Emerging Methods For Conservation Laws

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Conservation laws are systems of nonlinear partial differential equations (PDEs) on conservation (flux) form and can be written:

$$\frac{\partial}{\partial t} U(x, t) + \sum_{j=1}^{3} \frac{\partial}{\partial x_j} F_j(U, x, t) = S(U),$$

where

- $U(x, t)$ is a vector function in 3D space coordinate $x$ and time $t > 0$.
- $F_j$ are given flux vectors dependent on $(U, x, t)$ and include diffusive and convective effects.
- $S(U)$ is the source term.

E.g: **Navier-Stokes equations** for compressible and incompressible flows can be written in this form with $U$ representing mass, momentum and energy, $S(U)$ representing exterior forces.

A large class of atmospheric equations of motion can be cast in this form.

Scalar conservation law (e.g., mass continuity equation):

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0; \quad \rho_t + \text{div}(\rho \mathbf{V}) = 0$$
Finite-Volume (FV) methods are traditionally used for solving conservation laws.

Possible candidates for atmospheric modeling applications (high-order accurate) include:

- Weighed Essentially Non-Oscillatory (WENO) [Shu, 1997]
- Discontinuous Galerkin Method (DGM) [Cockburn, 2000]
- Spectral Finite Volume Methods (SFV) [Wang, JCP, 2003]

We are interested in "Local and Compact" high-order method such as DGM.

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1. Local and global conservation
2. High-order accuracy
3. High parallel efficiency
4. Geometric flexibility ("Any" type of grid)
5. Promise for monotonic (non-oscillatory) transport
Discontinuous Galerkin Method (DGM) in 1D

1D scalar conservation law:

\[
\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0 \quad \text{in} \quad \Omega \times (0, T),
\]

\[
U_0(x) = U(x, t = 0), \quad \forall x \in \Omega
\]

E.g., \( F(U) = c U \) (Linear advection), \( F(U) = U^2/2 \) (Burgers’ Equation)

The domain \( \Omega \) (periodic) is partitioned into \( N_x \) non-overlapping elements (intervals) \( I_j = [x_{j-1/2}, x_{j+1/2}], \ j = 1, \ldots, N_x \), and \( \Delta x_j = (x_{j+1/2} - x_{j-1/2}) \)
A weak formulation of the problem for the approximate solution $U_h$ is obtained by multiplying the PDE by a test function $\varphi_h(x)$ and integrating over an element $I_j$:

$$
\int_{I_j} \left[ \frac{\partial U_h}{\partial t} + \frac{\partial F(U_h)}{\partial x} \right] \varphi_h(x) dx = 0, \quad U_h, \varphi_h \in \mathcal{V}_h
$$

Integrating the second term by parts $\implies$

$$
\int_{I_j} \frac{\partial U_h(x, t)}{\partial t} \varphi_h(x) dx - \int_{I_j} F(U_h(x, t)) \frac{\partial \varphi_h}{\partial x} dx + 
F(U_h(x_{j+1/2}, t)) \varphi_h(x_{j+1/2}^-) - F(U_h(x_{j-1/2}, t)) \varphi_h(x_{j-1/2}^+) = 0,
$$

where $\varphi(x^-)$ and $\varphi(x^+)$ denote "left" and "right" limits.
DGM-1D: Flux term ("Gluing" the discontinuous element edges)

Flux function $F(U_h)$ is **discontinuous** at the interfaces $x_{j\pm 1/2}$.

$F(U_h)$ is replaced by a **numerical flux** function $\hat{F}(U_h)$, dependent on the left and right limits of the discontinuous function $U$. At the interface $x_{j+1/2}$,

$$\hat{F}(U_h)_{j+1/2}(t) = \hat{F}(U_h(x_{j+1/2}^-, t), U_h(x_{j+1/2}^+, t)).$$

**Typical flux formulae** (*Approx. Reimann Solvers*): Gudunov, Lax-Friedrichs, Roe, HLLC, etc.

**Lax-Friedrichs numerical flux formula:-**

$$\hat{F}(U_h) = \frac{1}{2} \left[ (F(U_h^-) + F(U_h^+)) - \alpha(U_h^+ - U_h^-) \right].$$
Map every element $I_j$ onto the reference element $[-1, +1]$ by introducing a local coordinate $\xi \in [-1, +1]$ s.t.,

$$\xi = \frac{2(x - x_j)}{\Delta x_j}, \quad x_j = \frac{(x_{j-1/2} + x_{j+1/2})}{2} \Rightarrow \frac{\partial}{\partial x} = \frac{2}{\Delta x_j} \frac{\partial}{\partial \xi}.$$ 

Use a high-order Gaussian quadrature such as the Gauss-Legendre (GL) or Gauss-Lobatto-Legendre (GLL) quadrature rule. The GLL quadrature is ‘exact’ for polynomials of degree up to $2N - 1$.

$$\int_{-1}^{1} f(\xi) d\xi \approx \sum_{n=0}^{N} w_n f(\xi_n); \quad \text{for GLL}, \quad \xi_n \Leftarrow (1 - \xi^2)P'_\ell(\xi) = 0$$
The **model** basis set for the $\mathcal{P}^k$ DG method consists of Legendre polynomials, $\mathcal{B} = \{P_\ell(\xi), \ell = 0, 1, \ldots, k\}$.

Test function $\varphi_h(x)$ and approximate solution $U_h(x)$ belong to $\mathcal{B}$

$$U_h(\xi, t) = \sum_{\ell=0}^k U_\ell^h(t) P_\ell(\xi) \quad \text{for} \quad -1 \leq \xi \leq 1,$$

where

$$U_\ell^h(t) = \frac{2\ell + 1}{2} \int_{-1}^1 U_h(\xi, t) P_\ell(\xi) \, d\xi \quad \ell = 0, 1, \ldots, k.$$

$$\int_{-1}^1 P_m(x)P_n(x)\,dx = \frac{2}{2m+1} \delta_{m,n} \quad \Leftarrow \text{Orthogonality}$$

$U_\ell^h(t)$ is the **degrees of freedom (dof)** evolves w.r.t time.
DGM-1D: Modal Basis Set for a “$P^2$” Method

- For the $P^2$ method, $B = \{P_0, P_1, P_2\} = \{1, \xi, (3\xi^2 - 1)/2\}$.
- Approximate solution:

$$U_h(\xi, t) = U_h^0(t) + U_h^1(t)\xi + U_h^2(t)[3\xi^2 - 1]$$

- The degrees of freedom to evolve in $t$ are:

$$U_h^0(t) = \frac{1}{2} \int_{-1}^{1} U_h(\xi, t) d\xi \Leftarrow \text{Average}$$

$$U_h^1(t) = \frac{3}{2} \int_{-1}^{1} U_h(\xi, t) \xi d\xi$$

$$U_h^2(t) = \frac{5}{2} \int_{-1}^{1} U_h(\xi, t) [3\xi^2 - 1] d\xi$$
The nodal basis set $\mathcal{B}$ is constructed using Lagrange-Legendre polynomials $h_i(\xi)$ with roots at Gauss-Lobatto quadrature points (physical space).

$$U_j(\xi) = \sum_{j=0}^{k} U_j h_j(\xi) \quad \text{for} \quad -1 \leq \xi \leq 1,$$

$$h_j(\xi) = \frac{(\xi^2 - 1) P_k'(\xi)}{k(k + 1) P_k(\xi_j)(\xi - \xi_j)}, \quad \int_{-1}^{1} h_i(\xi) h_j(\xi) = w_i \delta_{ij}.$$

Nodal version was shown to be more computationally efficient than the Modal version (see, Levy, Nair & Tufo, Comput. & Geos. 2007)

Modal version is more “friendly” with monotonic limiting.
Finally, the weak formulation leads the PDE to the time dependent ODE

\[
\int_{I_j} \left[ \frac{\partial U_h}{\partial t} + \frac{\partial F(U_h)}{\partial x} \right] \varphi_h(x) dx = 0 \Rightarrow \frac{d}{dt} U_h^\ell(t) = \mathcal{L}(U_h) \quad \text{in} \ (0, T) \times \Omega
\]

Example: For the \( P^1 \) case on an element \( I_j \), we need to solve:

\[
\frac{d}{dt} U_{h}^{0}(t) = \frac{-1}{\Delta x_j} \left[ F(\xi = 1, t) - F(\xi = -1, t) \right]
\]

\[
\frac{d}{dt} U_{h}^{1}(t) = \frac{-3}{\Delta x_j} \left( [F(\xi = 1, t) + F(\xi = -1, t)] - \int_{-1}^{1} U_h(\xi, t) d\xi \right)
\]

Solve the ODEs for the modes at new time level \( U_h^\ell(t + \Delta t) \) For the \( P^1 \) case,

\[
U_h(\xi, t + \Delta t) = U_h^{0}(t + \Delta t) + U_h(\xi, t + \Delta t) \xi
\]
For the ODE of the form,

$$\frac{d}{dt} U(t) = \mathcal{L}(U) \quad \text{in } (0, T) \times \Omega$$

Strong Stability Preserving third-order Runge-Kutta (SSP-RK) scheme (Gottlieb et al., SIAM Review, 2001)

$$U^{(1)} = U^n + \Delta t \mathcal{L}(U^n)$$

$$U^{(2)} = \frac{3}{4} U^n + \frac{1}{4} U^{(1)} + \frac{1}{4} \Delta t \mathcal{L}(U^{(1)})$$

$$U^{n+1} = \frac{1}{3} U^n + \frac{2}{3} U^{(2)} + \frac{2}{3} \Delta t \mathcal{L}(U^{(2)}).$$

where the superscripts $n$ and $n+1$ denote time levels $t$ and $t + \Delta t$, respectively.

The Courant number for the DG scheme is estimated to be $1/(2k + 1)$, where $k$ is the degree of the polynomial (Cockburn and Shu, 1989).

For the linear case, CFL limit is $1/3$. 
DG-2D Spatial Discretization for an Element $\Omega$

### 2D Scalar conservation law

\[
\frac{\partial U}{\partial t} + \nabla \cdot F(U) = S(U), \quad \text{in} \quad \Omega \times (0, T); \quad \forall (x^1, x^2) \in \Omega
\]

where $U = U(x^1, x^2, t)$, $\nabla \equiv (\partial/\partial x^1, \partial/\partial x^2)$, $F = (F, G)$ is the flux function, and $S$ is the source term.

- **Weak Galerkin formulation**: Multiplication of the basic equation by a test function $\varphi_h \in \mathcal{V}_h$ and integration over an element $\Omega$.

\[
\frac{\partial}{\partial t} \int_\Omega U_h \varphi_h \, d\Omega - \int_\Omega F(U_h) \cdot \nabla \varphi_h \, d\Omega + \int_\Gamma F(U_h) \cdot \vec{n} \varphi_h \, d\Gamma = \int_\Omega S(U_h) \varphi_h \, d\Omega
\]

where $U_h$ is an approximate solution in $\mathcal{V}_h$.

- Can be extended to a system of equations
Along the boundaries ($\Gamma$) of an element the solution $U_h$ is discontinuous ($U_h^-$ and $U_h^+$ are the left and right limits).

Therefore, the analytic flux $\mathbf{F}(U_h) \cdot \vec{n}$ must be replaced by a numerical flux such as the Lax-Friedrichs Flux:

$$\mathbf{F}(U_h) \cdot \vec{n} = \frac{1}{2} \left[ (\mathbf{F}(U_h^-) + \mathbf{F}(U_h^+)) \cdot \vec{n} - \alpha(U_h^+ - U_h^-) \right].$$

For the SW system, $\alpha$ is the upper bound on the absolute value of eigenvalues of the flux Jacobian $\mathbf{F}'(U)$; (Nair et al., 2005)

$$\alpha^1 = \max \left(|u^1| + \sqrt{\Phi G^{11}}\right), \quad \alpha^2 = \max \left(|u^2| + \sqrt{\Phi G^{22}}\right)$$
DGM is a hybrid approach (\( DG \leftrightarrow SE + FV \))

- The domain \( D \) is partitioned into non-overlapping elements \( \Omega_{ij} \) such that the element boundaries are discontinuous.
- Based on conservation laws but exploits the spectral expansion of SE method and treats the element boundaries using FV “tricks.”
High-order nodal basis set

The **nodal basis set** is constructed using a tensor-product of Lagrange-Legendre polynomials \( h_i(\xi) \) with roots at **Gauss-Lobatto** quadrature points.

\[
h_i(\xi) = \frac{(\xi^2 - 1) P'_N(\xi)}{N(N + 1) P_N(\xi_i)(\xi - \xi_i)}; \quad \int_{-1}^{1} h_i(\xi) h_j(\xi) \, d\xi = w_i \delta_{ij}.
\]

where \( P_N(\xi) \) is the \( N^{th} \) order Legendre polynomial, and \( w_i \) weights associated with the Gauss quadrature.

- The approximate solution \( (U_h) \) and test function \( (\varphi_h) \) are represented in terms of nodal basis set.

\[
U_{ij}(\xi, \eta) = \sum_{i=0}^{N} \sum_{j=0}^{N} U_{ij} h_i(\xi) h_j(\eta) \quad \text{for} \quad -1 \leq \xi, \eta \leq 1,
\]

- The nodal version was shown to be more computationally efficient than the modal in *(Dennis et al., 2006)*.
Final form for the nodal discretization leads to the ODE:
\[
\frac{d}{dt} U_{ij}(t) = \frac{4}{\Delta x_i \Delta x_j} w_i w_j \left[ I_{\text{Grad}} + I_{\text{Flux}} + I_{\text{Source}} \right],
\]

Evaluate the integrals (RHS) using GLL quadrature rule.

“Mass matrix” is diagonal (i.e., decoupled system of ODEs to solve)

For a system of conservation laws, solve the ODE system:
\[
\frac{d}{dt} \mathbf{U} = L(\mathbf{U}) \quad \text{in} \quad (0, T) \times \Omega
\]

Time integration: Explicit third-order Runge-Kutta (SSP) scheme (Gottlieb et al., 2001)
Scaling Plots

**Strong scaling** is measured by increasing the number of processes running while keeping the problem size constant.

**Weak scaling** is measured by scaling the problem along with the number of processes, so that work per process is constant.
Monotonic Limiter: Piecewise Linear Method (PLM) \([\text{van Leer (1979)}]\)

- **Godunov theorem (1959):** “Linear numerical schemes for solving PDE’s, having the property of not generating new extrema (monotone scheme), can be at most first-order accurate.”

- We consider monotonic limiter introduced by van Leer (1979) in MUSCL scheme (Monotone Upwind Schemes for Conservation Laws).

For the PLM, density distribution in a cell \(I_j = [x_{j-1/2}, x_{j+1/2}]\), with slope \(U_{x,j}\):

\[
U(x)_j = \bar{U}_j + (x - x_j)U_{x,j}, \quad \bar{U}_j = \frac{1}{\Delta x_j} \int_{x_{j-1/2}}^{x_{j+1/2}} U_j(x) dx,
\]
A \textbf{minmod} limiter essentially chooses the min of absolute value of the ‘left’ and ‘right’ slopes if both preserve the same sign, but sets to zero slope if the signs are opposite.

\[
U(x)_j = \overline{U}_j + (x - x_j) \tilde{U}_{x,j}, \quad \tilde{U}_{x,j} \leftarrow \text{minmod}(U_{x,j}, U_{x,j-1/2}, U_{x,j+1/2})
\]

\[
\text{minmod}(a, b, c) = \begin{cases} 
  s \min(|a|, |b|, |c|) & \text{if } s = \text{sign}(a) = \text{sign}(b) = \text{sign}(c); \\
  0 & \text{otherwise}
\end{cases}
\]

\[
U_{x,j-1/2} = \frac{\overline{U}_j - \overline{U}_{j-1}}{(\Delta x_j + \Delta x_{j-1})/2}, \quad U_{x,j+1/2} = \frac{\overline{U}_{j+1} - \overline{U}_j}{(\Delta x_{j+1} + \Delta x_j)/2}
\]
For DG method (modal version) minmod limiter can be applied in the case of MUSCL scheme. High-order modes ($U_{h}^{\ell}(t)$, $\ell > 1$) are set zero if the minmod limiter changes the slope ($U_{h}^{1}(t)$).

$$U_{h}(\xi, t) = \sum_{\ell=0}^{k} U_{h}^{\ell}(t) P_{\ell}(\xi) = U_{h}^{0}(t) + U_{h}^{1}(t) \xi + \sum_{\ell=2}^{k} U_{h}^{\ell}(t) P_{\ell}(\xi)$$

High-order ‘moment’ limiting (Biswa et al. 1994) is effective but expensive!
The minmod limiter can be applied in $x$ and $y$-direction sequentially, however it is very diffusive.

$$U_h(x, y, t) = \overline{U}_h(t) + U_x(t)\xi + U_y(t)\eta$$

For high-order DG method, a tensor-product of 1-D limiter such as the moment limiter (Krivonodova, 2008) or WENO limiter (Qui & Shu, 2005) may be applied. But in general, they do not preserve positivity.

A new limiter developed for DGM transport problems, selectively applies slope limiting employs a $3 \times 3$ element stencil and positivity as a constraint.
The minmod limiter preserves positivity, but too diffusive. Selective application of the slope limiter is desirable (preserves high-order nature of the solution).

DG 2-D solution with minmod limiter (left) and constrained slope limiter (right)
Solid-Body rotation of a cosine-cone and a square block after one revolution without limiting.
DG2D: Monotonic limiting (with positivity preservation)

Solid-Body rotation of a cosine-cone and a square block after one revolution with constrained limiting.
Flux-form SW equations (Vector invariant form)

\[ \begin{align*}
\frac{\partial u_1}{\partial t} + \frac{\partial}{\partial x^1} E &= \sqrt{G} u^2 (f + \zeta) \\
\frac{\partial u_2}{\partial t} + \frac{\partial}{\partial x^2} E &= -\sqrt{G} u^1 (f + \zeta) \\
\frac{\partial}{\partial t}(\sqrt{G} h) + \frac{\partial}{\partial x^1}(\sqrt{G} u^1 h) + \frac{\partial}{\partial x^2}(\sqrt{G} u^2 h) &= 0
\end{align*} \]

where \( G = \det(G_{ij}) \), \( h \) is the height, \( f \) Coriolis term; energy term and vorticity are defined as

\[ E = \Phi + \frac{1}{2} (u_1 u^1 + u_2 u^2), \quad \zeta = \frac{1}{\sqrt{G}} \left[ \frac{\partial u_2}{\partial x^1} - \frac{\partial u_1}{\partial x^2} \right] . \]
Flux is the only “communicator” at the element edges

- Each face of the cubed-sphere is partitioned into $N_e \times N_e$ rectangular non-overlapping elements (i.e., total $6 \times N_e^2$).
- Each element is mapped onto the Gauss-Lobatto-Legendre (GLL) grid defined by $-1 \leq \xi, \eta \leq 1$, for integration.
Nodal version of DGM is computationally more efficient (Dennis et al. 2006) as compared to the Modal version.

Cosine-Bell Movie
Cosine-Bell with monotonic limiter
Convergence Results - Advection (Cubed Sphere) [Levy, Nair & Tufo, 2007]

- **h-error**: Measured by leaving the number of nodes per element constant but increasing the number of elements.
- **p-error**: Measured by leaving the number of elements constant but increasing the number of nodes per element.
**Deformational Flow Test:** *Nair & Jablonowski (MWR, 2008)*

- The vortices are located at diametrically opposite sides of the sphere, the vortices deform as they move along a prescribed trajectory.

- Analytical solution is known and the trajectory is chosen to be a great circle along the NE direction ($\alpha = \pi/4$).
High-order accuracy and spectral convergence

Steady state geostrophic flow ($\alpha = \pi/4$). Max height error is $O(10^{-6})$ m.
No “spectral ringing” for the height fields

Flow over a mountain ($\approx 0.5^\circ$). Initial height field (left) initial and after 15 days of integration (right)

SW5 Movie
HOMME (High-Order Method Modeling Environment)

- The **Discontinuous Galerkin (DG)** model is a conservative option in the HOMME framework
- HOMME Grid: The sphere is decomposed into 6 identical regions, using the equiangular projection (Sadourny, 1972)
  - Local coordinate systems are free of singularities
  - Creates a non-orthogonal curvilinear coordinate system

**Cubed Sphere Geometry:** Logical cube-face orientation

![Cubed Sphere Diagram](image-url)
Metric Tensor $G_{ij}$, [Cubed-Sphere ⇐ Sphere] Transform

Central angles $x^1, x^2 \in [-\pi/4, \pi/4]$ are the independent variables.

$$G_{ij} = \frac{R^2}{\rho^4 \cos^2 x^1 \cos^2 x^2} \begin{bmatrix} 1 + \tan^2 x^1 & -\tan x^1 \tan x^2 \\ -\tan x^1 \tan x^2 & 1 + \tan^2 x^2 \end{bmatrix}$$

where $\rho^2 = 1 + \tan^2 x^1 + \tan^2 x^2$, $i, j \in \{1, 2\}$

- Metric tensor in terms of longitude-latitude $(\lambda, \theta)$:
  $$G_{ij} = A^T A; \quad A = \begin{bmatrix} R \cos \theta \partial \lambda / \partial x^1 & R \cos \theta \partial \lambda / \partial x^2 \\ R \partial \theta / \partial x^1 & R \partial \theta / \partial x^2 \end{bmatrix}$$

- The matrix $A$ is used for transforming spherical velocity $(u, v)$ to the covariant $(u_1, u_2)$ and contravariant $(u^1, u^2)$ vectors.
\[ \frac{\partial u_1}{\partial t} + \nabla_c \cdot E_1 + \dot{\eta} \frac{\partial u_1}{\partial \eta} = \sqrt{G} u^2 (f + \zeta) - R T \frac{\partial}{\partial x^1} (\ln p) \]

\[ \frac{\partial u_2}{\partial t} + \nabla_c \cdot E_2 + \dot{\eta} \frac{\partial u_2}{\partial \eta} = -\sqrt{G} u^1 (f + \zeta) - R T \frac{\partial}{\partial x^2} (\ln p) \]

\[ \frac{\partial}{\partial t} (m) + \nabla_c \cdot (U^i m) + \frac{\partial (m\dot{\eta})}{\partial \eta} = 0 \]

\[ \frac{\partial}{\partial t} (m\Theta) + \nabla_c \cdot (U^i \Theta m) + \frac{\partial (m\dot{\eta} \Theta)}{\partial \eta} = 0 \]

\[ \frac{\partial}{\partial t} (mq) + \nabla_c \cdot (U^i q m) + \frac{\partial (m\dot{\eta} q)}{\partial \eta} = 0 \]

\( m \equiv \sqrt{G} \frac{\partial p}{\partial \eta}, \nabla_c \equiv \left( \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2} \right), \eta = \eta(p, p_s), G = \det(G_{ij}), \frac{\partial \Phi}{\partial \eta} = -\frac{R T}{p} \frac{\partial p}{\partial \eta}. \)

Where \( m \) is the mass function, \( \Theta \) is the potential temperature and \( q \) is the moisture variable. \( U^i = (u^1, u^2), E_1 = (E, 0), E_2 = (0, E); E = \Phi + \frac{1}{2} (u^1 u^1 + u^2 u^2) \) is the energy term. \( \Phi \) is the geopotential, \( \zeta \) is the relative vorticity, and \( f \) is the Coriolis term.
A “vanishing trick” for vertical advection terms!

- Terrain-following Eulerian surfaces are treated as material surfaces.
- The resulting Lagrangian surfaces are free to move up or down direction.
Vertically moving Lagrangian Surfaces

- Over time, Lagrangian surfaces deform and thus must be remapped.
- The velocity fields \((u_1, u_2)\), and total energy \((\Gamma_E)\) are remapped onto the reference coordinates using the 1-D conservative cell-integrated semi-Lagrangian (CISL) method (Nair & Machenhauer, 2002).

\[ \Delta P = \text{Pressure thickness} \]

\[ \text{Lagrangian Surface} \]

Terrain-following Lagrangian control-volume coordinates

Remapping: Lauritzen & Nair, MWR, 2008; Norman & Nair, MWR, 2008)
To assess the evolution of an idealized baroclinic wave in the Northern Hemisphere.

The initial conditions are quasi-realistic and defined by analytic expressions. Analytic solutions do not exist.
Baroclinic waves are triggered by perturbing the velocity field at (20°E, 40°N)
This test case recommends up to 30 days of model simulation
Ne = Nv = 8 (approx. 1.6°) with 26 vertical levels and Δt = 30 Sec.
The HOMME-DG dynamical core successfully simulates baroclinic instability.

Simulated temperature (K) and surface pressure (hPa) at day 8 for a baroclinic instability test with the HOMME-DG model and the NCAR global spectral model (right). The horizontal resolution is approximately 1.4°.

Note that the DG solution is free of “spectral ringing”.

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Emerging Methods For Conservation Laws
Simulated surface pressure at day 11 for a baroclinic instability test with DG model, and NCAR global spectral model and a FV model. The models use 26 vertical levels and with approximate horizontal resolution of $0.7^\circ$. 

![Diagram showing surface pressure](image-url)
- **DG-3D parallel performance**: Sustained Mflops on IBM BG/L (1024 DP nodes, 700 MHz PPC 440s): Approx. 9% peak

- **Held-Suarez (preliminary) test**: 800 days idealized climate simulation ($1^\circ$ resolution, 26 vertical levels, $\Delta t = 10$ Sec)

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**Parallel performance (strong scaling) results for JW-Test**

**Held-Suarez test (800 days)**
The DG methods (third or fourth-order) is a good choice for solving conservation laws as applied in atmospheric sciences (local conservation and monotonic transport).

The preliminary idealized test results and parallel scaling results are impressive and comparable to the SE version in HOMME.

The explicit R-K time integration scheme is robust for the DG-3D model, but very time-step restrictive.

More efficient time integration schemes are required for practical climate simulations. Possible approaches: Semi-implicit, IMEX-RK, Rosenbrock with optimized Schwarz, etc.