From classical to optimized Schwarz

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(PART-2)

The quest for cheaper and faster preconditioning

Part 2:

The Robin method

- Lions (1990)
- Used to accelerate convergence of Schwarz
- Free positive parameter: how to find its correct value?
- Convergence rate not demonstrated theoretically

\[
\begin{align*}
\mathcal{L} u_j^{k+1} &= u_j^{k+1} - \Delta u_j^{k+1} = f_j \\
p u_j^{k+1} + \frac{\partial u_j^{k+1}}{\partial n_{ji}} &= p u_i^k + \frac{\partial u_i^k}{\partial n_{ji}} \quad \text{on } \partial \Omega_j \cap \partial \Omega_l \quad \text{for } l \in \mathcal{N}(\Omega_j) \\
u_j^{k+1} &= u_0 \quad \text{on } \partial \Omega_j \cap \partial \Omega
\end{align*}
\]
Convergence of the Robin method

Write the error as: $e_j^{k+1} = u_j^{k+1} - u|_{\Omega_j}$

Homogeneous case:
$e_j^{k+1} - \Delta e_j^{k+1} = 0$
$pe_j^{k+1} + \frac{\partial e_j^{k+1}}{\partial n_{\Omega_j}} = pe_l^{k} + \frac{\partial e_l^{k}}{\partial n_{\Omega_l}}$ on $\partial \Omega_j \cap \partial \Omega_l$ for $l \in N(\Omega_j)$
$e_j^{k+1} = 0$ on $\partial \Omega_j \cap \partial \Omega$

Multiplication by error term + integration by parts:
$0 = \int_{\Omega_j} e_j^{k+1} e_j^{k+1} + \int_{\Omega_j} \nabla e_j^{k+1} \cdot \nabla e_j^{k+1} - \sum_{l \in N(j)} \int_{\Gamma_{j,l}} e_j^{k+1} \frac{\partial e_j^{k+1}}{\partial n_{\Omega_l}}$
$= \|e_j^{k+1}\|_{0,\Omega_j}^2 + \|\nabla e_j^{k+1}\|_{0,\Omega_j}^2 - \sum_{l \in N(j)} \int_{\Gamma_{j,l}} e_j^{k+1} \frac{\partial e_j^{k+1}}{\partial n_{\Omega_l}}$
$= \|e_j^{k+1}\|_{1,\Omega_j}^2 - \sum_{l \in N(j)} \int_{\Gamma_{j,l}} e_j^{k+1} \frac{\partial e_j^{k+1}}{\partial n_{\Omega_l}}$

Convergence of the Robin method

Using: $AB = \frac{1}{4p}[(A+pB)^2 - (A-pB)^2]$ and summing over all elements and the first $M$ iterations:
$\sum_{k=0}^M \sum_{i=0}^{K} ||e_i^{k+1}||^2_{\Omega_i} + \frac{1}{4p} \sum_{(i,l)} \int_{\Gamma_{i,l}} \left\{ (\frac{\partial e_i^M}{\partial n_{\Omega_i}} + pe_i^M)^2 + (\frac{\partial e_l^M}{\partial n_{\Omega_l}} - pe_l^M)^2 \right\}$

Implying:
$\lim_{M \to \infty} \sum_{k=0}^{M} \sum_{i=0}^{K} ||e_i^{k+1}||^2_{\Omega_i} < C$

For any positive “p”: what is the best choice?

Fourier analysis

- Study simple 2D problem
- Only 2 subdomains
- Fourier transform in the tangent direction to the separating interface between domains
- Solve the remaining ODE
- Obtain convergence rate of the algorithm

Problem setting:

$(\eta - \Delta)u(x, y) = 0$, on $\Omega$

Boundary conditions: solution decays at infinity

Subdomains:
$\Omega_1 = [-\infty, L] \times \mathbb{R}$ and $\Omega_2 = [0, \infty] \times \mathbb{R}$
Fourier analysis

Two subproblems:

\[
(\eta - \Delta)u_1^{n+1} = 0 \quad \text{in} \quad \Omega_1, \quad (\eta - \Delta)u_2^{n+1} = 0 \quad \text{in} \quad \Omega_2, \\
u_1^{n+1}(L, y) = u_2^0(L, y) \quad \text{on} \quad \Gamma_{12}, \quad u_2^{n+1}(0, y) = u_1^n(0, y) \quad \text{on} \quad \Gamma_{21}.
\]

Fourier transforming in the y direction:

\[
(\eta + k^2 - \partial_x)\hat{u}_1^{n+1} = 0 \quad \text{in} \quad \Omega_1, \quad (\eta + k^2 - \partial_x)\hat{u}_2^{n+1} = 0 \quad \text{in} \quad \Omega_2, \\
\hat{u}_1^{n+1}(L, k) = \hat{u}_2^0(L, k) \quad \text{on} \quad \Gamma_{12}, \quad \hat{u}_2^{n+1}(0, k) = \hat{u}_1^n(0, k) \quad \text{on} \quad \Gamma_{21}.
\]

Solving in the x direction:

\[
\hat{u}_1^n(x, k) = \hat{u}_2^{n-1}(L, k)e^{-\sqrt{k^2+\eta}(x-L)}, \quad \hat{u}_2^n(x, k) = \hat{u}_1^{n-1}(0, k)e^{-\sqrt{k^2+\eta}x}
\]

Convergence rate of classical Schwarz (Gander 2006 SINUM):

\[
\rho_{cla} = \rho_{cla}(k, \eta, L) = e^{-\sqrt{k^2+\eta}L}
\]

 Remarks about convergence rate

- Converges for all frequencies
- Is a smoother: damps quickly high frequencies
- Convergence depends on eta and overlap size
- For no overlap the algorithm does not converge

Optimized approach

Inspired by the Robin problem:

\[
(\eta - \Delta)u_1^{n+1} = 0 \quad \text{in} \quad \Omega_1, \quad (\eta - \Delta)u_2^{n+1} = 0 \quad \text{in} \quad \Omega_2, \\
(\partial_x + S_1)u_1^{n+1} = (\partial_x + S_2)u_2^{n+1} \quad \text{on} \quad \Gamma_{12}, \quad u_2^{n+1}(0, y) = u_1^n(0, y) \quad \text{on} \quad \Gamma_{21}.
\]

We are looking for the best possible forms of in Fourier space

Proceeding as before leads to the solutions:

\[
\sigma_r(k) = \mathcal{F}(S_r)
\]

\[
\hat{u}_1^n(x, k) = \frac{\sigma_1(k) - \sqrt{k^2 + \eta}}{\sigma_1(k) + \sqrt{k^2 + \eta}}e^{-\sqrt{k^2+\eta}(x-L)}\hat{u}_2^{n-1}(L, k), \quad \hat{u}_2^n(x, k) = \frac{\sigma_2(k) + \sqrt{k^2 + \eta}}{\sigma_2(k) - \sqrt{k^2 + \eta}}e^{-\sqrt{k^2+\eta}x}\hat{u}_1^{n-1}(0, k)
\]

New convergence rate:

\[
\rho_{opt} = \rho_{opt}(k, \eta, L) = \frac{\sigma_1(k) - \sqrt{k^2 + \eta}}{\sigma_1(k) + \sqrt{k^2 + \eta}}\sigma_2(k) + \sqrt{k^2 + \eta}e^{-2\sqrt{k^2+\eta}L}
\]

Optimized approach

The choice

\[
\sigma_1(k) = \sqrt{k^2 + \eta}, \quad \sigma_2(k) = -\sqrt{k^2 + \eta}
\]

leads to the convergence of the algorithm in 2 iterations \(\rho_{opt} = 0\)

The operators are not local operators in physical space!

An approximation is sought such that all frequencies have an optimal decay rate:

\[
\sigma_{app}^1(k) = p_1 + q_1 k^2, \quad \sigma_{app}^2(k) = -p_2 - q_2 k^2
\]
Various choices (one sided)

Taylor zeroth order:
\[ \sigma_{0}^{app}(k) = \sqrt{\eta} \]

Taylor second order:
\[ \sigma_{1}^{app}(k) = \sqrt{\eta} + \frac{1}{2\sqrt{\eta}} k^2 \]

Zeroth order optimized:
\[ k(L, \eta, p) = \frac{\sqrt{L(2p + L(p^2 - \eta))}}{L} \]
\[ \rho_{OO0}(k_{min}, L, \eta, p^*) = \rho_{OO0}(k(p^*), L, \eta, p^*) \]

Zeroth order optimized (no overlap): \[ p^* = ((k_{min}^2 + \eta)(k_{max}^2 + \eta))^{\frac{1}{2}} \]

Second order optimized: very long and complex formulas for \( p \) and \( q \) ...

Details see Gander (SINUM 2006)

Examples for FDM

\[ u - \Delta u = 0, \text{ on } [0, 1] \times [0, 1], \quad u(0) = u(1) = 0 \]

2 subdomains
2nd order Laplacian
Mesh spacing \( h = 1/30 \)
Optimization done at the matrix level (SGT 2006 SISC)

Examples for FDM: ORAS

Convergence rates

Classical Schwarz
Examples for FDM: OMS

HOMs: spectral elements

Galerkin idea: identical to FEM

High-order basis on each element

Integration with Gauss-Legendre-Lobatto quadratures

\[ \mathbf{v}_h^k(r_1, r_2) = \sum_{i=0}^{N} \sum_{j=0}^{N} \mathbf{v}_{ij} h_i(r_1) h_j(r_2) \]

\[ \langle f, g \rangle_{GL} = \sum_{k=1}^{K} \sum_{i=0}^{N} \sum_{j=0}^{N} f^k(\xi_i, \xi_j) g^k(\xi_i, \xi_j) \rho_i \rho_j \]
**HOMs: spectral elements**

**Asymptotic behavior**

<table>
<thead>
<tr>
<th>(\kappa (M^{-1}A))</th>
<th>(h)</th>
<th>(N)</th>
</tr>
</thead>
<tbody>
<tr>
<td>AS, no overlap</td>
<td>(O(h^{-1}))</td>
<td>(O(N^2))</td>
</tr>
<tr>
<td>SS, no overlap</td>
<td>(O(h^{-1}))</td>
<td>(O(N^2))</td>
</tr>
<tr>
<td>OO0, no overlap</td>
<td>(O(h^{-1/2}))</td>
<td>(O(N))</td>
</tr>
<tr>
<td>OO2, no overlap</td>
<td>(O(h^{-1/2}))</td>
<td>(O(N^{1/2}))</td>
</tr>
</tbody>
</table>

- Number of subdomains dependance: \(1/H^2\)
- Removed by coarse solver.
- Optimal is the \(Q_1\) fem problem on GLL mesh (S.D. Kim 2006)

**Primitive equations**

- Momentum:
  \[
  \frac{d\mathbf{v}}{dt} + f \mathbf{k} \times \mathbf{v} + \nabla \Phi + RT \nabla \ln p = 0
  \]

- Thermodynamic:
  \[
  \frac{dT}{dt} - \frac{\kappa}{p} T \omega = 0
  \]

- Hydrostatic:
  \[
  \frac{\partial}{\partial t} \left( \frac{\partial p}{\partial \eta} \right) + \nabla \cdot \left( \mathbf{v} \frac{\partial p}{\partial \eta} \right) + \frac{\partial}{\partial \eta} \left( \frac{\dot{\eta}}{\partial \eta} \right) \frac{\partial p}{\partial \eta} = 0
  \]

**Cubed sphere**

Spectral element
Time discretization

Semi-implicit time discretization
Leads to positive definite Helmholtz problem to solve at each time step
Optimized Schwarz with tangential derivative used
Results on Blue Gene/L machine
Held-Suarez test case

Parallel performance BG/L

Conclusion

A simple modification to classical Schwarz leads to a faster converging solver
This is an easy intervention in a model
With coarse solver, optimized Schwarz is nearly optimal: no need to keep constant overlap (none is required!)
Good performance in a general circulation model
Future work: semi-discrete optimizations, rate of convergence for SEM and optimal control (S.D. Kim)