A Discontinuous Galerkin high-order Non-Hydrostatic model: Formulation and 2D experiments

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Abstract:

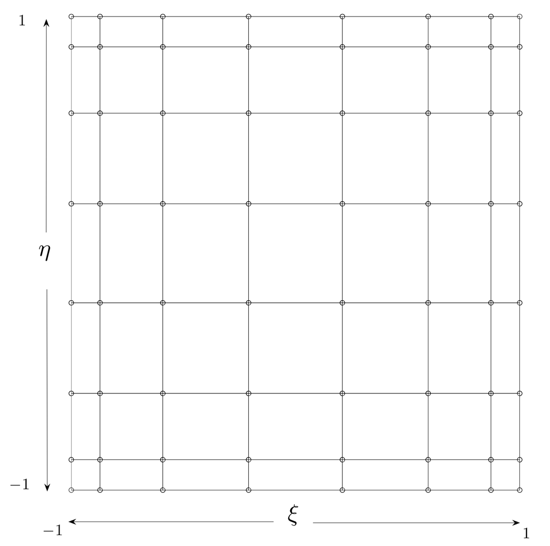
A discontinuous Galerkin (DG) prototype for the compressible Euler equations is presented in its explicit version. In order to avoid aliasing issues, all integrals appearing in the weak form are performed exactly. The resulting mass matrix of the DG method is block diagonal and therefore invertible once and for all. Initial numerical experiments without any filtering nor artificial diffusion demonstrate the feasibility of the method.

High-order Discontinuous Galerkin method:

High-order finite element methods are currently garnering a great deal of attention in the atmospheric modeling community due to their desirable numerical properties and inherent parallelism. The discontinuous Galerkin method combines ideas from spectral methods, finite-elements and finite volumes methods. It leads to a stable, high-order accurate and locally conservative finite element method whose approximate solution is discontinuous across inter-element boundaries: this property renders the method ideally suited for hp-adaptivity. The basis used to represent fields consist of Lagrange polynomials passing through Gauss-Legendre-Lobatto (GLL) or Gauss-Legendre quadrature points. The physical domain Ω is partitioned into K elements Ω_k. On each element unknowns, for instance velocity, are expanded in terms of the N-th degree Lagrangian <

$$\mathbf{v}_h^k(\mathbf{x}) = \sum_{i=0}^{N}\sum_{j=0}^{N} \mathbf{v}_{ij}^k \, h_i(\xi^k(\mathbf{x}))\, h_j(\eta^k(\mathbf{x}))$$

where $\mathbf{x} \to (\xi^k(\mathbf{x}), \eta^k(\mathbf{x}))$ is an afine transformation from element Ω_k to the reference element on $[-1,1]\times[-1,1]$:



Numerous equations modeling physical phenomenon can be written as systems of conservation laws:

$$\mathbf{U}_t + \sum_{i=1}^{d} \mathbf{F}_i(\mathbf{U})_{x^i} = \mathbf{S}(\mathbf{u})$$

For simplicity consider the scalar case

$$u_t + \nabla\cdot\mathcal{F}(u) = S(u)$$

A weak form on each element is obtained by multiplying the last equation by a suitable test function

$$\frac{d}{dt}\int_{\Omega_k} \varphi_h u_h \, d\Omega = \int_{\Omega_k} \varphi_h S(u_h)\, d\Omega + \int_{\Omega_k} \mathcal{F}(u_h)\cdot\nabla\varphi_h \, d\Omega - \int_{\partial\Omega_k} \varphi_h \mathcal{F}\cdot\hat{n}\, ds$$

At inter-element boundaries, the solution is multi-valued and a Riemann problem needs to be approximately solved. This is accomplished using the simple Lax-Friedrichs numerical flux:

$$\widehat{\mathcal{F}}(u_h^+, u_h^-) = \frac{1}{2}\left[(\mathcal{F}(u_h^+) + \mathcal{F}(u_h^-))\cdot\hat{n} - \alpha(u_h^+ - u_h^-)\right]$$

The spatial discretization, performed using Gauss quadrature, results in a system of ordinary differential equations

$$\frac{d\mathbf{U}_h}{dt} = \mathbf{L_h}(\mathbf{U}_h).$$

Time-stepping:

Strong-stability preserving (SSP) time discretization methods were developed for semi-discrete method of lines approximation (13) of hyperbolic PDE's in conservative form, Gottlieb et al (2001). SSP methods ensure stability in an arbitrary norm once the forward Euler time discretization is shown to be strongly stable. In its general form, a SSP Runge-Kutta ODE solver is written as

$$u^{(0)} = u^n$$
$$u^{(i)} = \sum_{k=0}^{i-1} \alpha_{ik} u^{(k)} + \Delta t \beta_{ik} L(u^{(k)}), \quad i = 1,\ldots,m,$$
$$u^{n+1} = u^{(m)}.$$

Compressible Euler equations:

Atmospheric models rarely use the full meteorological equations. Reasons for this include stringent Courant condition imposed by meteorologically irrelevant sound waves and savings imposed by solving a series of 2D problems instead of 3D coupled equations. The equations,

$$\underline{\mathrm{U}} \equiv (\rho, \rho u, \rho w, \Theta)^T = (\rho, U, W, \Theta)^T$$
$$\mathbf{F}(\underline{\mathrm{U}}) \equiv (F, G)$$
$$F = (U, \frac{UU}{\rho} + p, \frac{WU}{\rho}, \frac{U\Theta}{\rho})^T$$
$$G = (W, \frac{UW}{\rho}, \frac{WW}{\rho} + p, \frac{W\Theta}{\rho})^T$$
$$S(\underline{\mathrm{U}}) = (0, 0, -g\rho, 0)^T$$
$$p = p_0(\frac{R\Theta}{p_0})^\gamma$$

include a conserved generalized potential temperature which is the potential temperature multiplied by density. A hydrostatically balanced state is removed numerically from the equations once they are discretized.

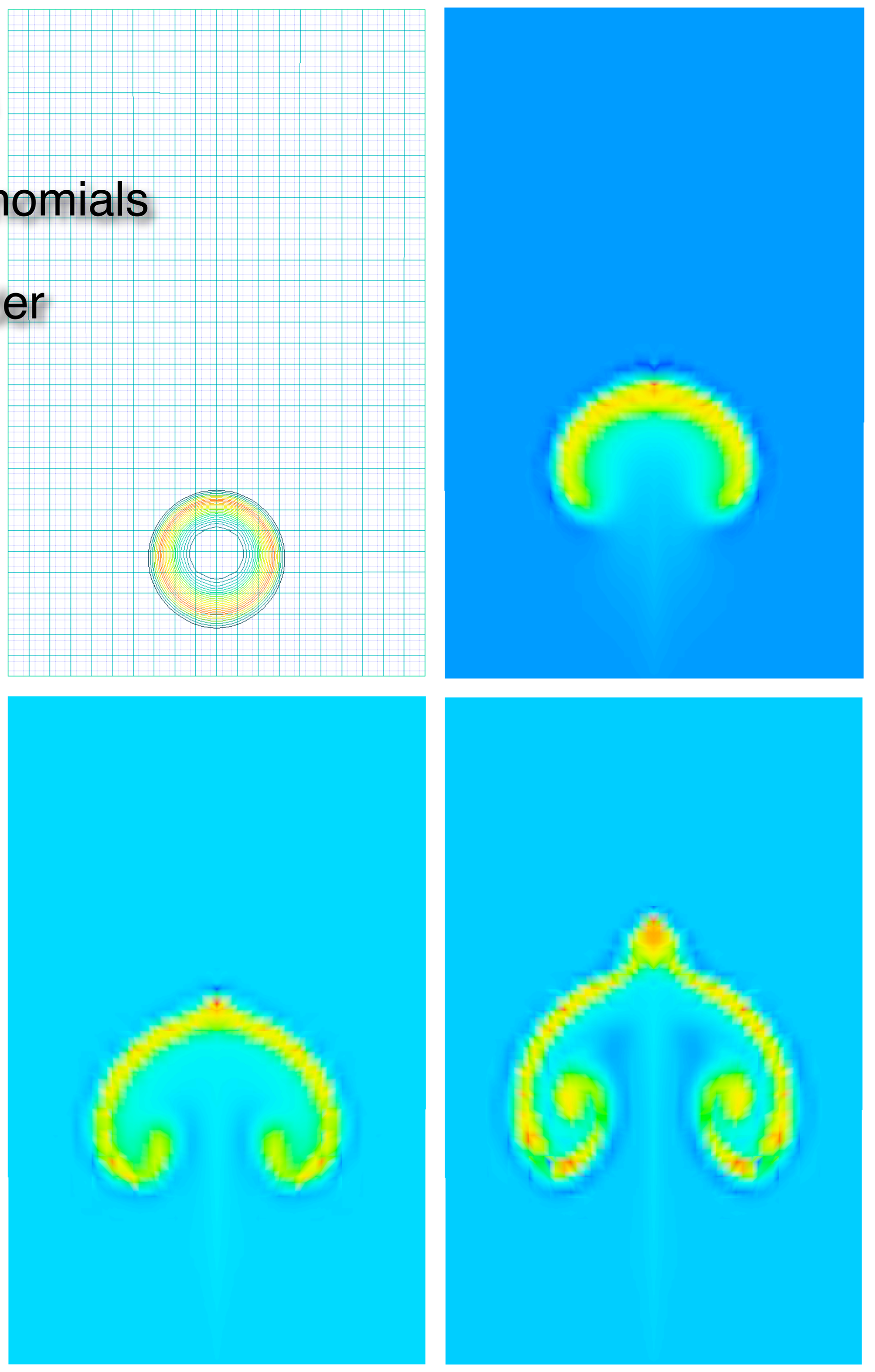
Numerical experiment:

A warm bubble is introduced in a dry isentropic atmosphere by perturbing an hydrostatically balanced state. The perturbation is set to 0.5 Celsius uniformly inside a radius of 50 meters and decays from this circle following a smooth Gaussian profile. Solid, reflective boundaries are imposed on all walls. This is a difficult test since the entrainment is a function of the numerical viscosity built into the scheme. The DG method is diffusive at element boundaries and solid walls. Initially, the bubble is 'momentum-dominated' and becomes 'buoyancy-dominated'.

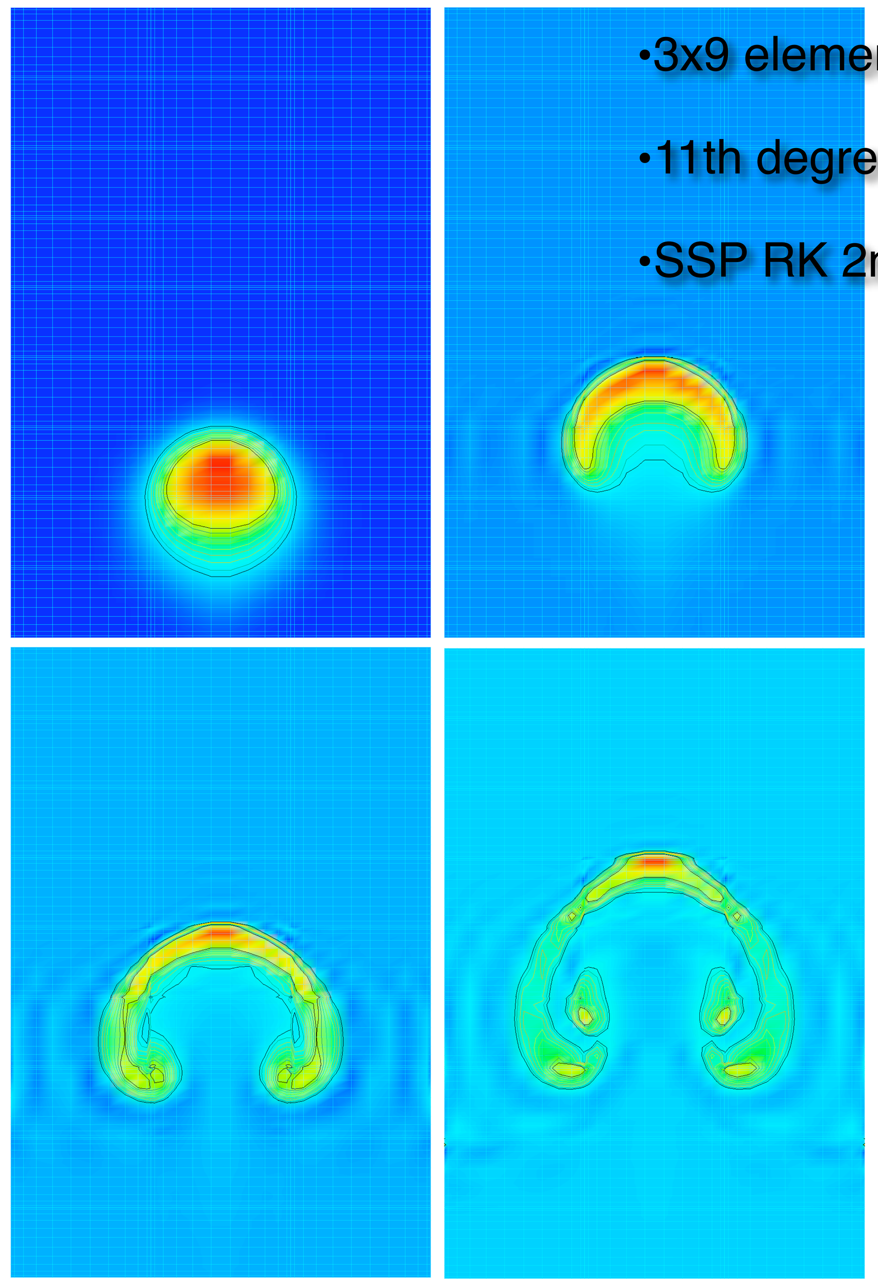
Conclusions:

Initial results of a high-order DG model for the compressible Euler equations were shown. With comparable number of unknowns, a high polynomial degree within each element seems to give beter results. No filtering is employed during the simulations of a warm bubble experiment suggested by Robert (1993). From the experiments, it seems that a less diffusive numerical flux is required. To be efficient, a better time-stepping strategy is required: this will be the subject of future work.

- 20x30 elements
- 3rd degree polynomials
- SSP RK 2nd order



- 3x9 elements
- 11th degree polynomials
- SSP RK 2nd order



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