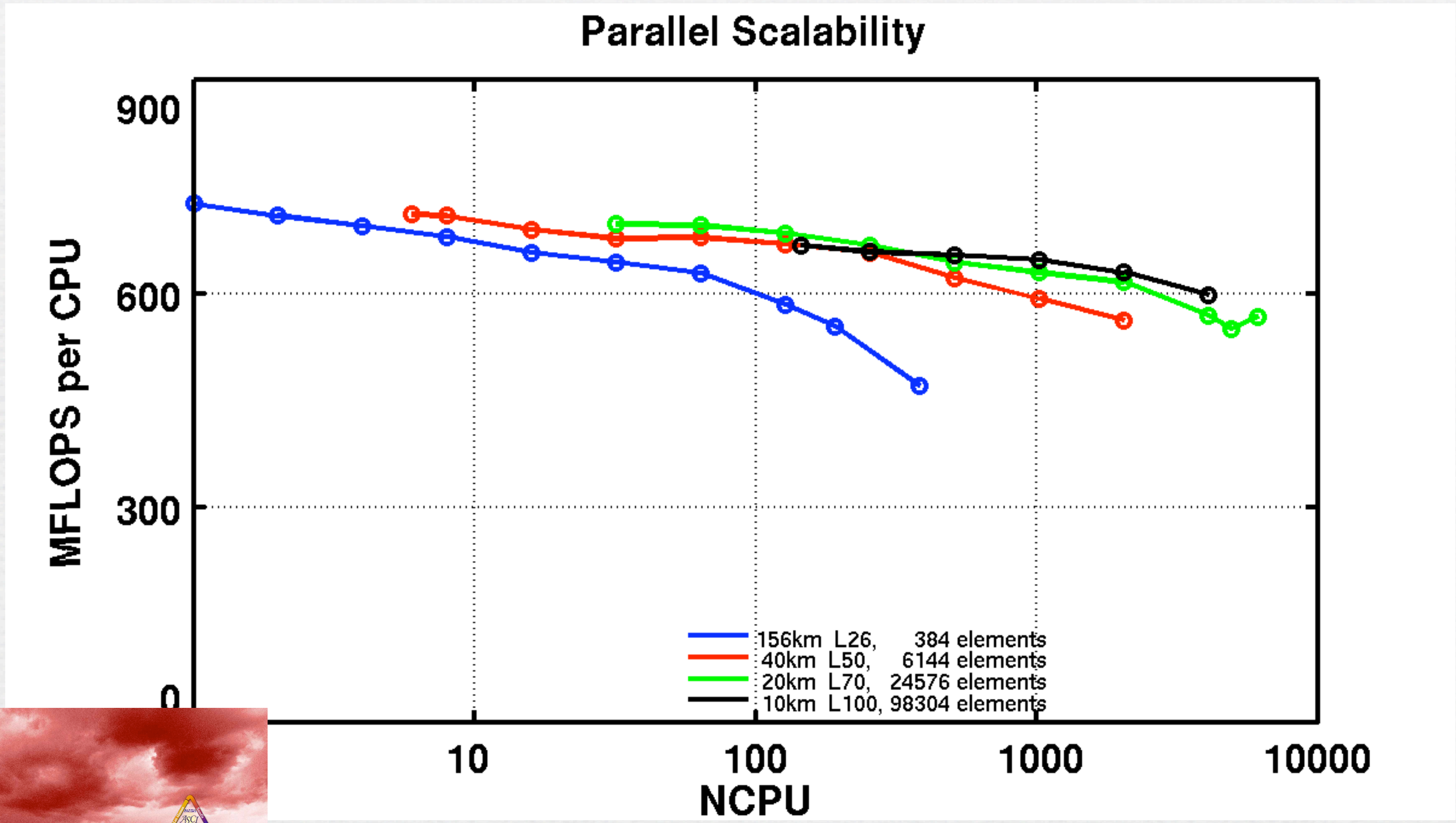
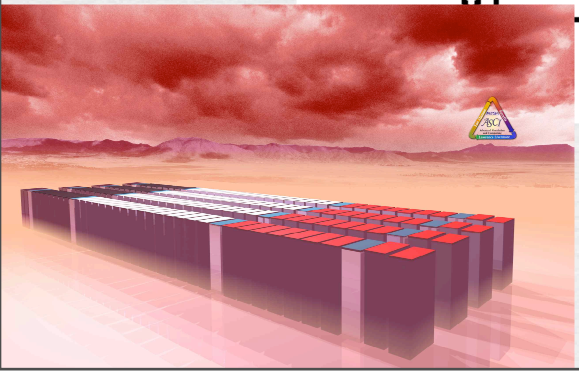


Large scale runs

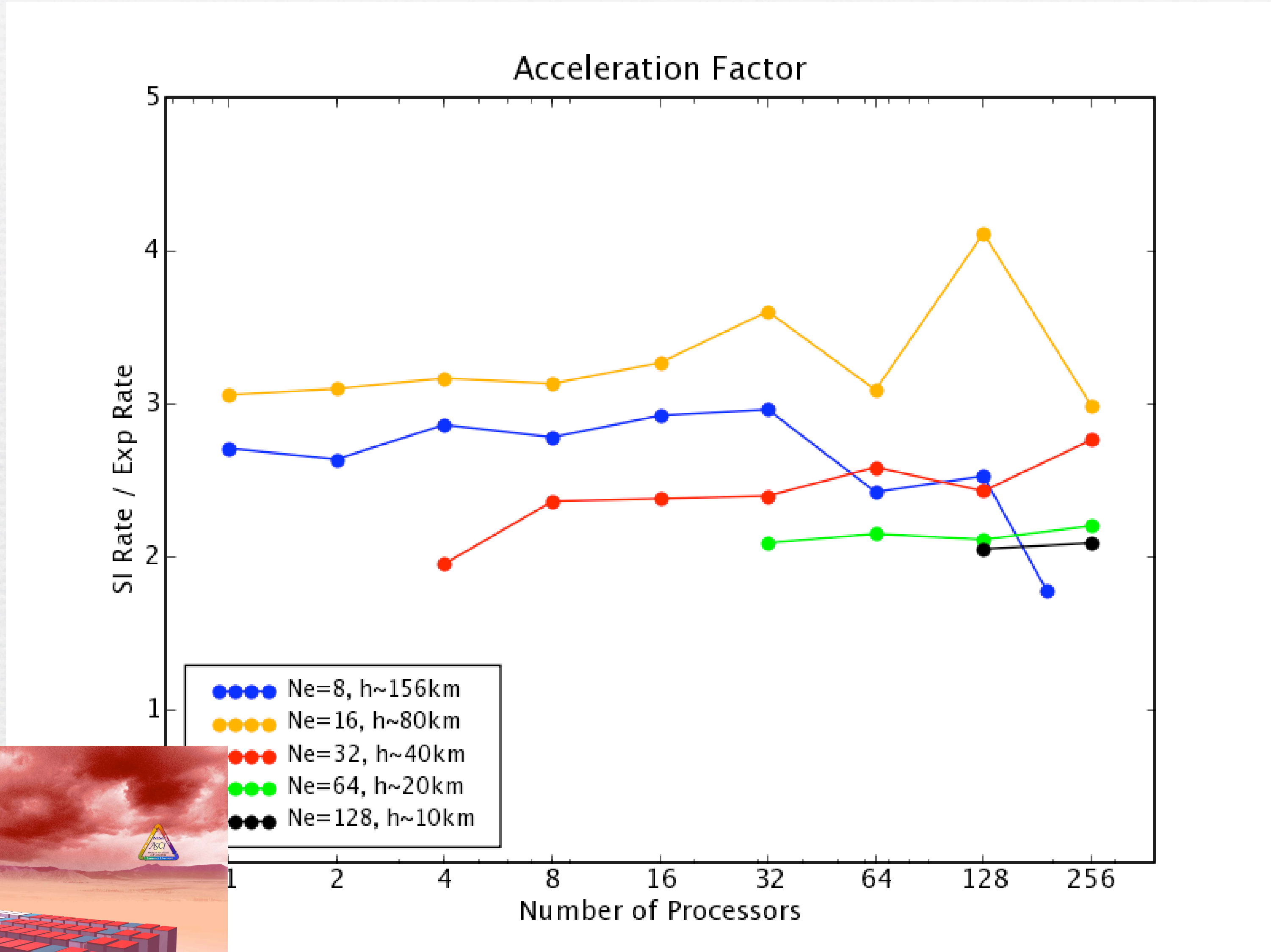


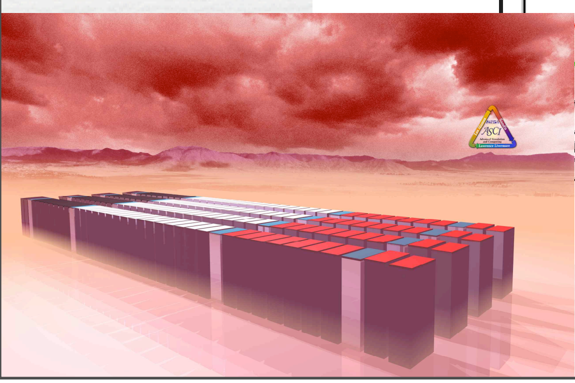


Max 5TF

W. Spotz and M. Taylor: Sandia National Labs

Si vs Exp: Red Storm



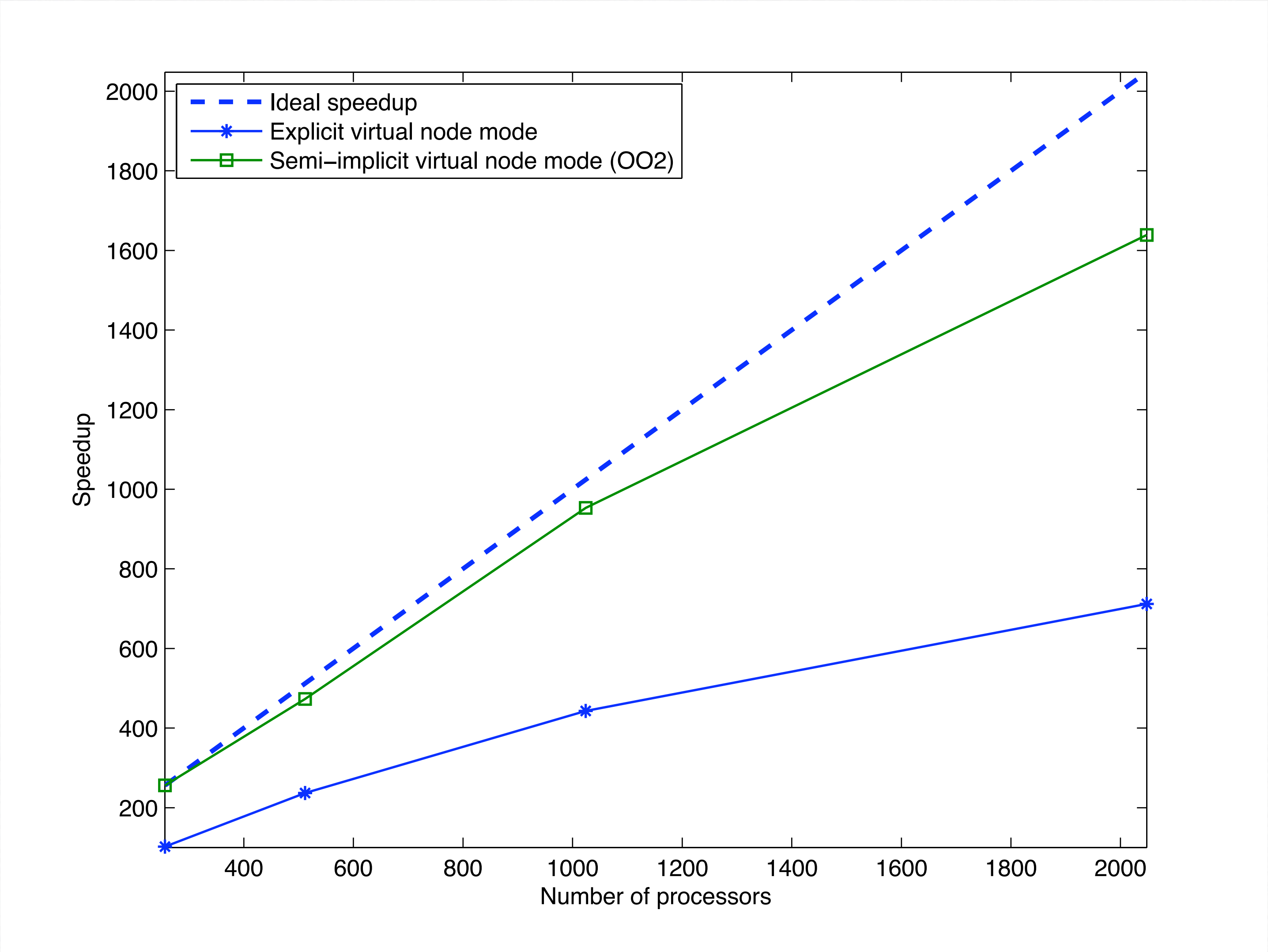


W. Spotz : Sandia National Labs

Si vs Exp: Blue gene

ne=32, 40km

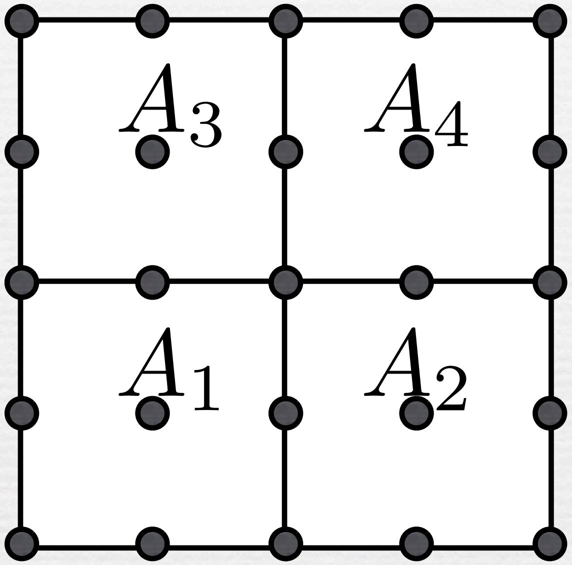
Si vs Exp: Blue gene



ne=32, 40km

Adaptive Mesh Refinements

Conforming SEM

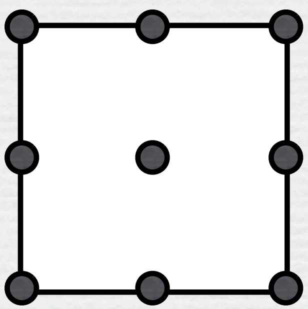


Linear problem associated with elliptic problem discretized with SEM

$$Au = f$$

$$A_L = block\{A_1, A_2, A_3, A_4\}$$

$$v^T Au = v^T Q^T A_L Q u = v^T Q^T M_L Q f = v^T f$$

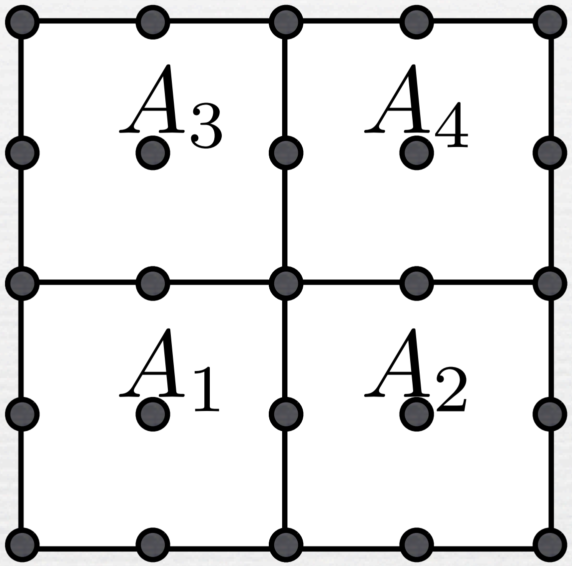


A_k
f_k
u_k

$$v^T Q^T A_L u_L = v^T Q^T M_L f_L$$

$$QQ^T A_L u_L = QQ^T M_L f_L$$

Conforming SEM



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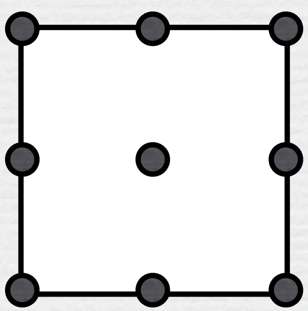


A_k
f_k
u_k

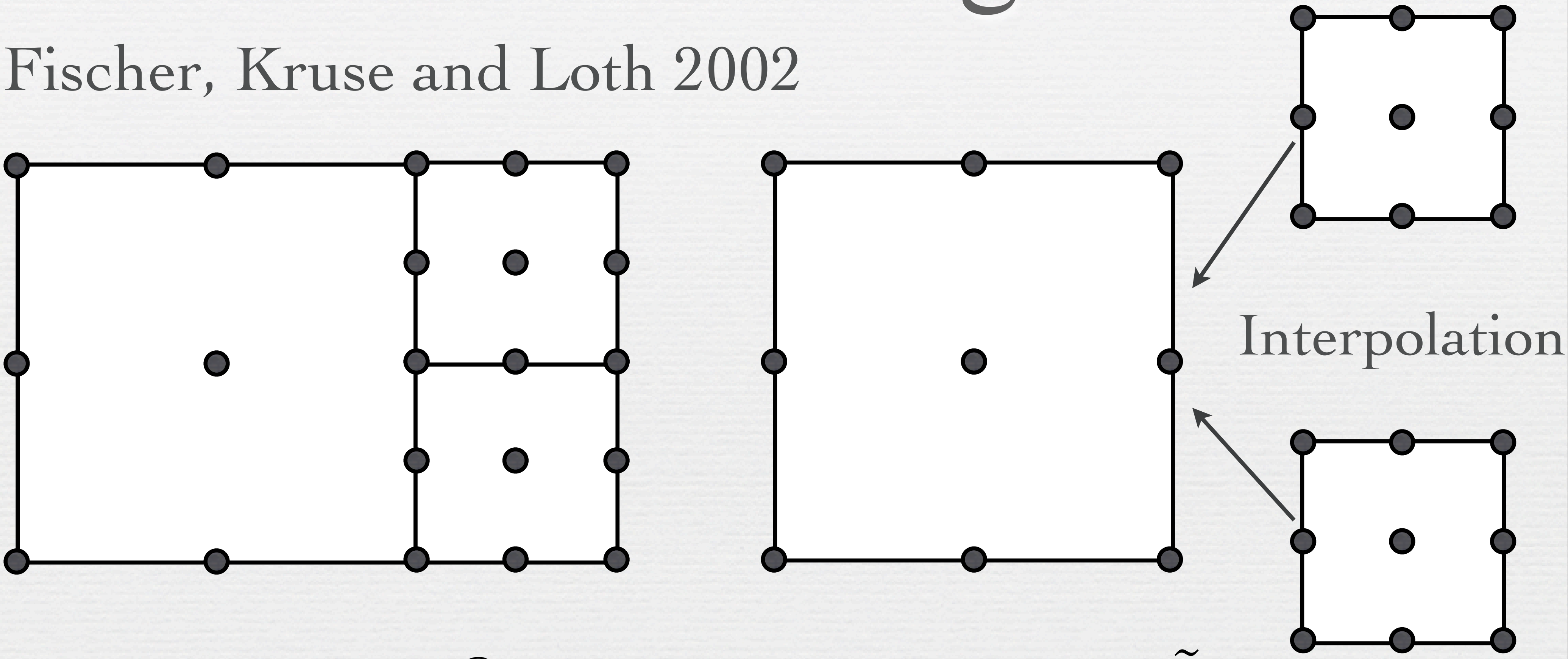
$$v^T Q^T A_L u_L = v^T Q^T M_L f_L$$

$$\boxed{QQ^T} A_L u_L = QQ^T M_L f_L$$

Direct stiffness summation: represents boolean operations

Nonconforming SEM

Fischer, Kruse and Loth 2002

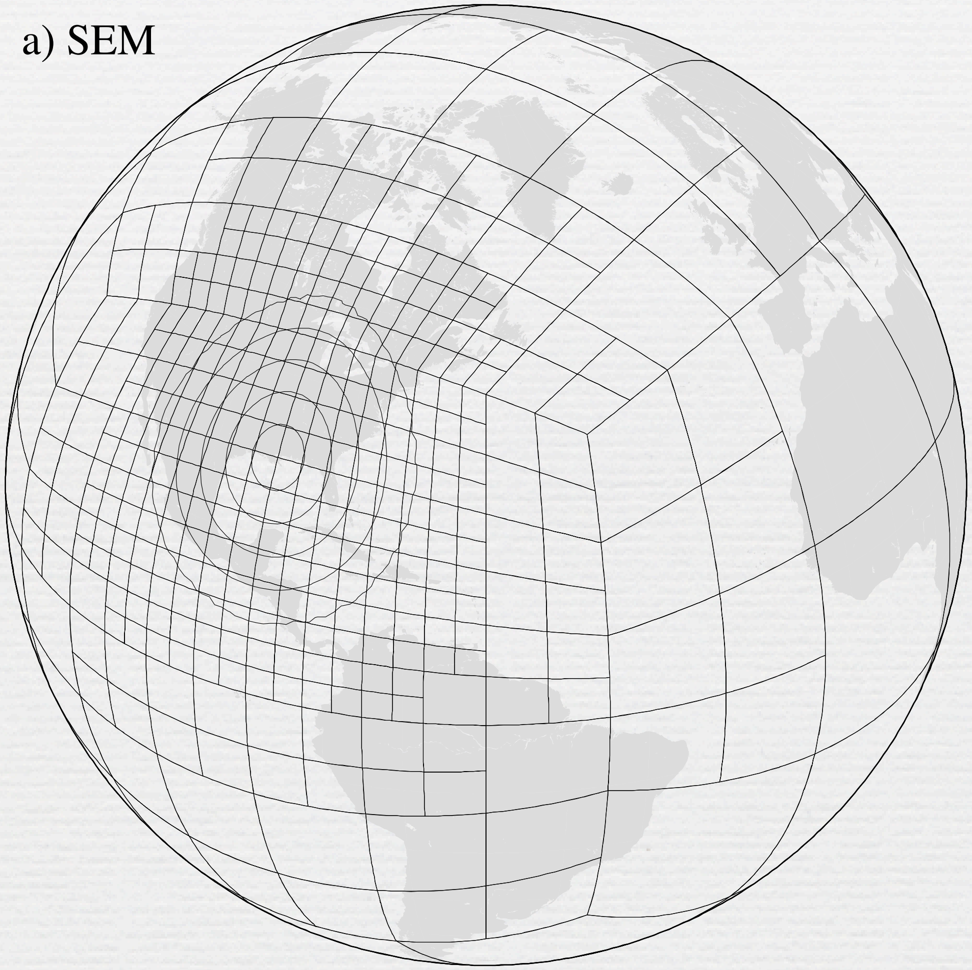


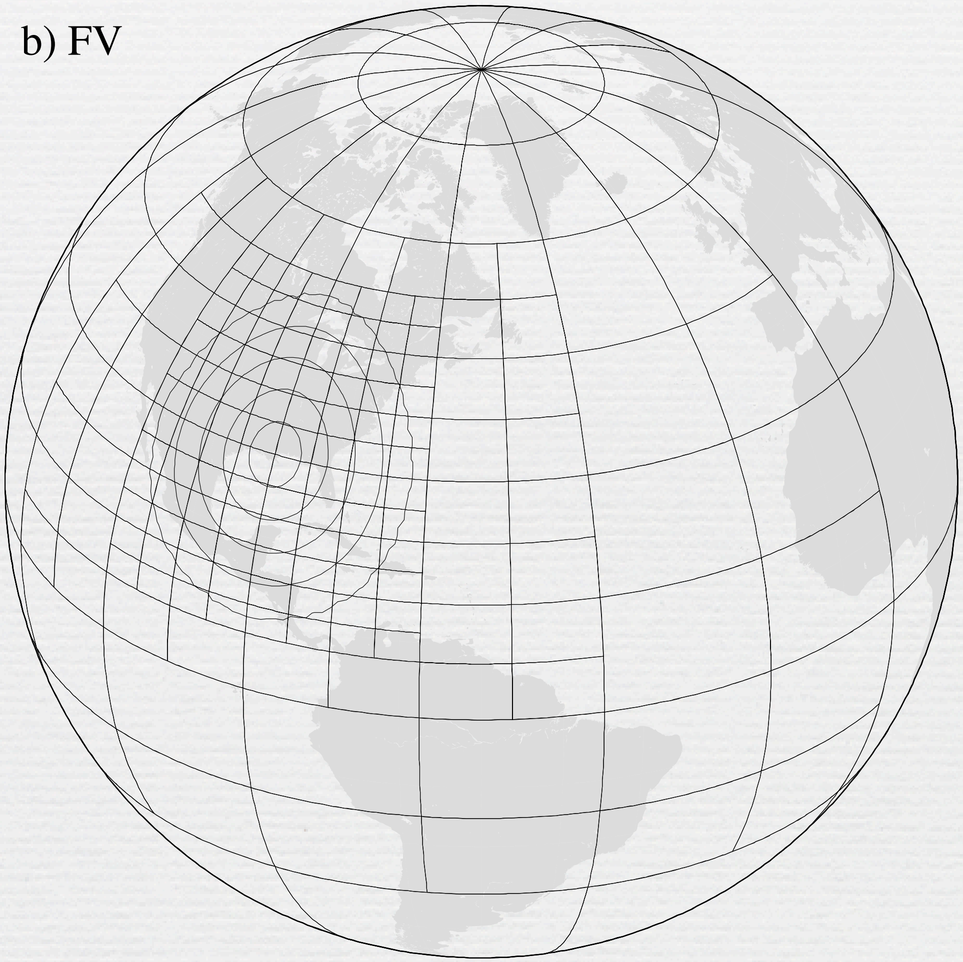
Boolean matrix Q is redefined as $Q = J_L \tilde{Q}$

DSS is conceptually the same:

$$v^T A u = v^T (\tilde{Q}^T J_L^T) A_L (J_L \tilde{Q}) u = v^T Q^T A_L Q u.$$

SEM vs FVM





Standard test suite is employed

Cosine bell advection

Alpha = 0

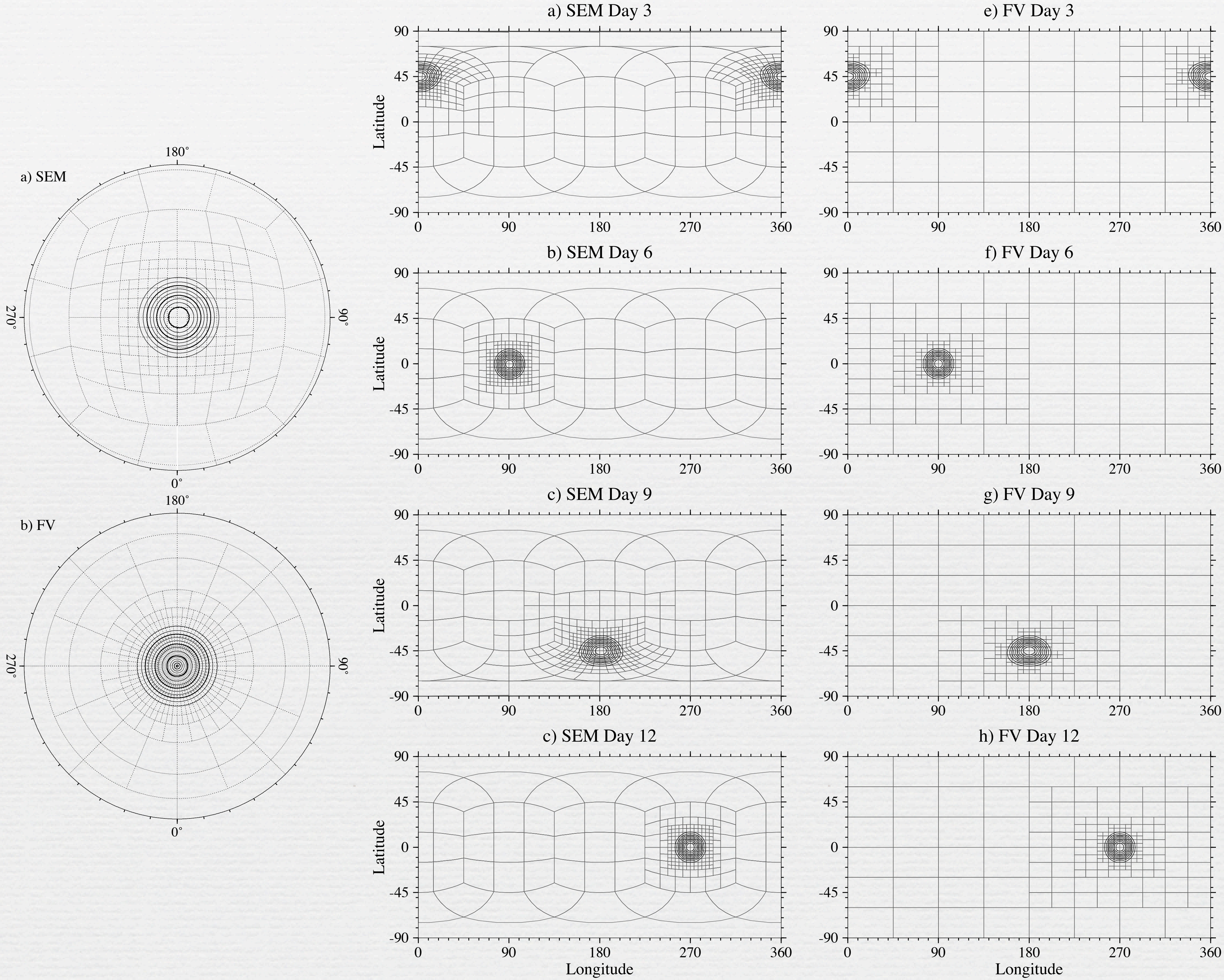
FV

Resolution	l_1	l_2	l_∞	$h(m)$ max/min
2.5	0.0341	0.0301	0.0317	949.1/0
1.25	0.0097	0.0103	0.0150	984.2/0
0.625	0.0016	0.0021	0.0044	995.0/0
0.3125	0.0003	0.0005	0.0014	998.4/0

SEM

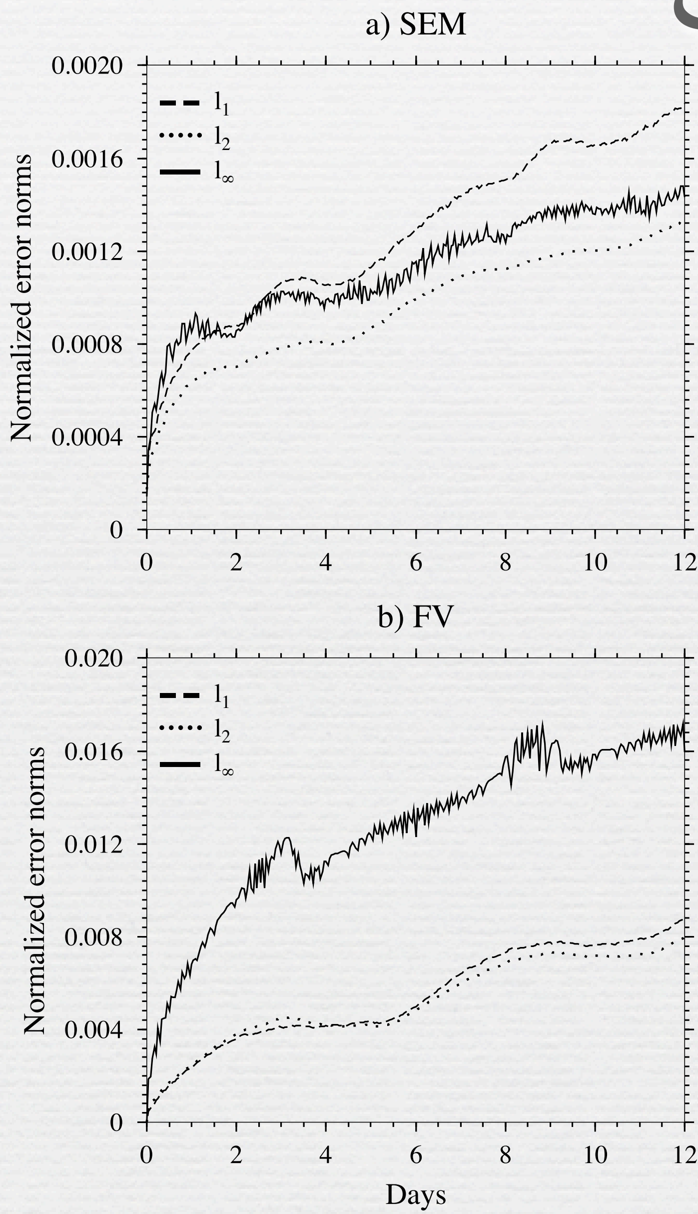
Resolution	l_1	l_2	l_∞	$h(m)$ max/min
2.5	0.0503	0.0269	0.0195	991.6/-15.1
1.25	0.0085	0.0056	0.0057	997.5/-4.2
0.625	0.0019	0.0014	0.0019	999.1/-1.1
0.3125	0.0008	0.0006	0.0015	999.7/-0.9

- For alpha different than 0 : FV has undershoots
- Conservation of mass breaks monotonicity

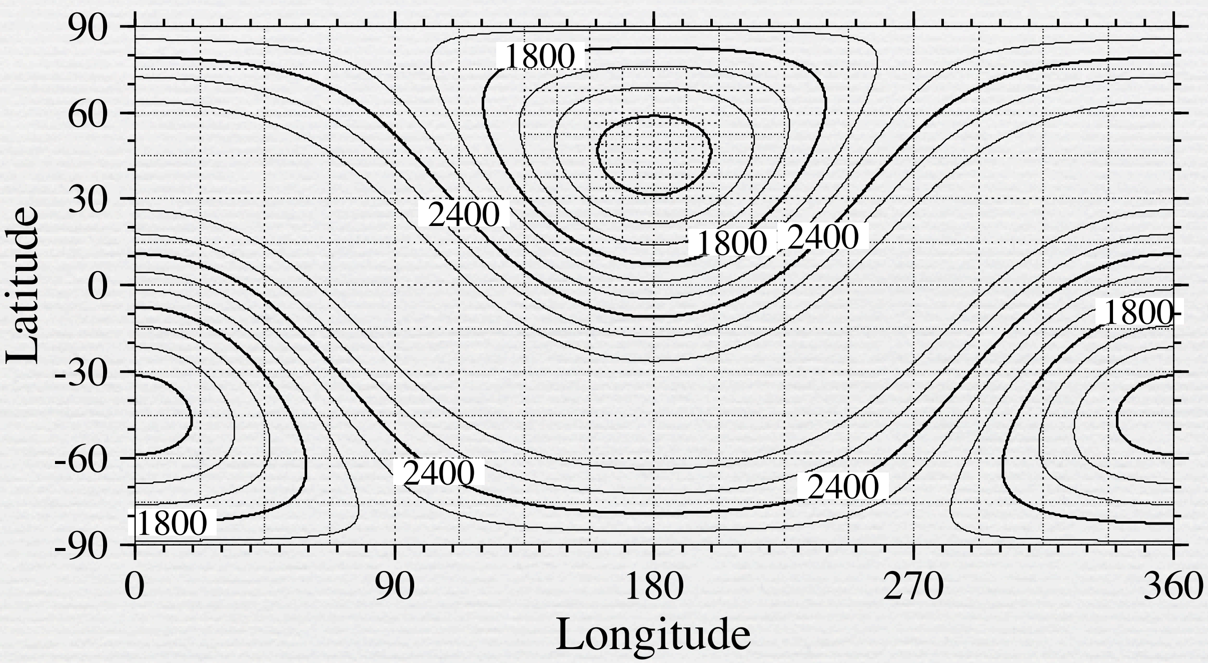


a) SEM
180˚
270˚
90˚
0˚
b) FV
180˚
270˚
90˚
0˚
a) SEM Day 3
e) FV Day 3
b) SEM Day 6
f) FV Day 6
c) SEM Day 9
g) FV Day 9
c) SEM Day 12
h) FV Day 12
Latitude
Longitude
90
45
0
-45
-90
0
90
180
270
360

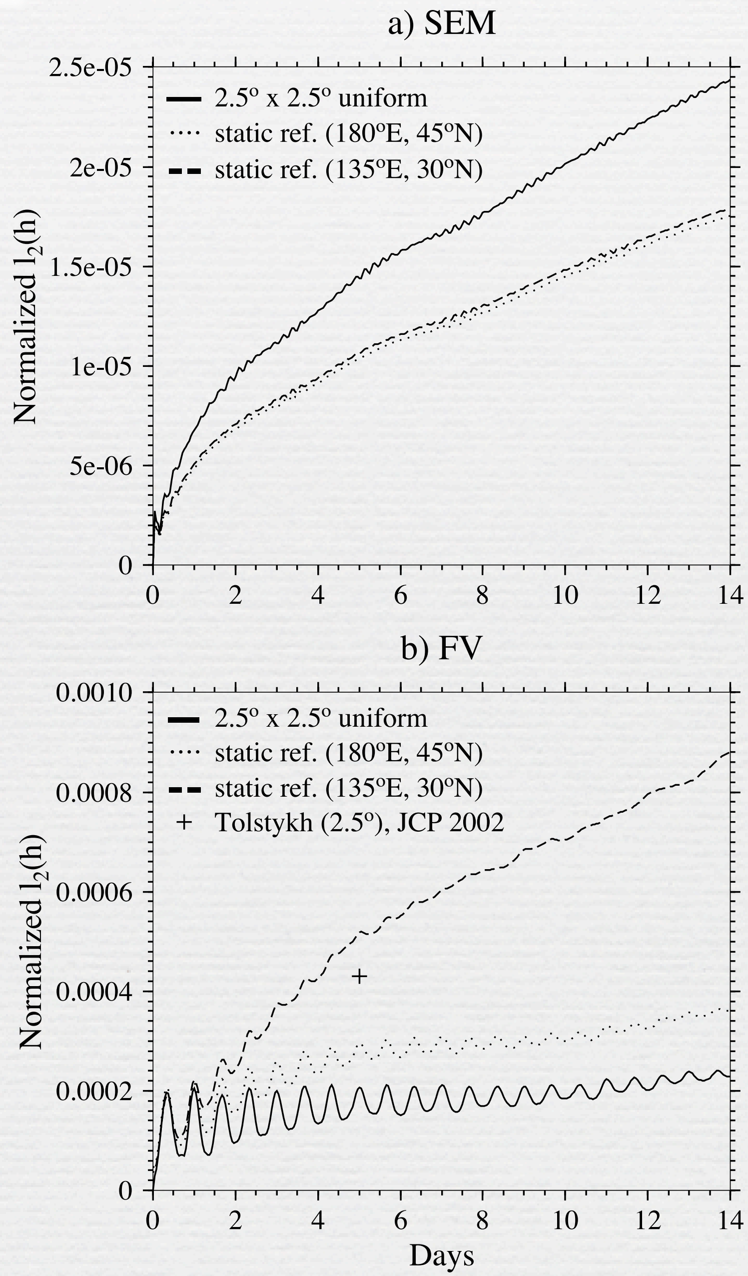
SWTC1 45 degrees



SWTC2

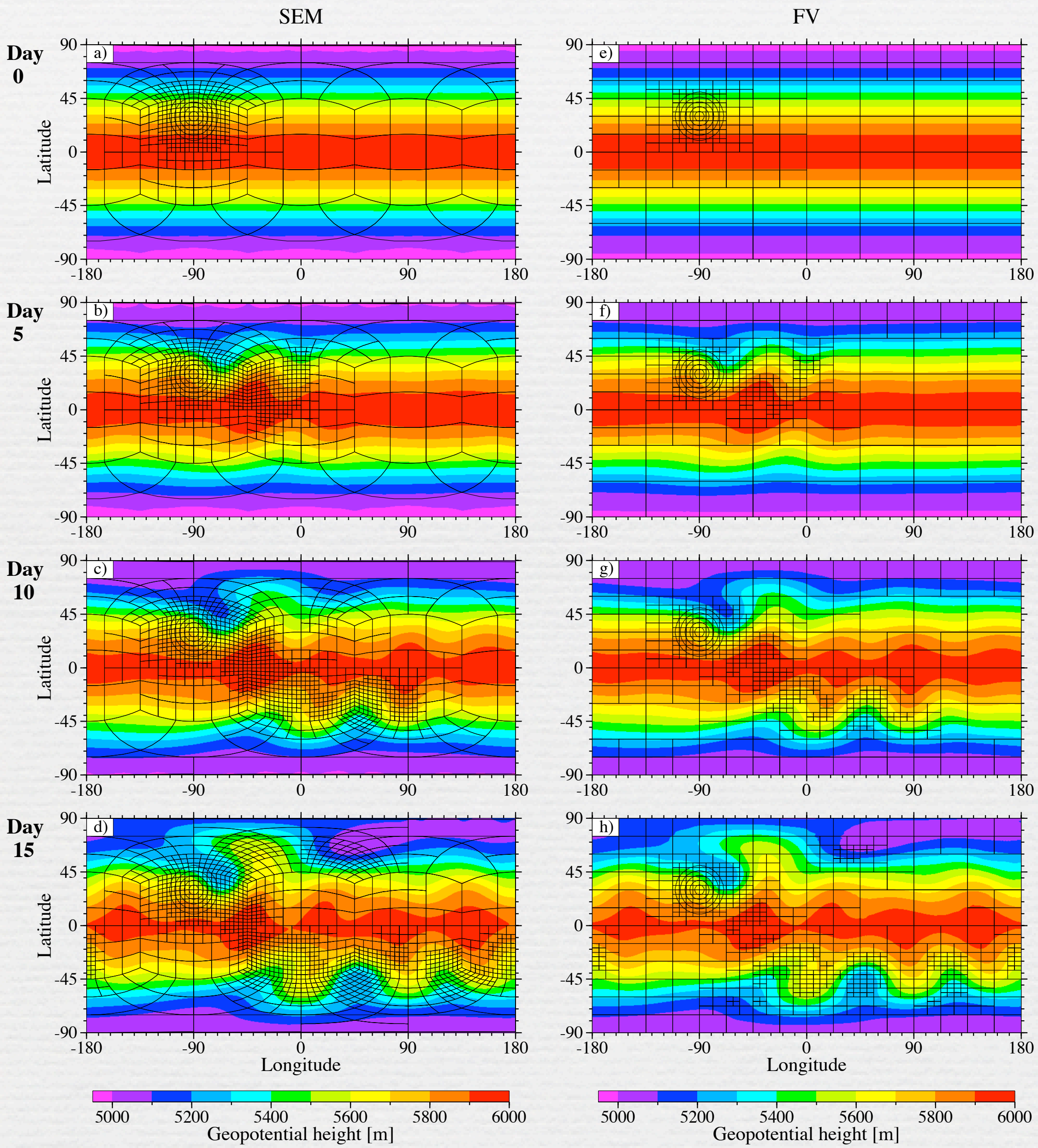


1800
2400
1800
2400
1800
2400
2400
1800
Latitude
Longitude
90
60
30
0
-30
-60
-90
0
90
180
270
360

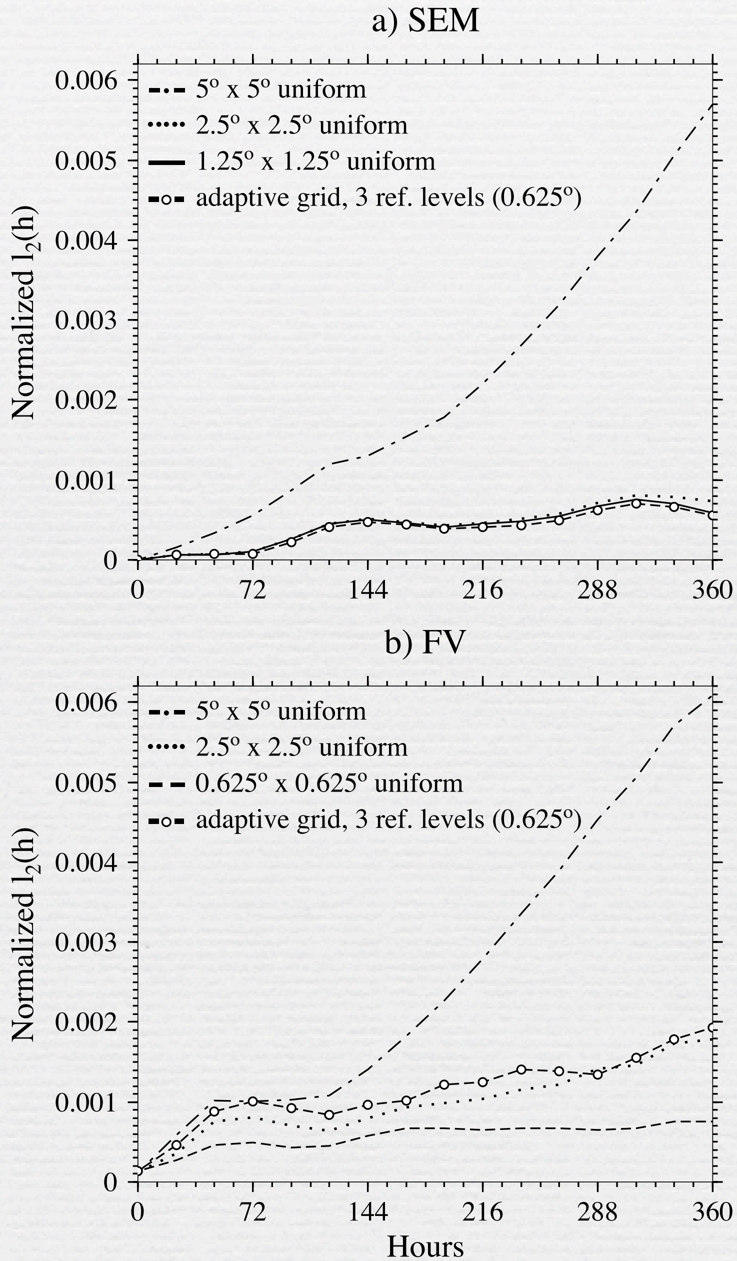


a) SEM
2.5° x 2.5° uniform
static ref. (180°E, 45°N)
static ref. (135°E, 30°N)
Normalized l2(h)
2.5e-05
2e-05
1.5e-05
1e-05
5e-06
0
b) FV
2.5° x 2.5° uniform
static ref. (180°E, 45°N)
static ref. (135°E, 30°N)
Tolstykh (2.5°), JCP 2002
Normalized l2(h)
0.0010
0.0008
0.0006
0.0004
0.0002
0
0
2
4
6
8
10
12
14
Days

SWTC5



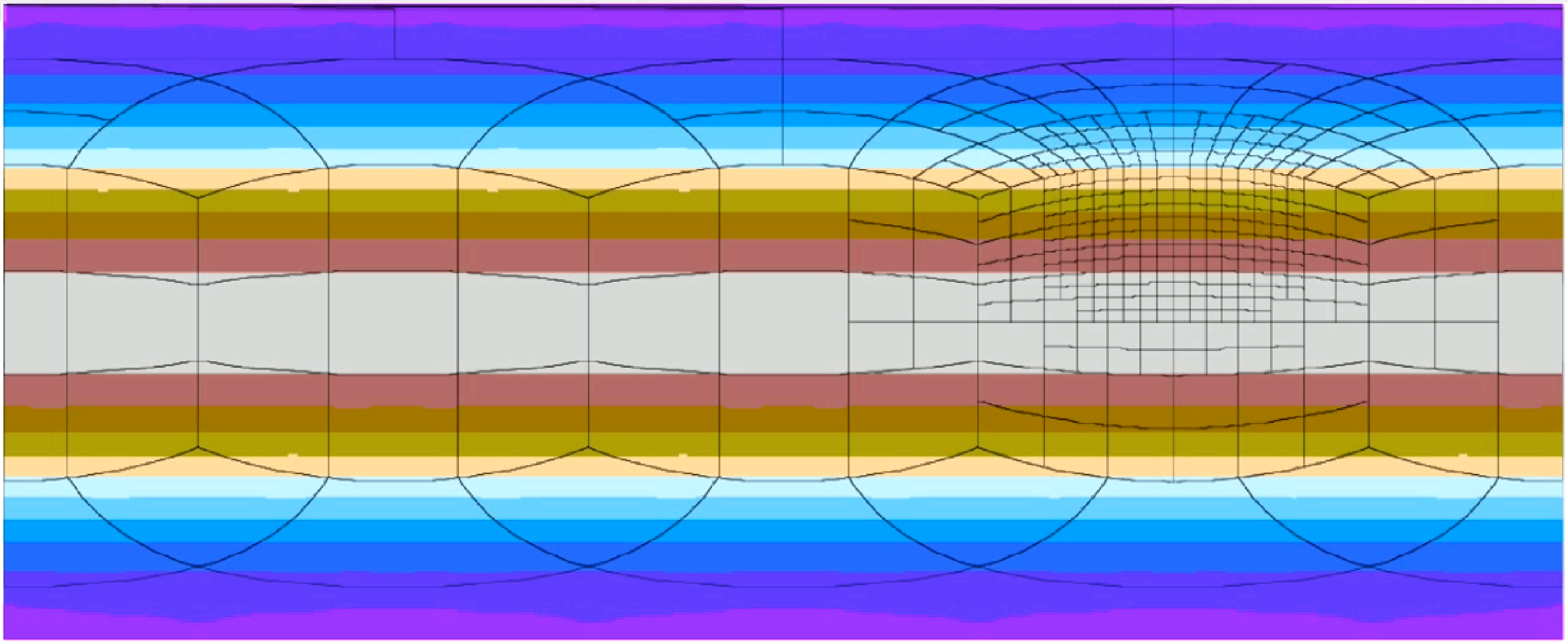
SEM
FV
Day 0
Day 5
Day 10
Day 15
a)
b)
c)
d)
e)
f)
g)
h)
Latitude
Longitude
Geopotential height [m]
5000 5200 5400 5600 5800 6000



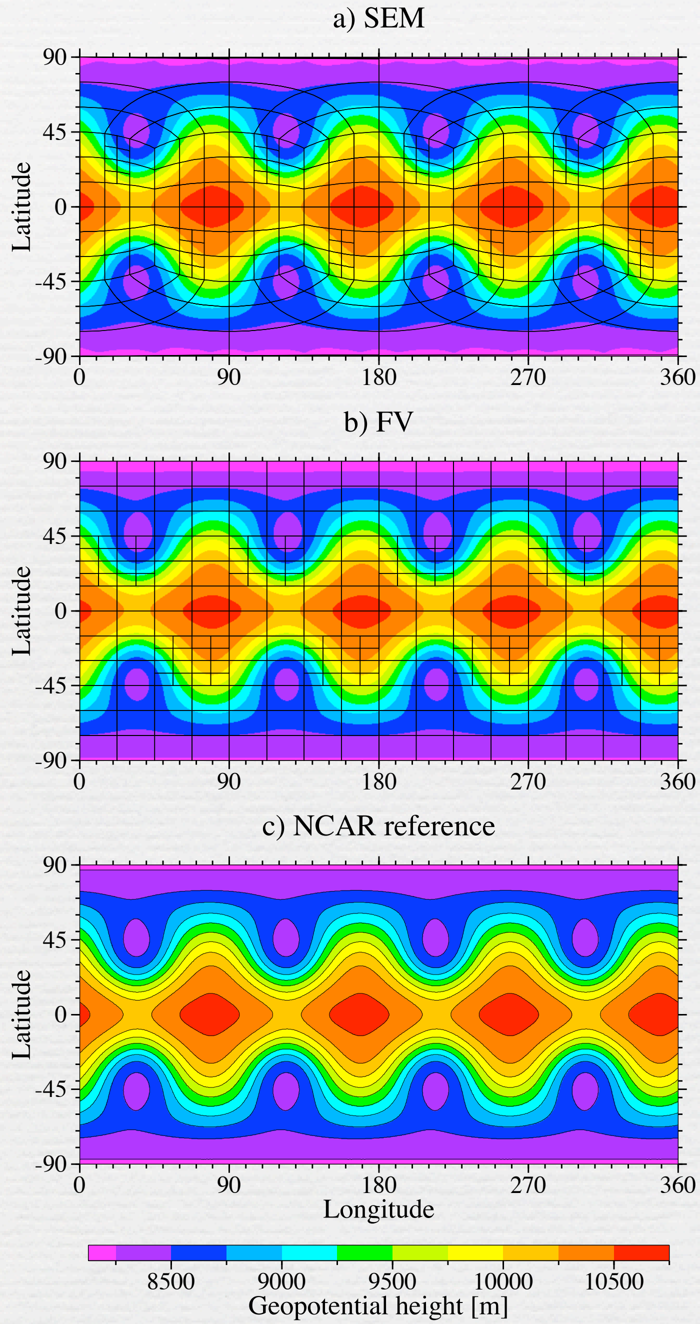
a) SEM
5° x 5° uniform
2.5° x 2.5° uniform
1.25° x 1.25° uniform
adaptive grid, 3 ref. levels (0.625°)
b) FV
5° x 5° uniform
2.5° x 2.5° uniform
0.625° x 0.625° uniform
adaptive grid, 3 ref. levels (0.625°)
Normalized l2(h)
Hours
0 72 144 216 288 360

SWTC5

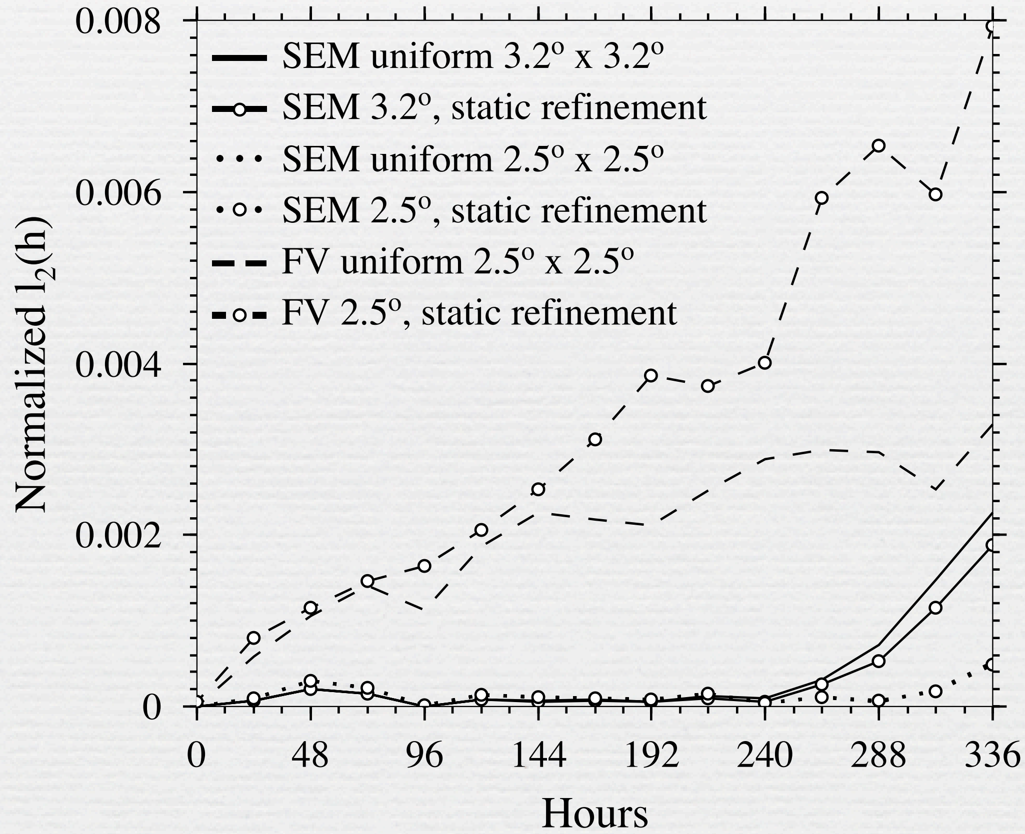
SWTC5



SWTC6



a) SEM
b) FV
c) NCAR reference
Latitude
Longitude
90
45
0
-45
-90
0
90
180
270
360
8500
9000
9500
10000
10500
Geopotential height [m]



SEM uniform 3.2° x 3.2°
SEM 3.2°, static refinement
SEM uniform 2.5° x 2.5°
SEM 2.5°, static refinement
FV uniform 2.5° x 2.5°
FV 2.5°, static refinement
Normalized $l_2(h)$
Hours
0.008
0.006
0.004
0.002
0
0
48
96
144
192
240
288
336

Time-stepping

OIFS+AMR

- Goal: SEM based AMR for primitive equations
- Oliger and Sundstrom show ill posedness for any kind of boundary conditions
- Cannot use local time stepping: Berger Oliger (84)
- Semi-implicit semi-Lagrangian approach?(Robert 81)

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- Cannot use local time stepping: Berger Oliger (84)
- Semi-implicit semi-Lagrangian approach? (Robert 81)
- **SISL is rather inefficient on modern computers**
- Attempts were made by Berhens: shmem only (96)

SISL

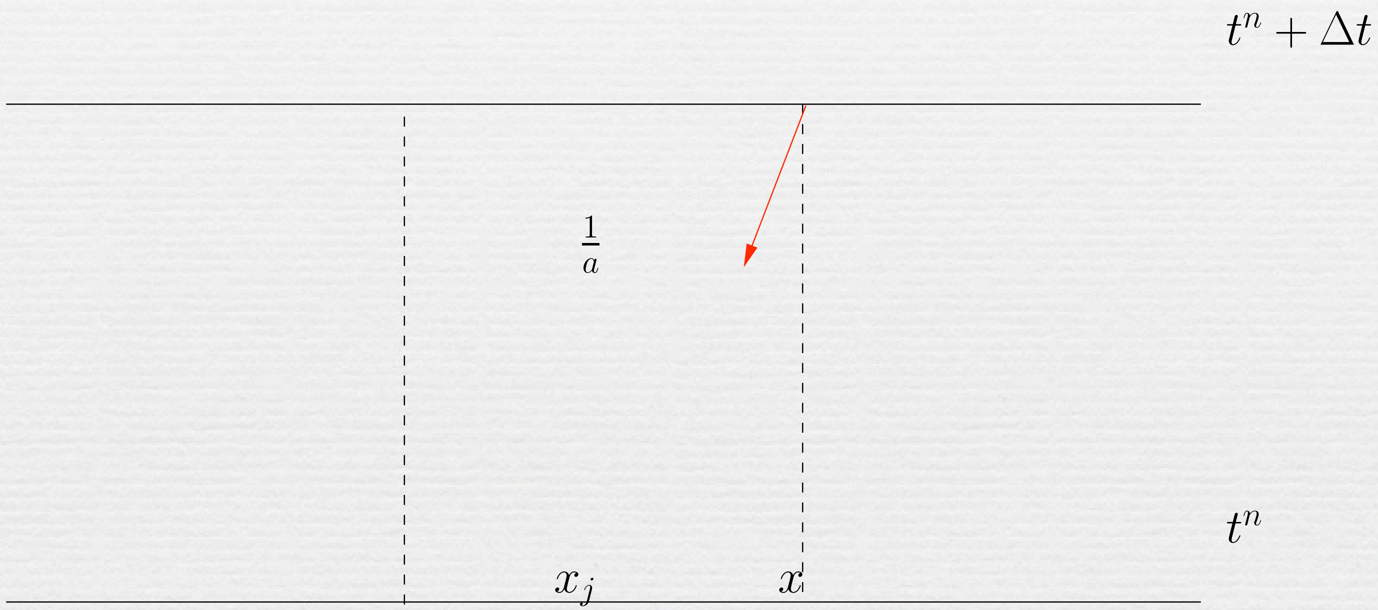
$$\frac{d\phi(X,t)}{dt} = \frac{\partial \phi}{\partial t} + \mathbf{u} \cdot \nabla \phi = f(\phi(X,t))$$

- Material derivative (hide advective term)
- Spatial position is now a function of time in the Lagrangian frame

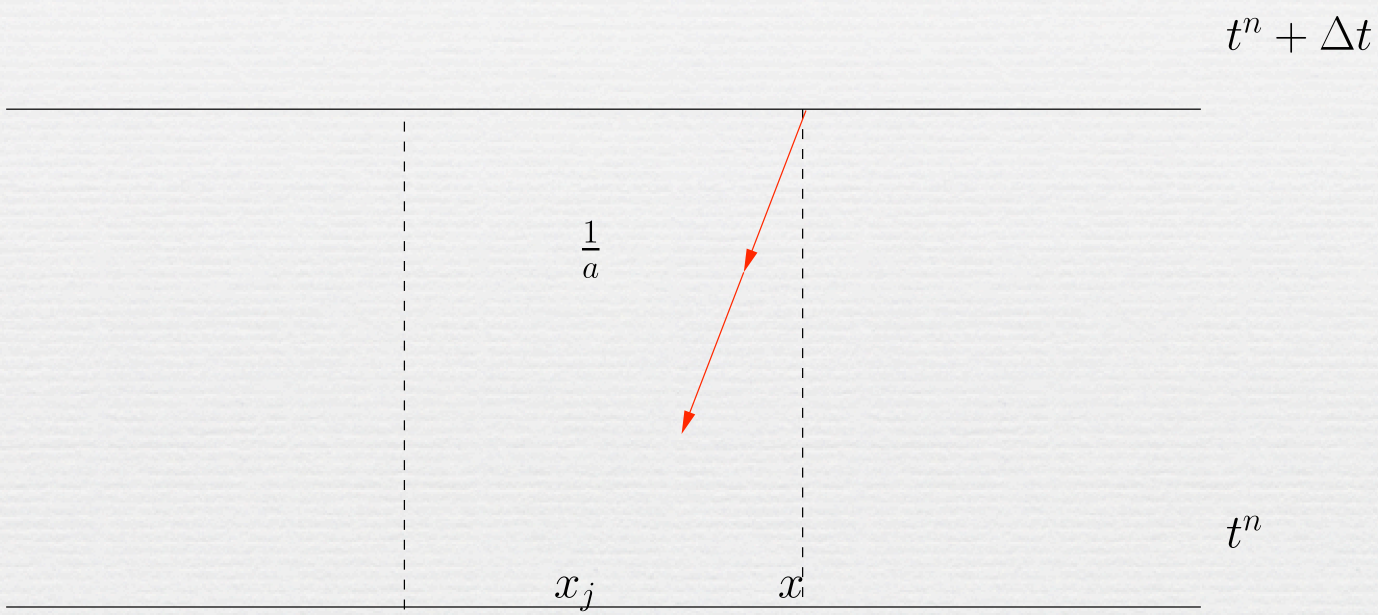
Characteristic equation: $$\frac{dX}{dt} = u(X,t)$$

$$X^n = X^{n+1} - \frac{\Delta t}{2}(\mathbf{u}(X^n, t^n) + \mathbf{u}(X^{n+1}, t^{n+1})), \text{ with } X(t^{n+1}) = x$$

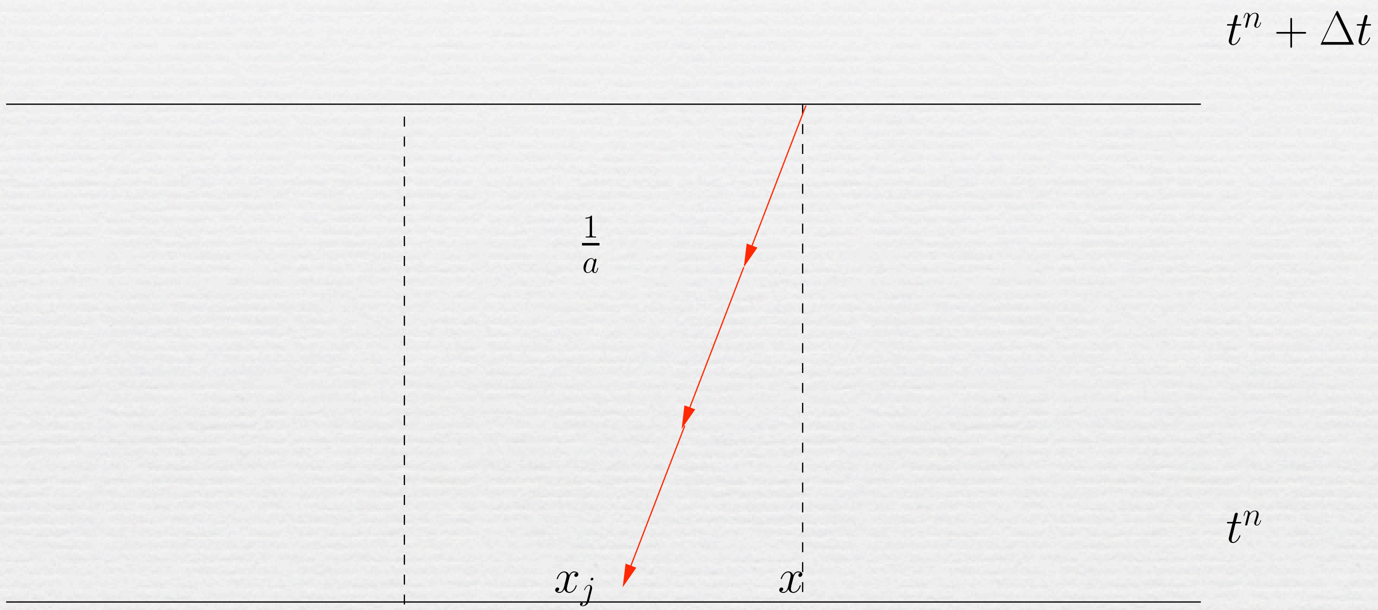
SISL



SISL



SISL



Operator integrating factor splitting

- Maday, Patera, Ronquist (90): OIFS.
- K elements of order N, KN^d grid points
- Interpolation KN^{2d}
- Scalar advection requires dKN^{d+1}
- OIFS more efficient if sub-step $< N^{d-1}$ "times"
- Purely Eulerian: regular communication patterns
- Nonlinear OIFS: St-Cyr and Thomas (05)
- Euler (MC2): Girard, Thomas and St-Cyr (07)

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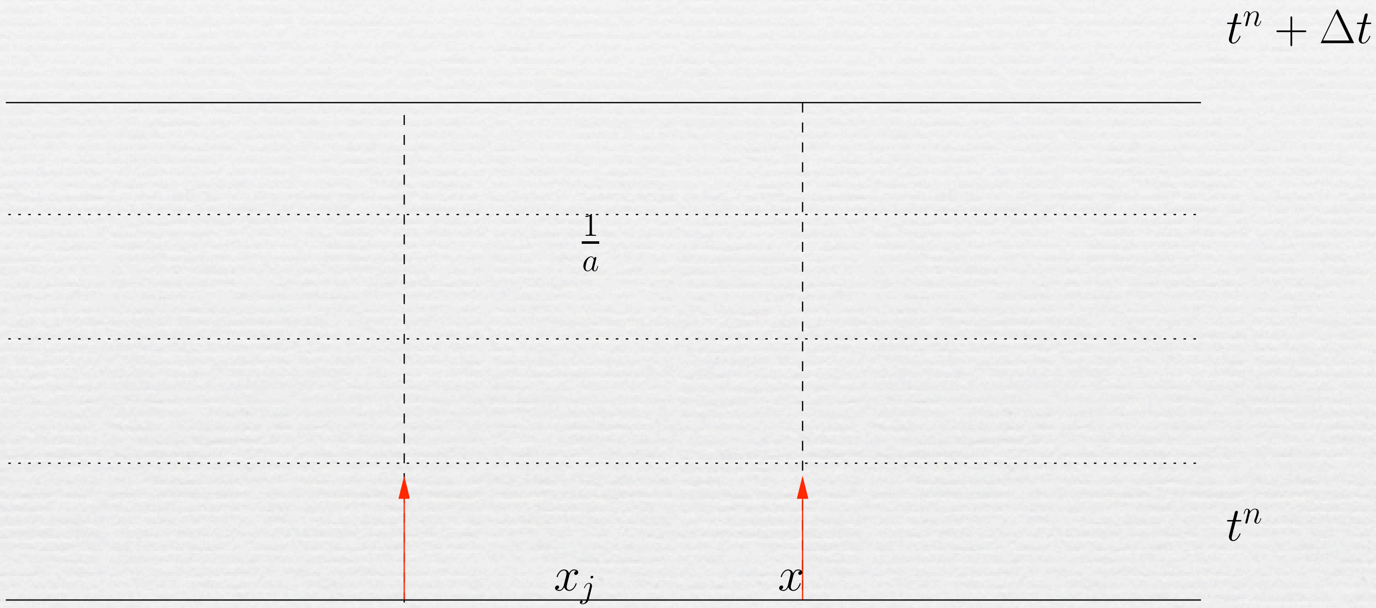
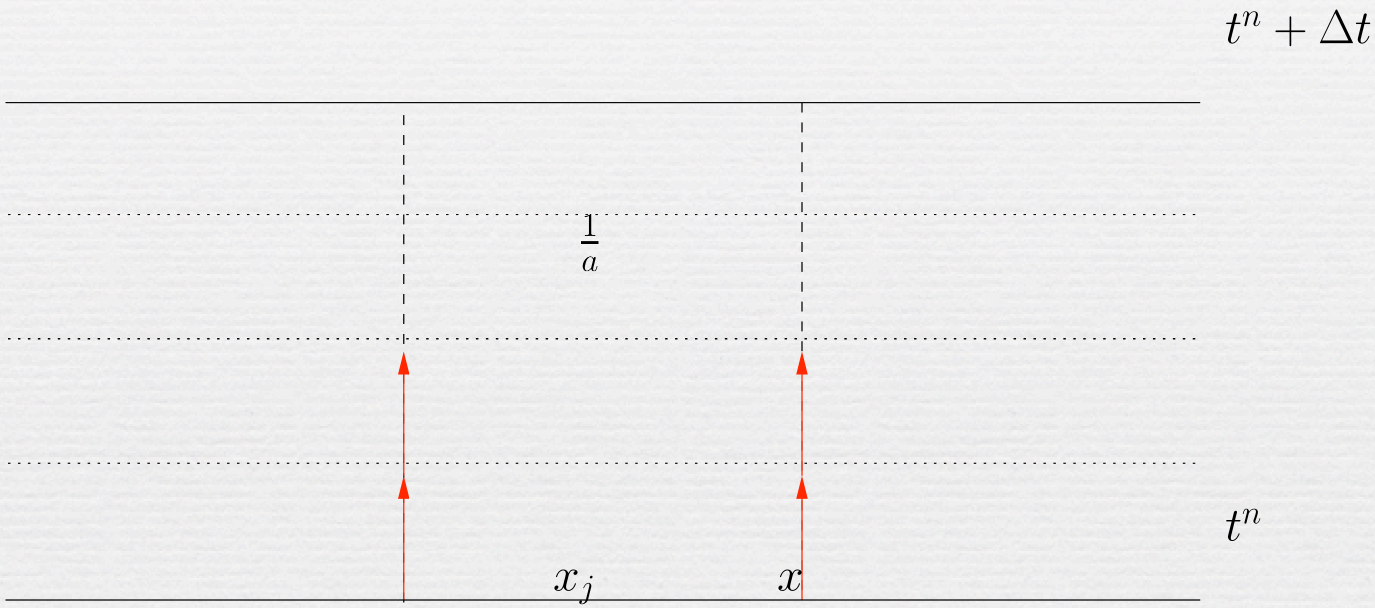
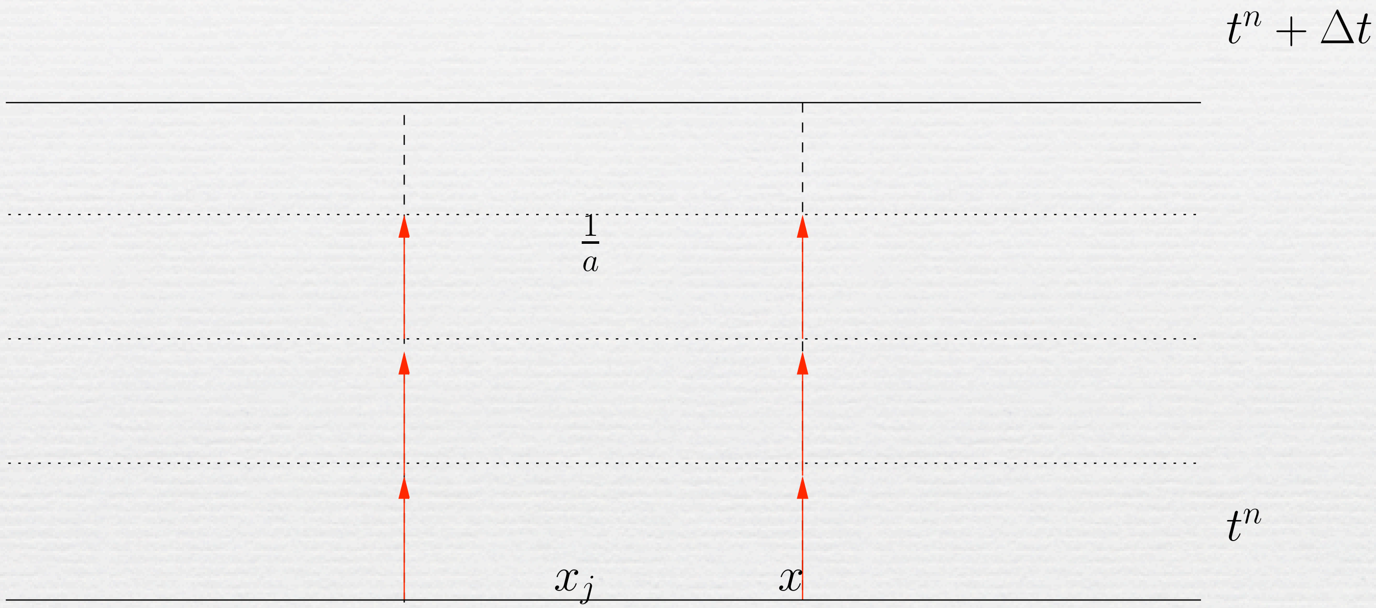


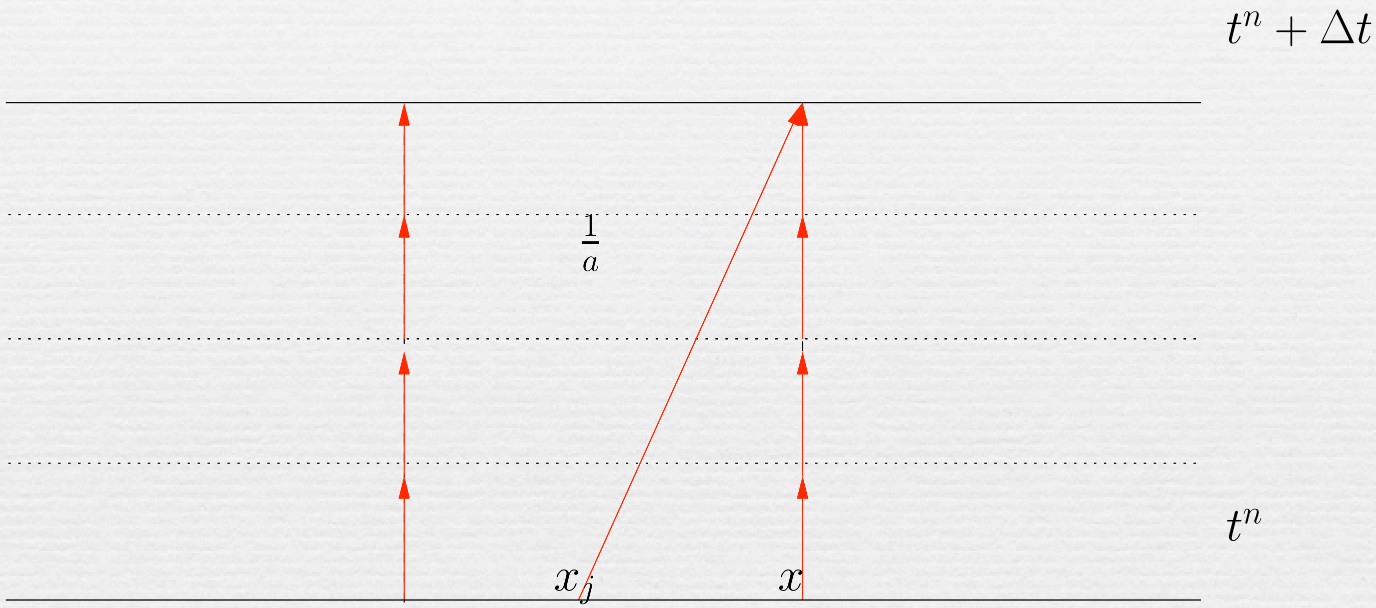
Operator integrating factor splitting



Operator integrating factor splitting



Operator integrating factor splitting



Operator integrating factor splitting

ODE resulting from SEM discretization (MOL)

$$\frac{du(t)}{dt} = S(u(t)) + F(u(t)), \quad t \in [0, T]$$

with initial condition $u(0) = u_0$

Problem: find integrating factor, $Q_S^{t^*}(t)$ such that $Q_S^{t^*}(t^*) = I$,

$$\frac{d}{dt} Q_S^{t^*}(t) \cdot u = Q_S^{t^*}(t) \cdot F(u).$$

To find the action of $Q_S^{t^*}(t)$ solve:

$$\frac{dv^{(t^*,t)}(s)}{ds} = S(v^{(t^*,t)}), \quad 0 \leq s \leq t - t^*$$

with initial condition $v^{(t^*,t)}(0) = u(t)$

Nonlinear OIFS

St-Cyr and Thomas (2005) sub-step

$$\frac{\partial \tilde{\mathbf{v}}}{\partial s} + \tilde{\zeta}\, \mathbf{k} \times \tilde{\mathbf{v}} + \frac{1}{2}\nabla\left(\tilde{\mathbf{v}} \cdot \tilde{\mathbf{v}}\right) = 0$$

$$\frac{\partial \tilde{\Phi}}{\partial s} + \nabla \cdot (\tilde{\Phi}\, \tilde{\mathbf{v}}) = 0$$

with initial conditions $\tilde{\mathbf{v}}(\mathbf{x}, t^{n-q}) = \mathbf{v}(\mathbf{x}, t^{n-q})$, $\tilde{\Phi}(\mathbf{x}, t^{n-q}) = \Phi(\mathbf{x}, t^{n-q})$.

Nonlinear OIFS

Integration factor applied to the SWE's

$$\frac{d}{dt} Q_S^{t^*}(t) \begin{bmatrix} \mathbf{v} \\ \Phi \end{bmatrix} = -Q_S^{t^*}(t) \begin{bmatrix} f\,\mathbf{k} \times \mathbf{v} + \nabla\Phi \\ \Phi_0\, \nabla \cdot \mathbf{v} \end{bmatrix}$$

Terms responsible for

Backward Differentiation Formula (BDF-2):

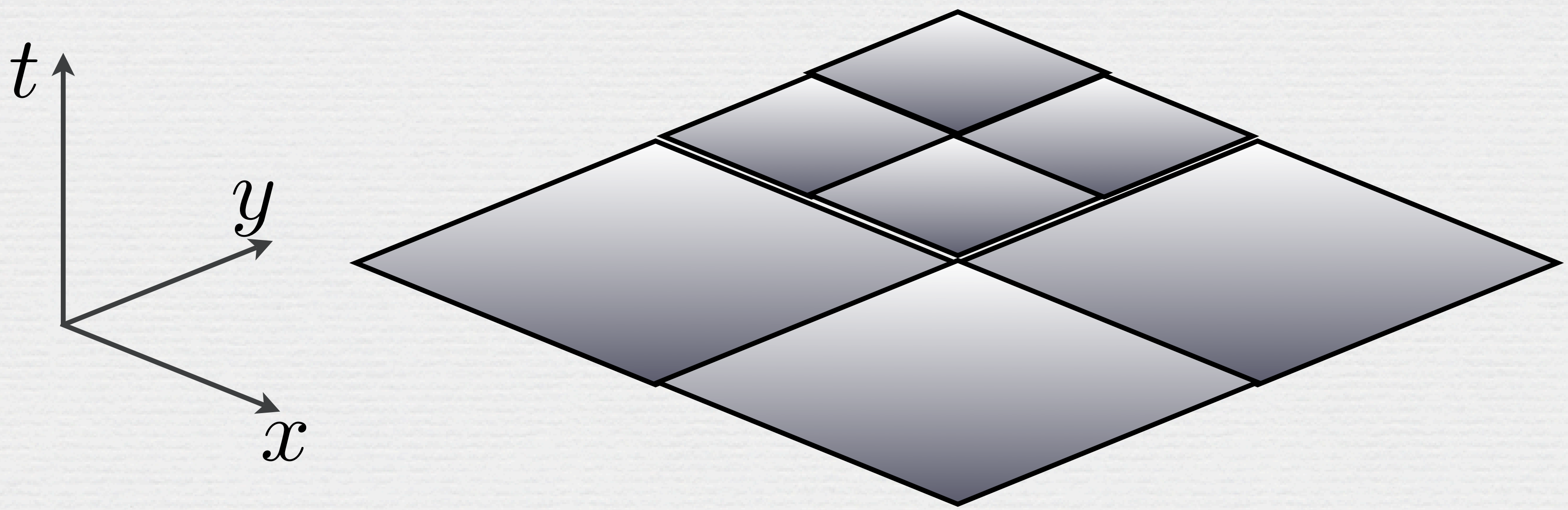
$$\frac{3\mathbf{v}^n - 4\tilde{\mathbf{v}}^{n-1} + \tilde{\mathbf{v}}^{n-2}}{2\Delta t} = -\mathbf{M} f \mathbf{v}^n - \nabla\Phi^n$$

$$\frac{3\Phi^n - 4\tilde{\Phi}^{n-1} + \tilde{\Phi}^{n-2}}{2\Delta t} = -\Phi_0\, \nabla \cdot \mathbf{v}^n$$

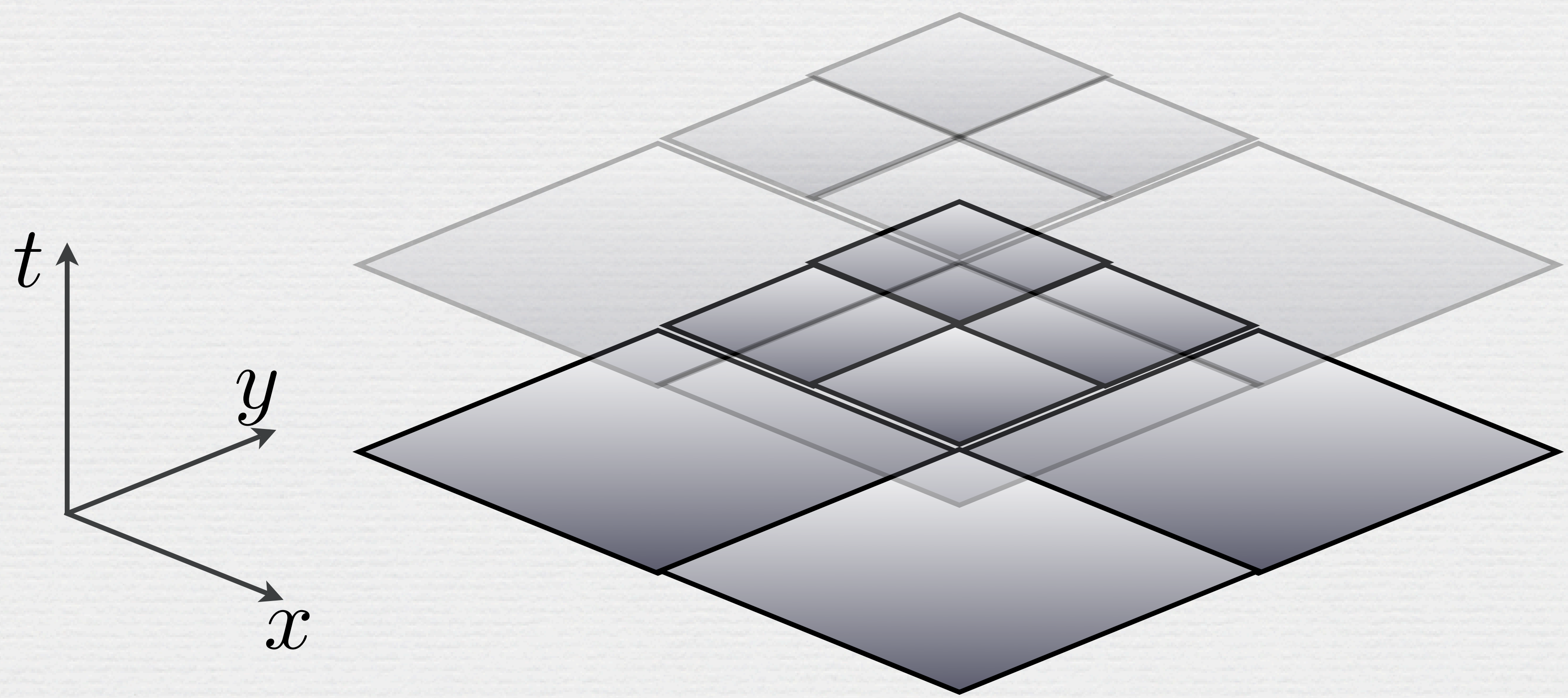
Coriolis inverted

Non-symmetric due to implicit Coriolis: CGS.

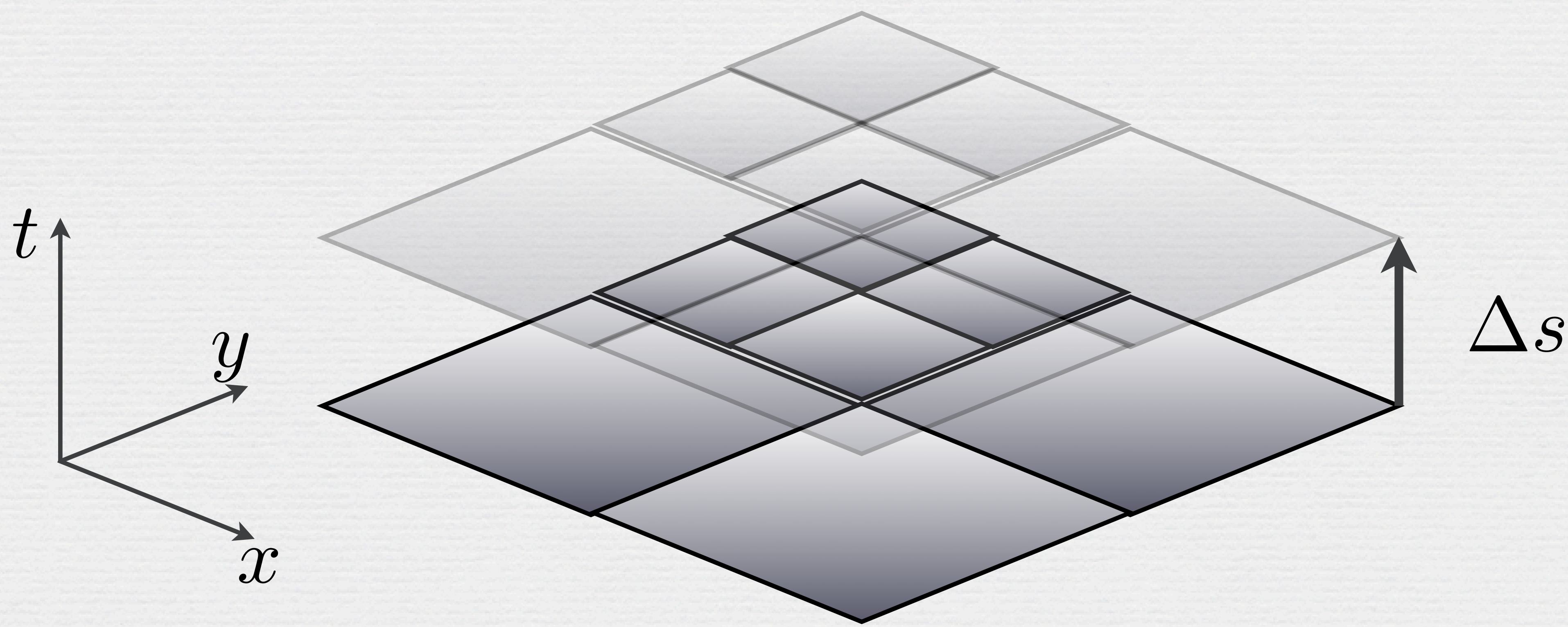
Implicit/Explicit



Implicit/Explicit



Implicit/Explicit



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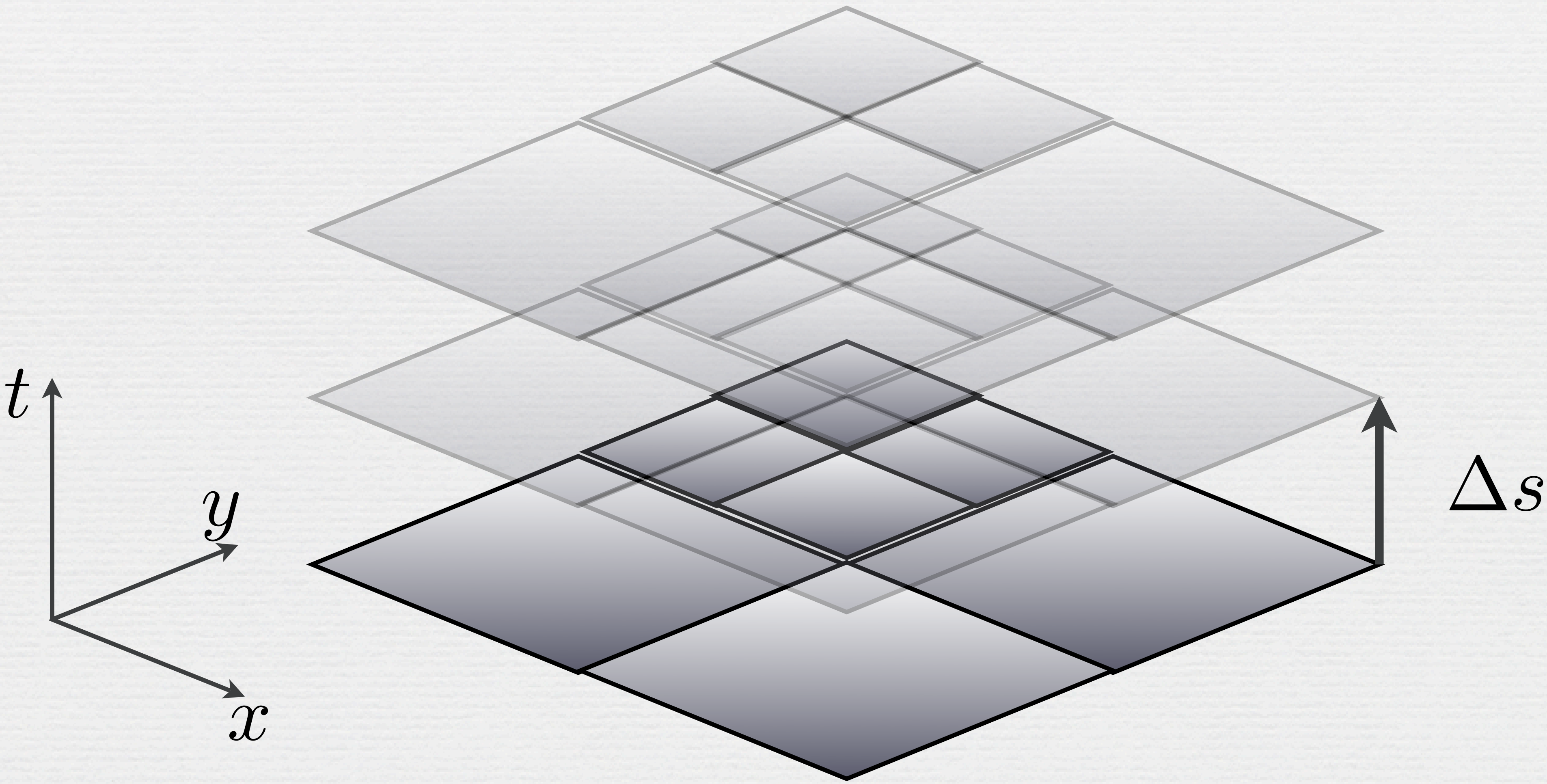
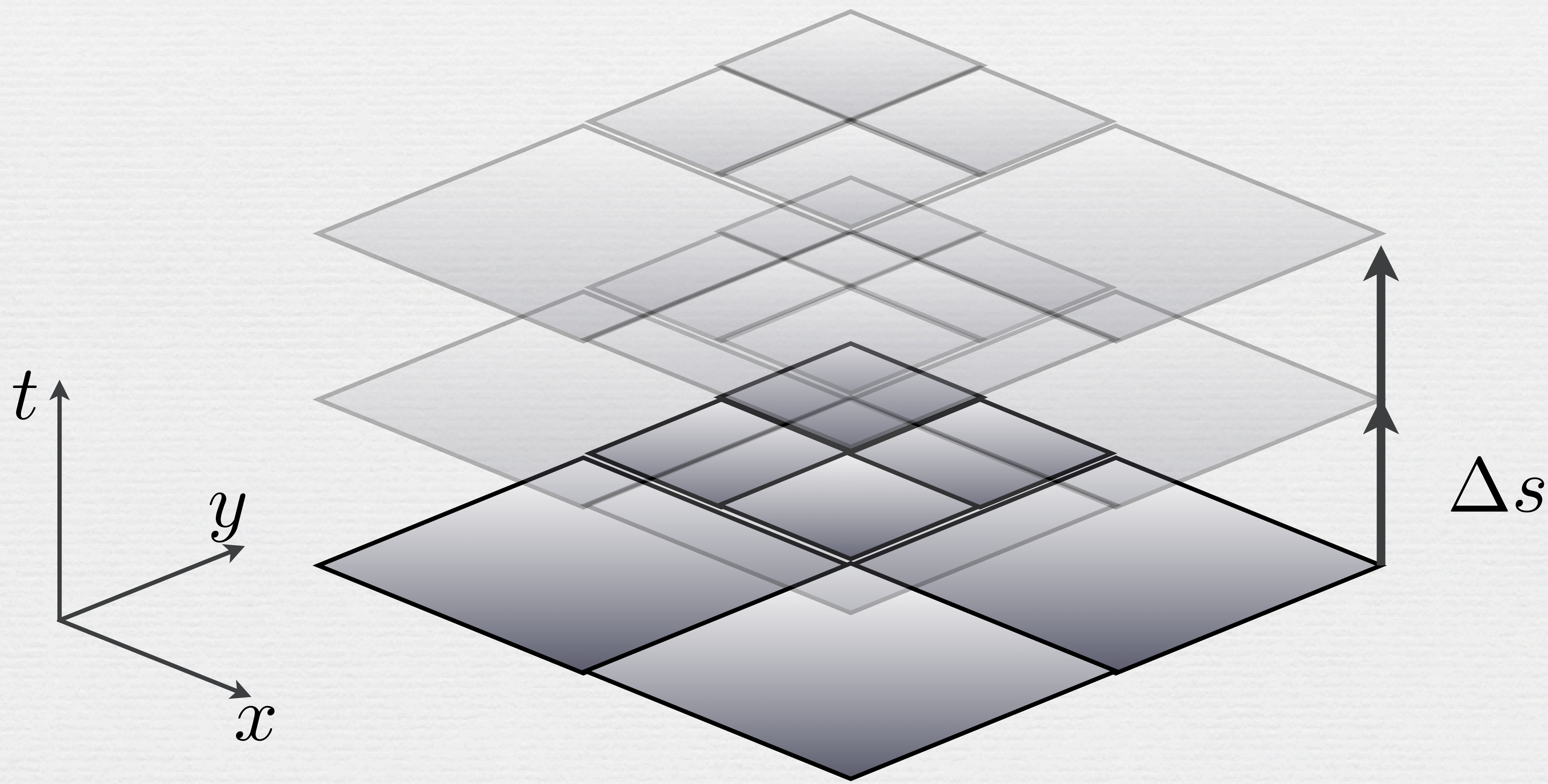
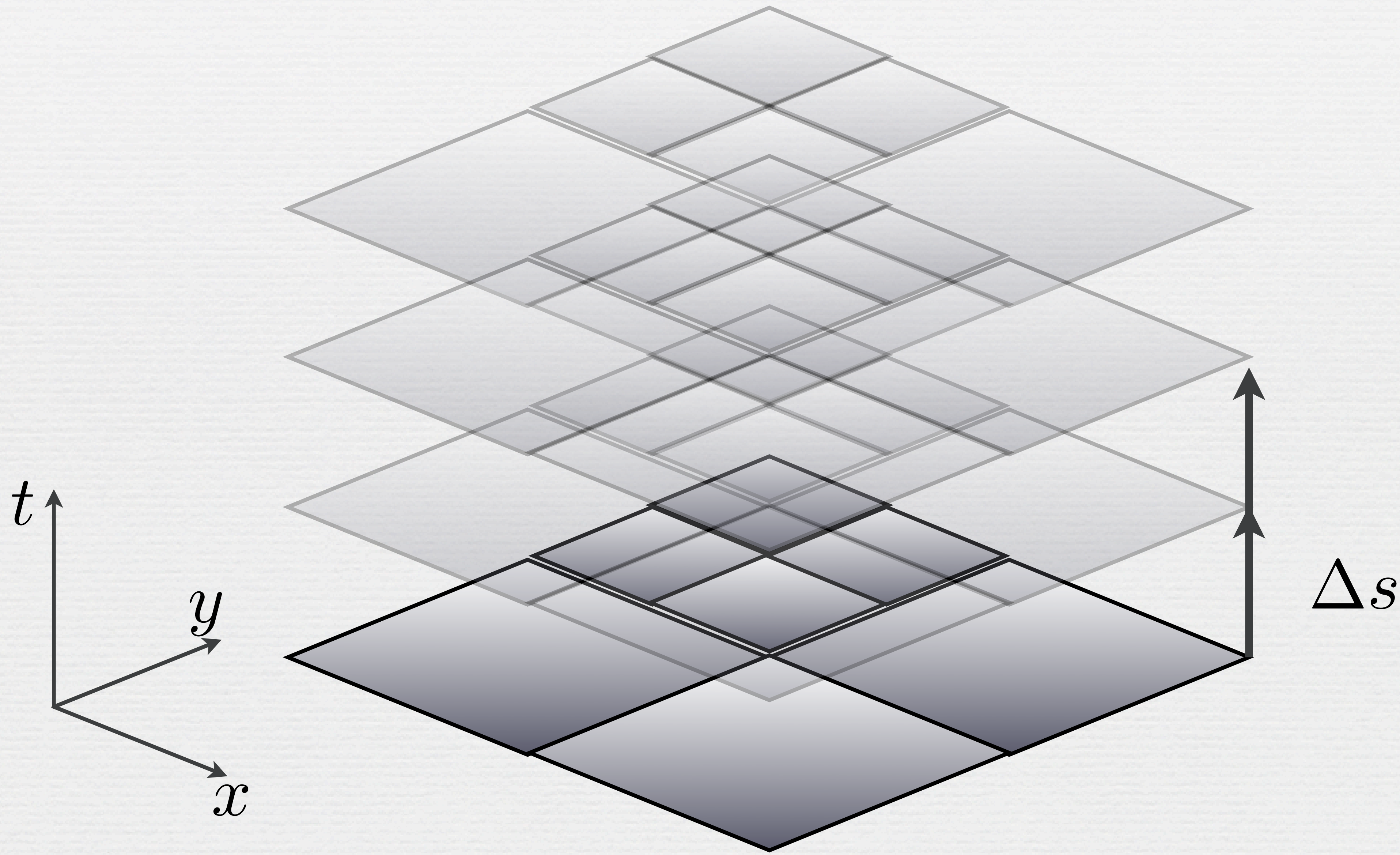


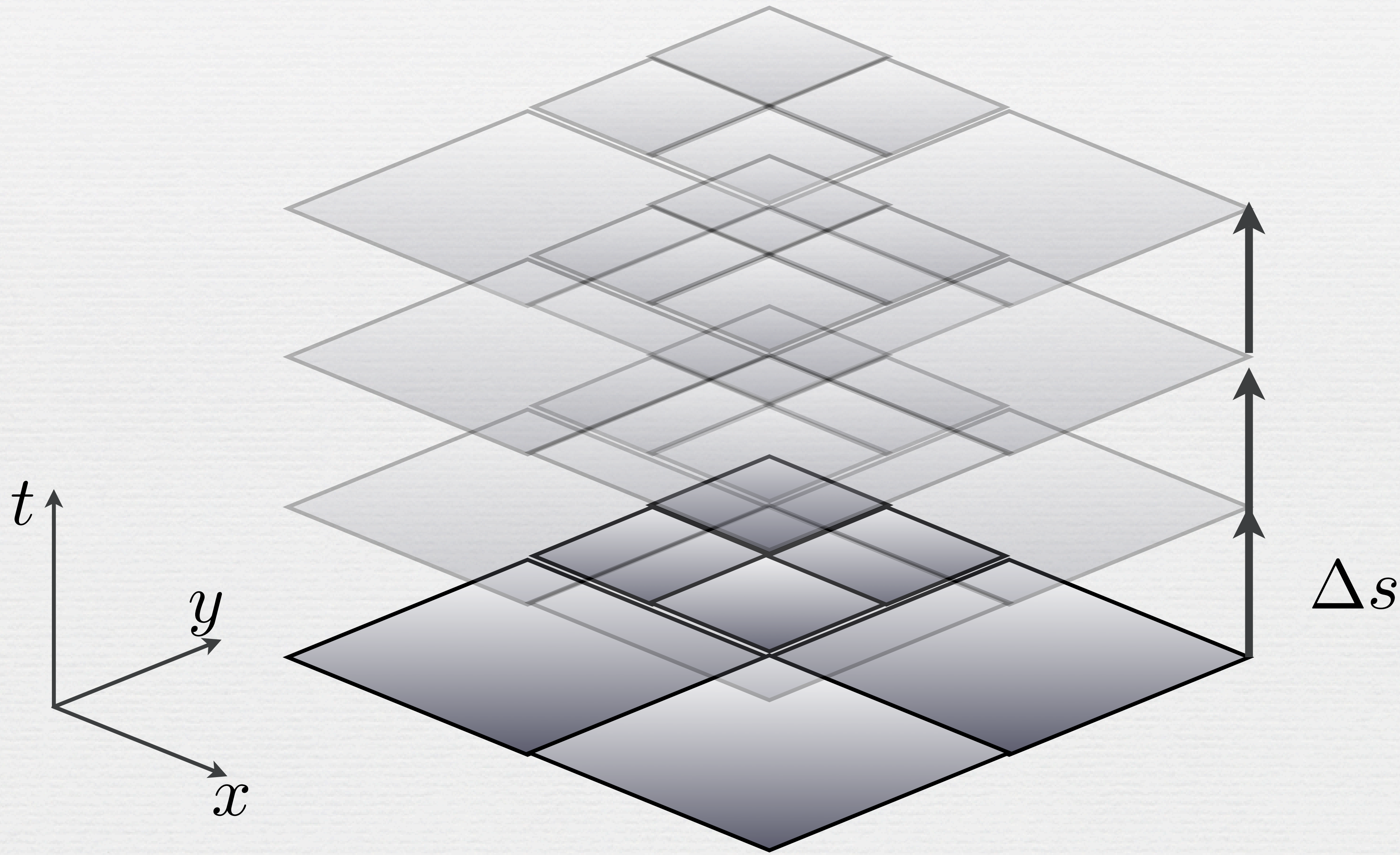
Implicit/Explicit



Implicit/Explicit



Implicit/Explicit



Implicit/Explicit

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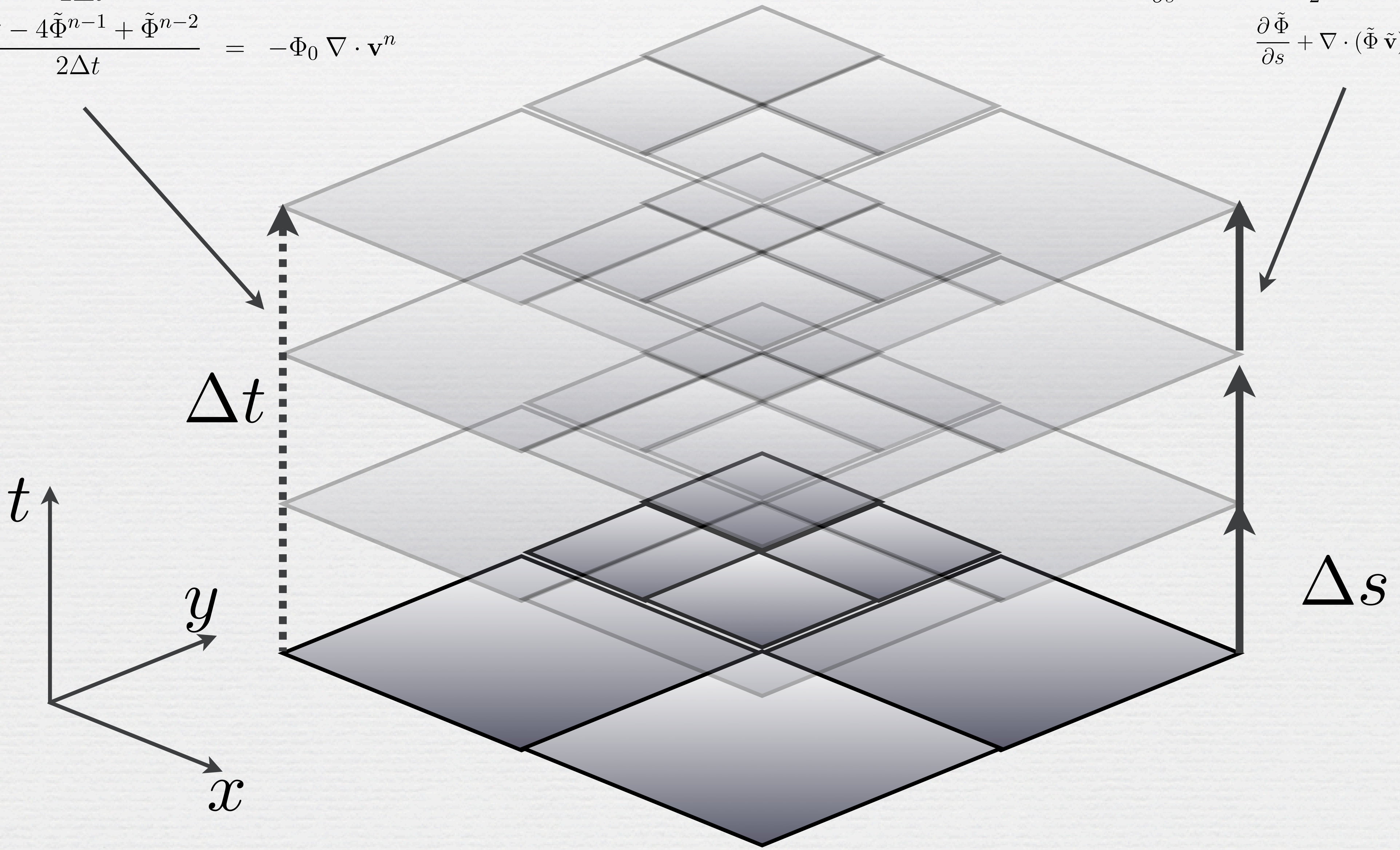
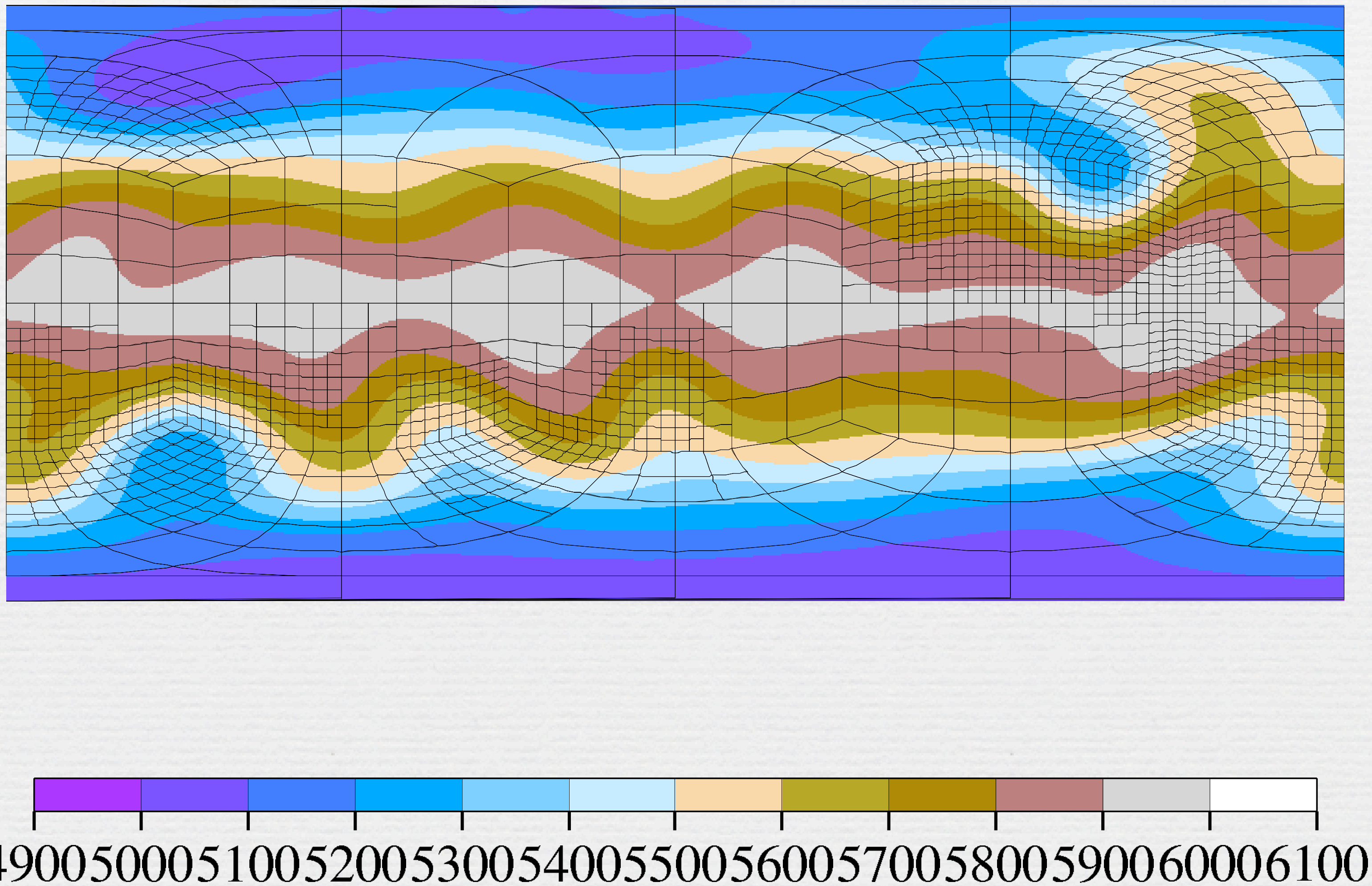
$$\frac{3\Phi^n - 4\tilde{\Phi}^{n-1} + \tilde{\Phi}^{n-2}}{2\Delta t} = -\Phi_0\,\nabla\cdot\mathbf{v}^n$$

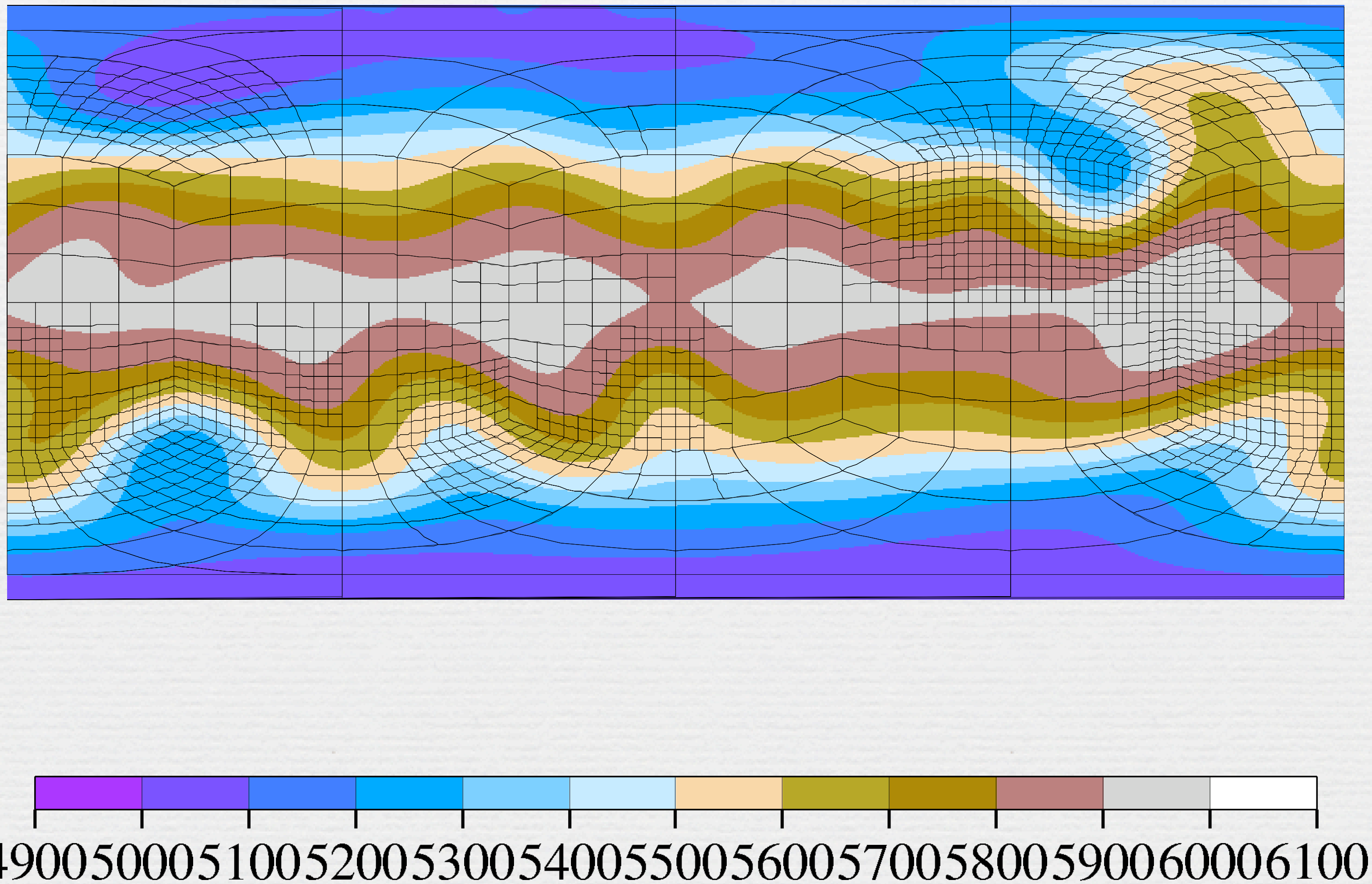
$$\frac{\partial\,\tilde{\mathbf{v}}}{\partial s} + \tilde{\zeta}\,\mathbf{k}\times\tilde{\mathbf{v}} + \frac{1}{2}\nabla\left(\tilde{\mathbf{v}}\cdot\tilde{\mathbf{v}}\right) = 0$$
$$\frac{\partial\,\tilde{\Phi}}{\partial s} + \nabla\cdot(\tilde{\Phi}\,\tilde{\mathbf{v}}) = 0$$



dt=120s

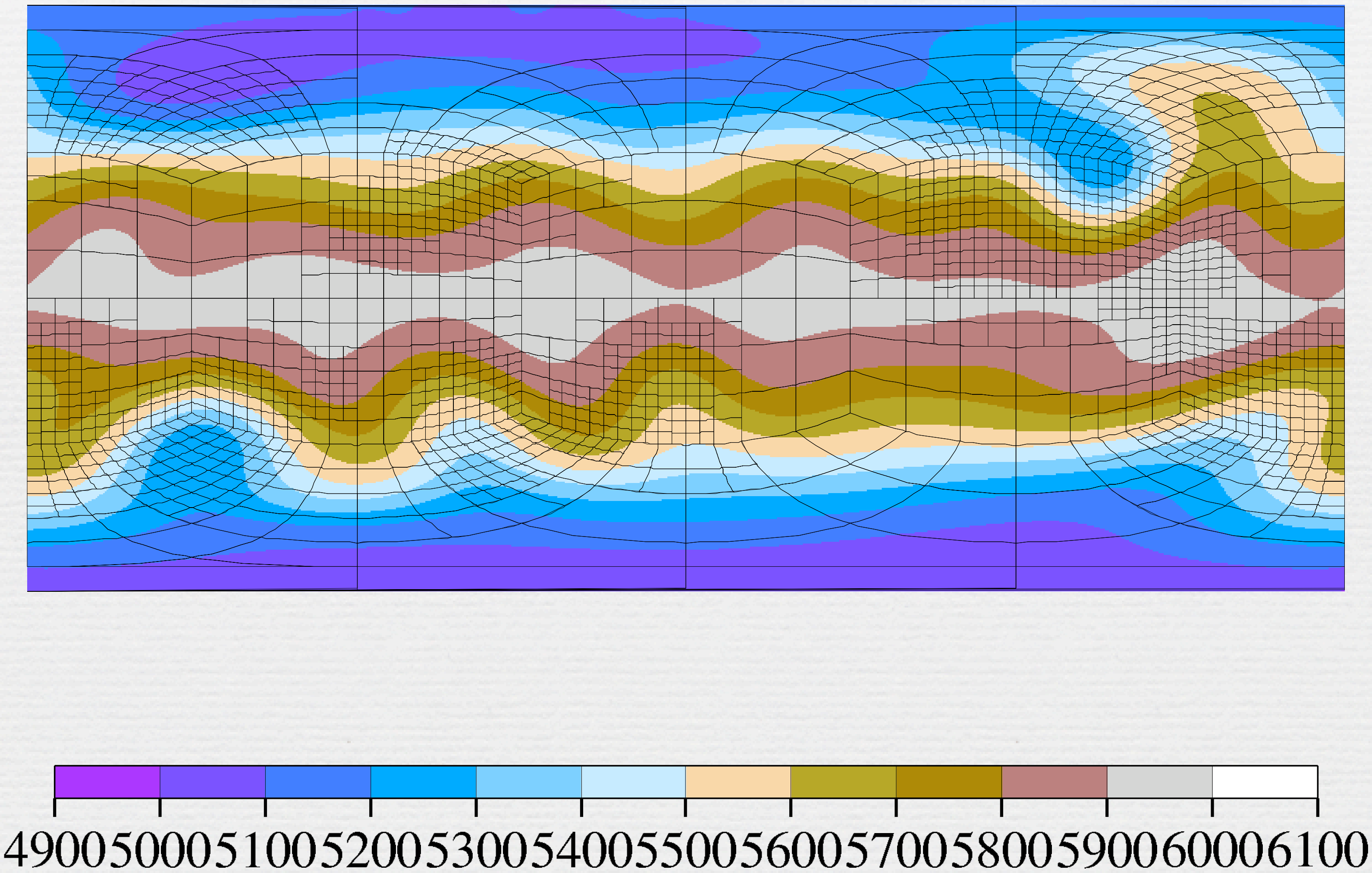

4900
5000
5100
5200
5300
5400
5500
5600
5700
5800
5900
6000
6100

dt=360s

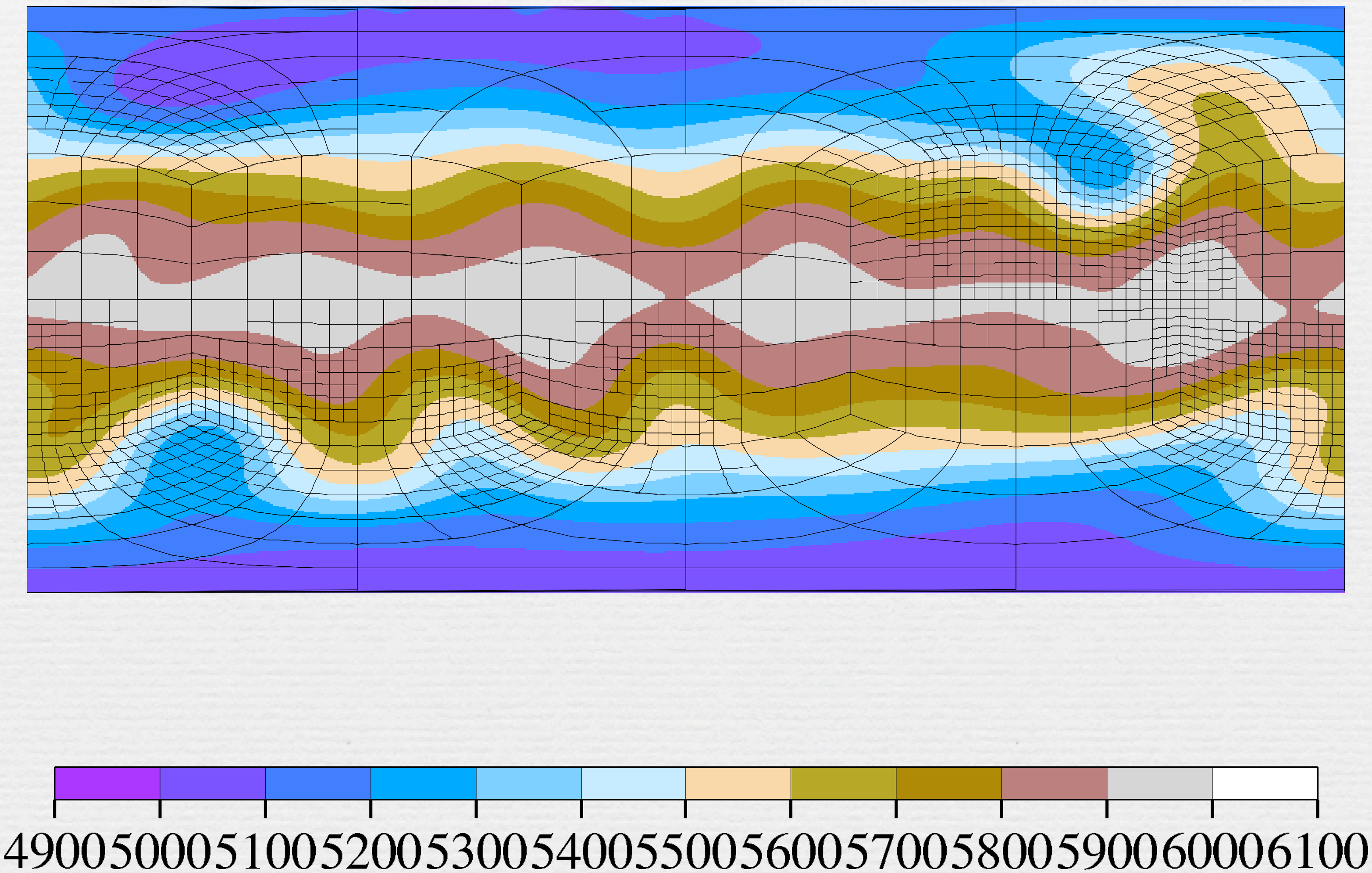


4900 5000 5100 5200 5300 5400 5500 5600 5700 5800 5900 6000 6100

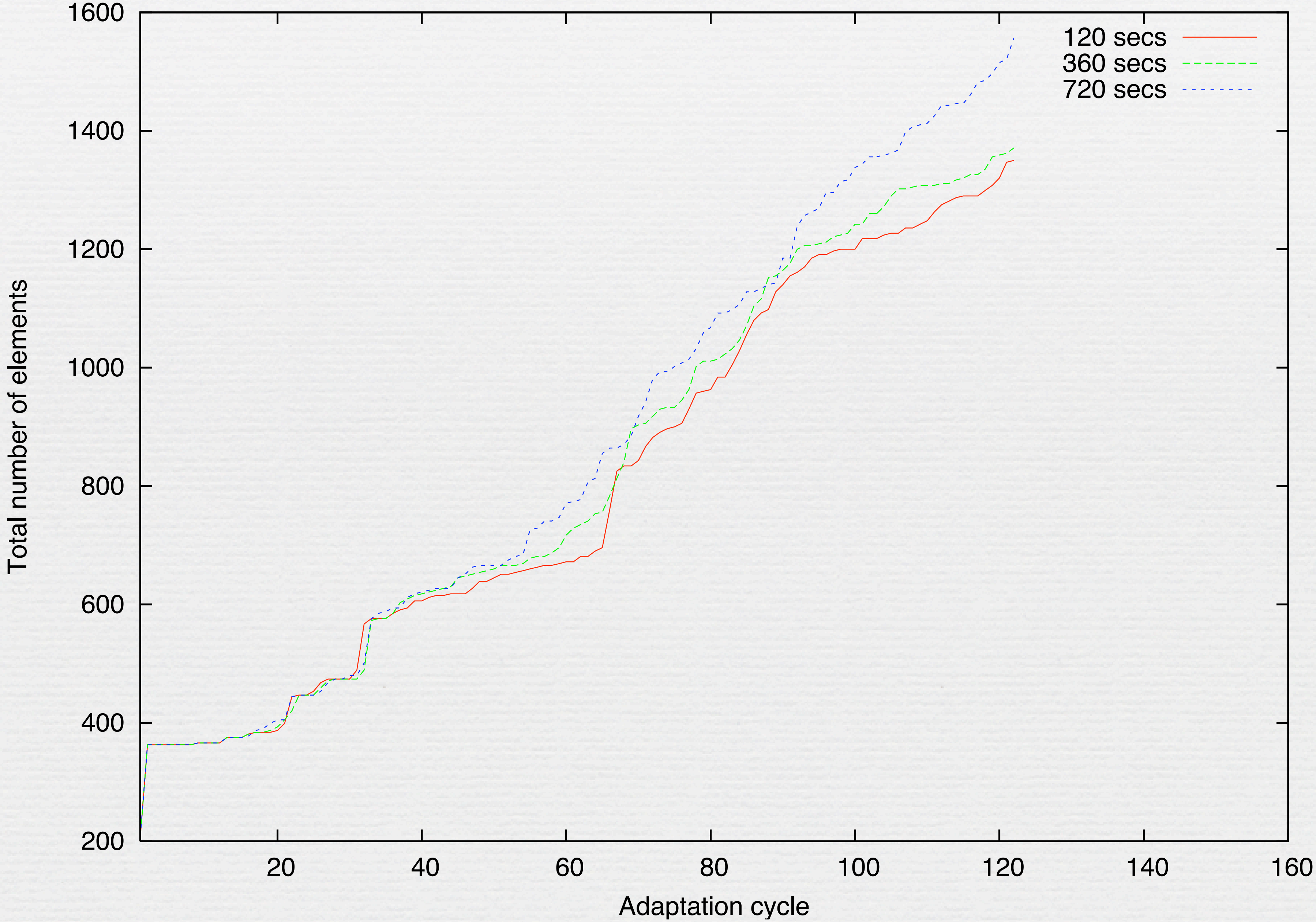
dt=720s


4900 5000 5100 5200 5300 5400 5500 5600 5700 5800 5900 6000 6100

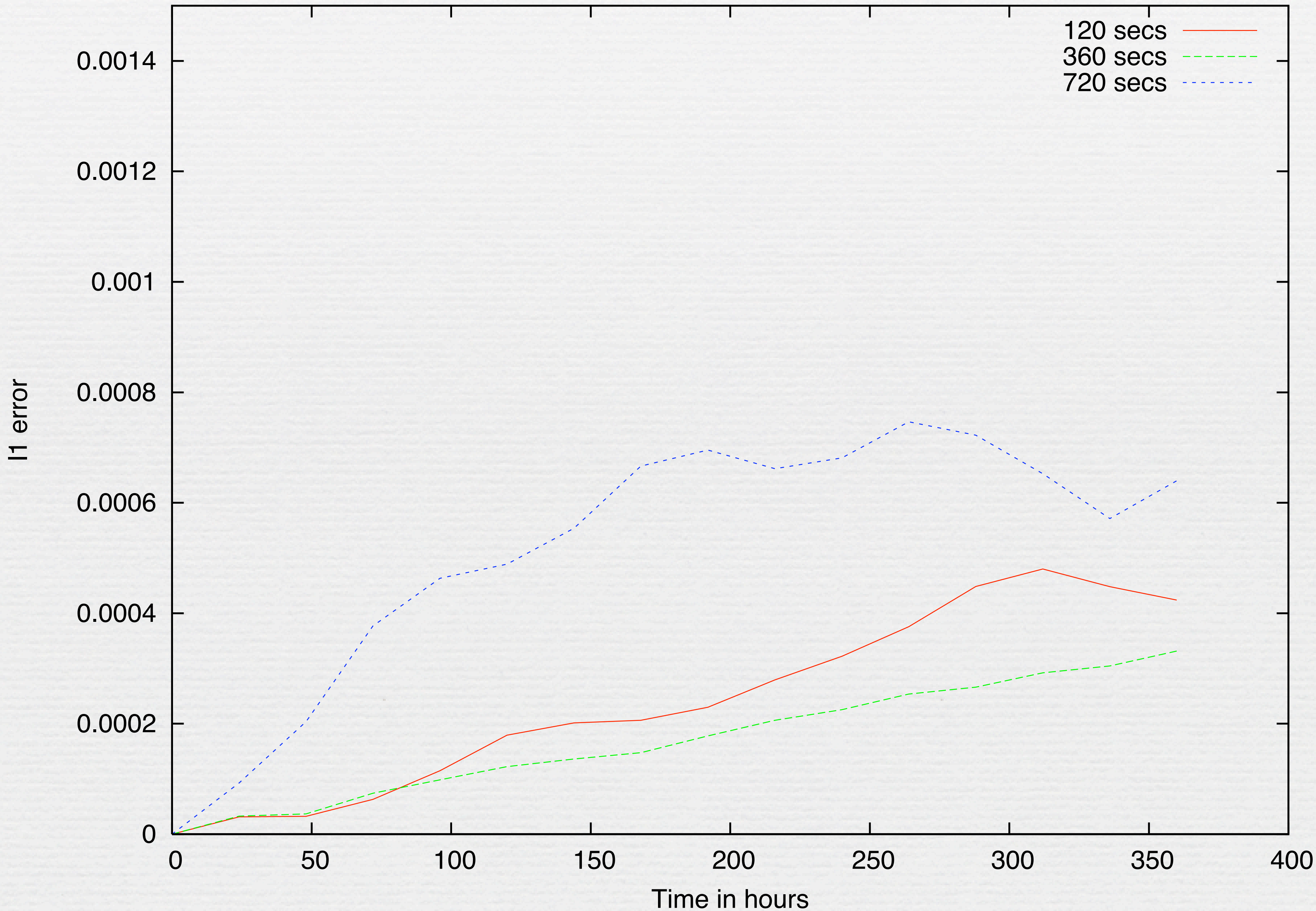
dt=120s


4900 5000 5100 5200 5300 5400 5500 5600 5700 5800 5900 6000 6100

Total number of elements



Comparison with reference solution from NCAR pseudo spectral



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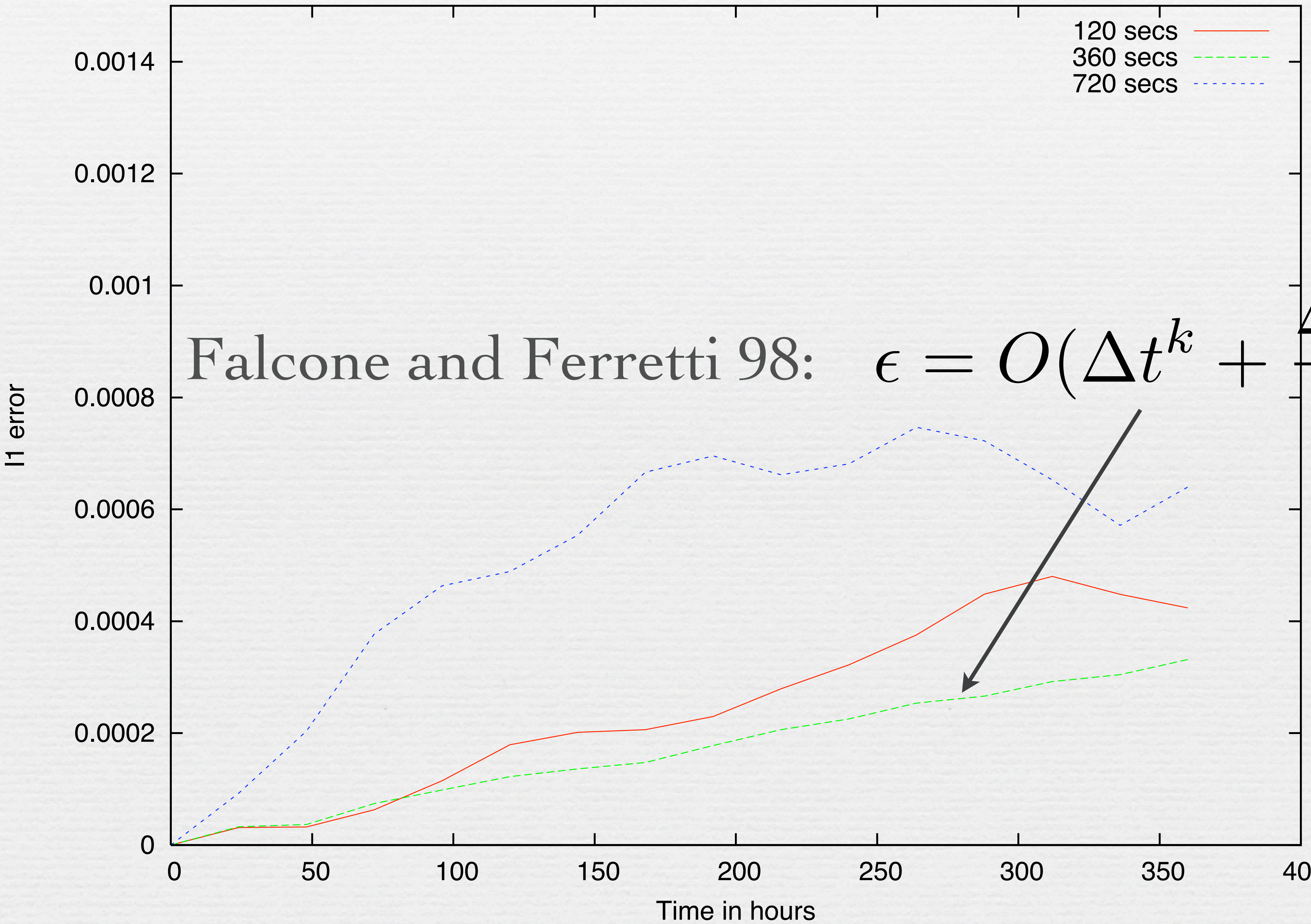


Falcone and Ferretti 98: $\epsilon = O(\Delta t^k + \frac{\Delta x^p}{\Delta t})$

Discontinuous Galerkin

Conservation law:

$$u_t + \nabla \cdot \mathcal{F}(u) = S(u)$$

Weak form:

$$\frac{d}{dt}\int\limits_{\Omega_k} \varphi_h u_h \, d\Omega = \int\limits_{\Omega_k} \varphi_h S(u_h)\, d\Omega + \int\limits_{\Omega_k} \mathcal{F}(u_h) \cdot \nabla \varphi_h \, d\Omega - \int\limits_{\partial\Omega_k} \varphi_h \mathcal{F} \cdot \hat{n} \, ds$$

Numerical flux:

$$\widehat{\mathcal{F}}(u_h^+, u_h^-) = \frac{1}{2}\left[(\mathcal{F}(u_h^+) + \mathcal{F}(u_h^-)) \cdot \hat{n} - \alpha(u_h^+ - u_h^-)\right]$$

SSP Runge-Kutta with extended linear stability region

$$\frac{d\mathbf{U}_h}{dt} = \mathbf{L_h}(\mathbf{U}_h).$$

$$u^{(0)} = u^n$$

$$u^{(i)} = \sum_{k=0}^{i-1} \alpha_{ik} u^{(k)} + \Delta t \beta_{ik} L(u^{(k)}), \quad i = 1, \ldots, m,$$

$$u^{n+1} = u^{(m)}.$$

Higueras 2004, JSC

Compressible Euler:

$$\underline{U} \equiv (\rho, \rho u, \rho w, \Theta)^T = (\rho, U, W, \Theta)^T$$

$$\mathbf{F}(\underline{U}) \equiv (F, G)$$

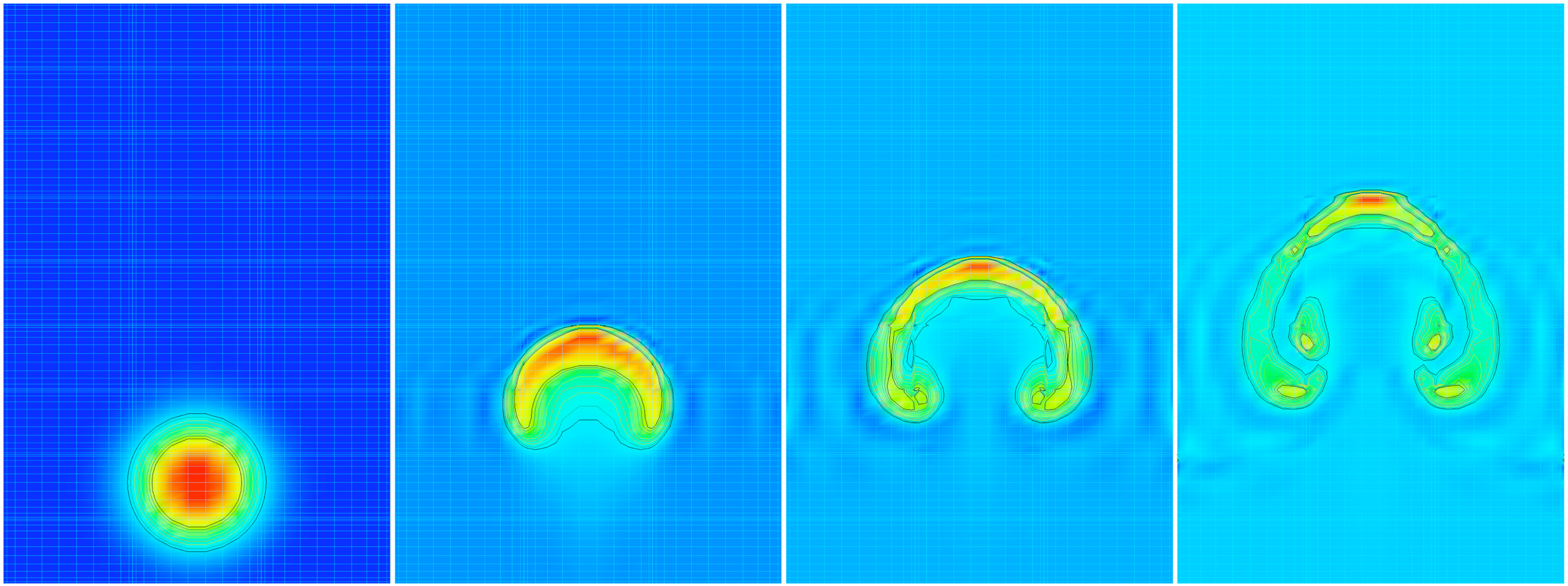
$$F = (U, \frac{UU}{\rho} + p, \frac{WU}{\rho}, \frac{U\Theta}{\rho})^T$$

$$G = (W, \frac{UW}{\rho}, \frac{WW}{\rho} + p, \frac{W\Theta}{\rho})^T$$

$$S(\underline{U}) = (0, 0, -g\rho, 0)^T$$

$$p = p_0(\frac{R\Theta}{p_0})^\gamma$$

Warm bubble (Robert 93)



To use or not to use HOMs?

- Low order: 2nd or 3rd
- Very dissipative
- Large dispersion error
- Oscillation control: grid point level
- Limiters - not - cache efficient
- CFL: dt = O(dx)
- Costly halos...

- Spectrally accurate
- No dissipation
- No dispersion error: with filtering
- Oscillation control: questionable
- Matrix-Matrix tensor operations: cache friendly
- Halos are minimal

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