Supplementary material to the paper *"Nonstationary covariance modeling for incomplete data : Monte Carlo EM approach"*

Tomoko Matsuo, Doug Nychka and Debashis Paul

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1 Outline

In this supplementary material we give theoretical justifications for the method described in that paper that relies on a wavelet transform of the spatial process and decimation of the resulting wavelet coefficients. In Sections 2-4, we deal with the simpler case of one dimensional stationary processes with the Matérn class covariance function. The basic notations involved with a wavelet decomposition of such processes are given in Section 2. In Section 5 we illustrate how the results described in Section 4 regarding the Matérn class covariance can be extended when the wavelet has a certain number of vanishing moments and a few other properties that generalize well beyond the Meyer wavelets and the Matérn family. In Section 6 we extend the decay properties described in Section 4 to finite mixture of Matérn class covariances. In Section 7 we describe the behavior of the covariances of the wavelet coefficients when the processes are on $\mathbb{R}^2$ instead of on $\mathbb{R}$. Finally, in Section 8 we describe the decay characteristics of the covariances of the wavelet coefficients when the observed process can be expresses as a spatially varying nonnegative, smooth, scale parameter times a stationary Matérn process.

2 Setup

Suppose $X(t)$ is a stationary process on $\mathbb{R}$ with covariance kernel $K(s,t) = K(s-t), s,t \in \mathbb{R}$. Suppose $\{\psi_{jk} : j \in \mathbb{Z}, k \in \mathbb{Z}\}$ denote an orthonormal wavelet basis for $L^2(\mathbb{R})$, where ψ is the mother wavelet and let ϕ be the corresponding scaling function (father wavelet). Let

$$a_{jk} = \langle \phi_{jk}, X \rangle = \int X(t)\phi_{jk}(t)dt, \qquad b_{jk} = \langle \psi_{jk}, X \rangle = \int X(t)\psi_{jk}(t)dt \, . \tag{1}$$

We are interested in the behaviour of the correlation coefficients $\text{cov}(a_{jk}, a_{j'k'})$, $\text{cov}(a_{jk}, b_{j'k'})$ and $\text{cov}(b_{jk}, b_{j'k'})$ for arbitrary indices j, k, j', k'.

3 Analysis with Meyer wavelets

The analysis we shall carry out will be easier if we choose a special class of wavelets called Meyer wavelets, named after its originator Yves Meyer. For detailed discussion about its properties one is referred to Meyer (1992), Mallat (1998) and Daubechies (1992). We list below the properties of Meyer wavelet that will be most useful for us. Throughout $\widehat{f}$ will denote the Fourier transform of f for any function $f \in L^2(\mathbb{R})$.

(i) We can express $\widehat{\psi}(\xi)$ as $\widehat{\psi}(\xi) = e^{-i\xi/2}\theta(\xi)$ where $\theta(\cdot)$ is symmetric about zero.

(ii) $\theta(\cdot)$ is real valued (in fact ≥ 0) and can be chosen to be infinitely differentiable on $\mathbb{R}$.

(iii) $\theta(\cdot)$ is supported on $[-\frac{8\pi}{3}, -\frac{2\pi}{3}] \cup [\frac{2\pi}{3}, \frac{8\pi}{3}]$.

(iv) $\widehat{\phi}(\cdot)$ is a, nonnegative, even function, and is supported on $[-\frac{4\pi}{3}, \frac{4\pi}{3}]$.

(v) $\widehat{\phi}(\cdot)$ can be chosen to be $\mathbf{C}^\infty$; and $\widehat{\phi}(0) = 1$, $\widehat{\psi}^{(\ell)}(0) = 0$ for all $\ell \geq 0$.

The fact that the wavelet is compactly supported in the frequency domain and the recursive relation in defining ψ_{jk} ($\psi_{jk}(x) = 2^{j/2}\psi(2^j x - k)$) imply that

$$\widehat{\psi}_{jk}(\xi) = 2^{-j/2} e^{i2^{-j}k\xi}\widehat{\psi}(2^{-j}\xi) \tag{2}$$

is supported on $[-2^j\frac{8\pi}{3}, -2^j\frac{2\pi}{3}] \cup [2^j\frac{2\pi}{3}, 2^j\frac{8\pi}{3}]$ for all j. In particular, this means that the support of ψ_{jk} and $\psi_{j'k'}$ are disjoint whenever $|j - j'| \geq 2$.

4 Decay of covariances

We first study the covariances $\mathrm{cov}(b_{jk}, b_{j'k'})$. By elementary computations,

$$B_{jk,j'k'} := \mathrm{cov}(b_{jk}, b_{j'k'}) = \int_{\mathbb{R}}\int_{\mathbb{R}} \psi_{jk}(s)K(s-t)\psi_{j'k'}(t)dsdt\ . \tag{3}$$

Recalling that if $f * g$ denotes the convolution of f and g, functions in L^2, then $\widehat{f * g} = \widehat{f}\widehat{g}$, and further, by *Perseval relation*, $\int_{\mathbb{R}} f(t)g(t)dt = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{f}(\xi)\overline{\widehat{g}}(\xi)d\xi$, we get from (3) and (2) that

$$\begin{aligned}
B_{jk,j'k'} &= \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{K}(\omega)\widehat{\psi}_{jk}(\omega)\overline{\widehat{\psi}_{j'k'}}(\omega)d\omega \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{K}(\omega)2^{-(j+j')/2}e^{i(2^{-j}k-2^{-j'}k')\omega}\widehat{\psi}(2^{-j}\omega)\overline{\widehat{\psi}}(2^{-j'}\omega)d\omega \\
&= 2^{(j-j')/2}\frac{1}{2\pi}\int_{\mathbb{R}} \widehat{K}(2^j\xi)\widehat{\psi}(\xi)\overline{\widehat{\psi}}(2^{j-j'}\xi)e^{i(k-2^{j-j'}k')\xi}d\xi, \qquad (\text{setting } \xi = 2^{-j}\omega)
\end{aligned} \tag{4}$$

Equation (4) allows us to analyze the decay of $B_{jk,j'k'}$ for various covariance kernels. First important observation, however, is that $B_{jk,j'k'} = 0$ if $|j - j'| \geq 2$ which directly follows from the remark about the supports of the functions $\widehat{\psi}_{jk}$ and $\widehat{\psi}_{j'k'}$ made in the previous section. Also, from now onwards we shall without loss of generality assume $j' \geq j$ and denote the quantity $k - 2^{j-j'}k'$ by $r = r(j,k,j',k')$.

Also, define $A_{jk,jk'} = \mathrm{cov}(a_{jk}, a_{jk'})$ (observe that the coefficients a_{jk} correspond to a single scale j. And let $C_{jk,j'k'} = \mathrm{cov}(a_{jk}, b_{j'k'})$.

4.1 Matérn class covariance

Let us concentrate now on the Matérn class covariance (cf. Stein (1999)). Fourier transform of the kernel $K_{\alpha,\nu,\beta}$ for parameters $\alpha > 0, \nu \geq 1/2, \beta > 0$ is given by

$$\widehat{K}_{\alpha,\nu,\beta}(\xi) = \beta(\alpha^2 + \xi^2)^{-\nu-1/2}, \quad \xi \in \mathbb{R} \,. \tag{5}$$

Since β is just a scaling constant we shall henceforth assume that it is 1 and drop from the subscript. Substituting this expression in (4) we get

$$\begin{aligned} B_{jk,j'k'} &= 2^{(j-j')/2}\frac{1}{2\pi}\int_{\mathbb{R}} \widehat{K}_{\alpha,\nu}(2^j\xi)\widehat{\psi}(\xi)\overline{\widehat{\psi}}(2^{j-j'}\xi)e^{ir\xi}d\xi \\ &= 2^{(j-j')/2}\frac{1}{2\pi}2^{-j(2\nu+1)}\int_{\mathbb{R}} \widehat{K}_{\alpha 2^{-j},\nu}(\xi)\widehat{\psi}(\xi)\overline{\widehat{\psi}}(2^{j-j'}\xi)e^{ir\xi}d\xi \,. \end{aligned} \tag{6}$$

Observe that, for Meyer wavelet, $\widehat{\psi}(\xi)\overline{\widehat{\psi}}(2^{j-j'}\xi) = e^{i\xi(2^{j-j'}-1)/2}\theta(\xi)\theta(2^{j-j'}\xi)$. This implies that

$$B_{jk,j'k'} = 2^{-j(2\nu+1)-(j'-j)/2}\frac{1}{2\pi}\int_{\mathbb{R}} \widehat{K}_{\alpha 2^{-j},\nu}(\xi)\theta(\xi)\theta(2^{j-j'}\xi)e^{i\overline{r}\xi}d\xi, \tag{7}$$

where $\overline{r} = \overline{r}(j,j',k,k') = (k-1/2) - 2^{j-j'}(k'-1/2)$. Since $\widehat{K}_{\alpha 2^{-j},\nu}$ and θ are both even functions, (7) simplifies to

$$B_{jk,j'k'} = 2^{-j(2\nu+1)-(j'-j)/2}\frac{1}{\pi}\int_0^\infty \widehat{K}_{\alpha 2^{-j},\nu}(\xi)\theta(\xi)\theta(2^{j-j'}\xi)\cos(\overline{r}\xi)d\xi. \tag{8}$$

Moreover, from property (iv) of Meyer wavelets we have the following expressions for $A_{jk,jk'}$ and $C_{jk,j'k'}$.

$$A_{jk,jk'} = 2^{-j(2\nu+1)}\frac{1}{\pi}\int_0^\infty \widehat{K}_{\alpha 2^{-j},\nu}(\xi)|\widehat{\phi}(\xi)|^2\cos((k-k')\xi)d\xi, \tag{9}$$

$$C_{jk,j'k'} = 2^{-j(2\nu+1)-(j'-j)/2}\frac{1}{\pi}\int_0^\infty \widehat{K}_{\alpha 2^{-j},\nu}(\xi)\widehat{\phi}(\xi)\theta(2^{j-j'}\xi)\cos(\widetilde{r}\xi)d\xi, \tag{10}$$

where $\widetilde{r} = \widetilde{r}(j,k,j',k') = k - 2^{j-j'}(k-1/2)$. From the support properties of $\widehat{\phi}(\cdot)$ and $\widehat{\psi}(\cdot)$ it follows that $B_{jkj'k'} = 0$ and $C_{jkj'k'} = 0$ if $|j-j'| \geq 2$.

Now define $g_{jj'}(\xi) = \widehat{K}_{\alpha 2^{-j},\nu}(\xi)\theta(\xi)\theta(2^{j-j'}\xi)I\{\xi \geq 0\}$. Then, from the properties (ii) and (iii) of θ it can be deduced that $g_{jj'}$ is supported on $[2^{j'-j}\frac{2\pi}{3}, \frac{8\pi}{3}]$ and is infinitely differentiable on $\mathbb{R}$. In particular,

$$g_{jj'}^{(\ell)}(2^{j'-j}2\pi/3) = g_{jj'}^{(\ell)}(8\pi/3) = 0 \qquad \text{for all} \;\; \ell = 0, 1, \ldots, \tag{11}$$

where $g_{jj'}^{(\ell)}$ is the ℓ-th derivative of $g_{jj'}$. Thus, integrating by parts once, the expression (8) becomes (when $k \neq k'$ or $j \neq j'$ so that $\overline{r} \neq 0$),

$$
\begin{aligned}
&= 2^{-j(2\nu+1)-(j'-j)/2}\frac{1}{\pi}\left[\overline{r}^{-1}\left[\sin(\overline{r}\xi)g_{jj'}(\xi)\right]_{2^{j'-j}2\pi/3}^{8\pi/3} - \overline{r}^{-1}\int_{2^{j'-j}2\pi/3}^{8\pi/3}\sin(\overline{r}\xi)g_{jj'}^{(1)}(\xi)d\xi\right] \\
&= 2^{-j(2\nu+1)-(j'-j)/2}\frac{1}{\pi}[\overline{r}^{-1}\sin(8\pi\overline{r}/3)g_{jj'}(8\pi/3) - \sin(2^{j'-j}2\pi\overline{r}/3)g_{jj'}(2^{j'-j}2\pi/3)) \\
&\qquad -\overline{r}^{-1}\int_{2^{j'-j}2\pi/3}^{8\pi/3}\sin(\overline{r}\xi)g_{jj'}^{(1)}(\xi)d\xi] \\
&= -2^{-j(2\nu+1)-(j'-j)/2}\overline{r}^{-1}\frac{1}{\pi}\int_{2^{j'-j}2\pi/3}^{8\pi/3}\sin(\overline{r}\xi)g_{jj'}^{(1)}(\xi)d\xi \qquad \text{(by (11))}.
\end{aligned}
$$

Integrating by parts repeatedly and using (11), we get

$$
B_{jk,j'k'} = c_\ell 2^{-j(2\nu+1)-(j'-j)/2}\overline{r}^{-\ell}\frac{1}{\pi}\int_{2^{j'-j}2\pi/3}^{8\pi/3}h_\ell(\overline{r}\xi)g_{jj'}^{(\ell)}(\xi)d\xi, \qquad j = 0,1,2,\dots, \tag{12}
$$

where $c_0 = 1, c_1 = c_2 = -1, c_3 = c_4 = 1, \dots$ and $h_\ell(\cdot) = \sin(\cdot)$ if ℓ is odd and $h_\ell(\cdot) = \cos(\cdot)$ if ℓ is even.

This shows that for $j \neq j'$

$$
|B_{jk,j'k'}| \leq c_{\alpha 2^{-j},\nu,L}2^{-j(2\nu+1)-(j'-j)/2}|(k-1/2) - 2^{j-j'}(k'-1/2)|^{-L}, \qquad \text{for all } L \geq 0. \tag{13}
$$

As a special case, if $j = j'$ and $k \neq k'$, then $B_{jk,jk'} = B_{j0,j(k-k')}$ (note that this holds for all stationary covariances and all wavelets), and

$$
|B_{j0,j(k-k')}| \leq c_{\alpha 2^{-j},\nu,L}2^{-j(2\nu+1)}|k-k'|^{-L}, \qquad \text{for all } L \geq 0. \tag{14}
$$

Moreover, if $j = j'$ and $k = k'$ then

$$
B_{jk,jk} = B_{j0,j0} \leq c_{\alpha 2^{-j},\nu,L}2^{-j(2\nu+1)}. \tag{15}
$$

Similar calculations yield, after using properties (ii)-(v) of Meyer wavelets,

$$
|A_{jk,jk'}| \leq c'_{\alpha 2^{-j},\nu,L}2^{-j(2\nu+1)}|k-k'|^{-L} \tag{16}
$$

and

$$
|C_{jk,j'k'}| \leq c''_{\alpha 2^{-j},\nu,L}2^{-j(2\nu+1)-(j'-j)/2}|(k-1/2) - 2^{j-j'}(k'-1/2)|^{-L}, \quad \text{for all} \quad L \geq 0. \tag{17}
$$

Thus (16), (15) and (14) show both the decay of the variances of wavelet and scaling coefficients across different levels (scales) as well as the fast decay of covariances of wavelet coefficients away from the diagonal within each level. (13) shows similar decay of covariances of wavelet coefficients across scales. Moreover, as remarked earlier, if $|j - j'| \geq 2$, these across-scale covariances are identically zero. However, since the function $\widehat{\phi}(\xi)$ does not vanish at zero, the constant appearing in the bound for the covariances of scaling coefficients, namely $A_{jk,jk'}$, tends to explode as j increases. This is due to the fact that as j increases the function $\widehat{K}_{\alpha 2^{-j},\nu}(\xi)$

becomes more spiky at zero. The same happens to $\widehat{K}^{(L)}_{\alpha 2^{-j},\nu}(\xi)$ if j is kept fixed and L increases. Hence $c'_{\alpha 2^{-j},\nu,L}$ tends to increase rapidly as j or L is increased independently of one another. Thus, even though from expression (9) it is clear that there is geometric decay of $A_{jk,jk'}$ in $|k-k'|$, the effect is offset by the magnitude of the constant appearing in the bound. Similarly, (17) shows that the cross-covariances between the wavelet and scaling coefficients have a hyperbolic decay away from the diagonal. In this situation the effect of increase in j or L on the constant $c''_{\alpha 2^{-j},\nu,L}$ is less pronounced due to the fact that the function $\widehat{\phi}(\xi)\widehat{\theta}(2^{j-j'}\xi)$ at its derivatives all vanish at 0. This issue is elaborated in *Remark 2*.

Remark 1 : From our analysis it is clear that this sort of behavior of covariances of the wavelet coefficients is not restricted to the Matérn class alone. In fact whenever $\widehat{K}$ has any sort of decay, one would expect to have fast decay away from diagonal. For a different example involving fractional Brownian motion, and its relationship with *linear inverse problems* and *wavelet vaguelet decomposition (WVD)* one may look in Johnstone (1999). In fact the concept of WVD is somewhat more general and the general idea is that wavelets with suitable regularity nearly diagonalize singular integral operators. One may refer to Meyer and Coifman (1997) and Donoho (1995) for more details on WVD.

5 Wavelets with vanishing moments

From the derivations above it is clear that it is not absolutely essential that we perform the analysis with Meyer wavelets, though properties (ii) and (iii) (smoothness and compact support in frequency domain) do help. In fact any wavelet with enough vanishing moments and fast decay in the frequency domain will have similar decay properties. To be specific, suppose that ψ is a mother wavelet with $L+1$ vanishing moments and $\widehat{\psi}$ has L continuous, bounded, derivatives. Then the observation that

$$\widehat{\psi}^{(\ell)}(0) = 0 = \int x^\ell \psi(x)dx, \quad \ell = 0, 1, \dots, L$$

implies, by a Taylor series expansion of $\widehat{\psi}(\omega)$ around 0, that $|\widehat{\psi}(\omega)| = o(\omega^L)$ in a neighborhood of 0. Therefore, if we define, for $j' \geq j$, $h_{jj'}(\xi) = \widehat{K}_{\alpha 2^{-j},\nu}(\xi)\widehat{\psi}(\xi)\overline{\widehat{\psi}}(2^{j-j'}\xi)$, then from equation (6), it follows by mimicking the arguments used in the derivation of (12), that

$$B_{jk,j'k'} = i^L r^{-L} 2^{-j(2\nu+1)-(j'-j)/2} \frac{1}{2\pi} \int_{\mathbb{R}} e^{ir\xi} h^{(L)}_{jj'}(\xi) d\xi, \tag{18}$$

where $r = |k - 2^{j-j'}k'|$. In the derivation we used the following:

(a) $\widehat{K}^{(\ell)}_{\alpha',\nu'}(\omega) \to 0$, as $|\omega| \to \infty$, for all $\ell = 0, 1, \dots, L$

(b) $\widehat{K}^{(\ell)}_{\alpha',\nu'}(\omega)$ is bounded for all $\ell = 0, 1, \dots, L$

(c) $|\widehat{\psi}^{(L)}(\omega)|$ is bounded on $\mathbb{R}$.

Equation (18) readily gives geometric decay in scale and polynomial decay (with arbitrary power) in the difference between locations (more precisely, r) of the covariances of wavelet coefficients, similar to the ones described through the inequalities (13), (14) and (15) in the context of Meyer wavelets. Note however that, the constants $c_{\alpha 2^{-j},\nu,L}$ will depend on the specific wavelet, and its magnitude will depend on how fast the function $\widehat{\psi}(\omega)$ decays as $|\omega| \to \infty$. Note however, that unlike in the case of Meyer wavelet, the across-scale covariance does not vanish in general. However, if $\widehat{\psi}$ decays rapidly away from zero, then for moderately large $|j - j'|$, the integral in (18) is very small since the product $\widehat{\psi}(\xi)\overline{\widehat{\psi}}(2^{j-j'}\xi)$ is uniformly small in such cases. This condition is satisfied if the wavelet ψ is rather smooth, in terms of having enough derivatives.

Remark 2 : An important issue is how the constant $c_{\alpha 2^{-j},\nu,L}$, that serves as a bound for the integral appearing in (18) behaves as j increases. Since $h_{jj'}^{(L)}(\xi)$ decays rapidly to zero as $|\xi| \to \infty$, and this function is continuous, the bound is determined essentially by the behavior of the function $|h_{jj'}^{(L)}(\xi)|$ near 0. Since for large enough j, $\alpha 2^{-j}$ is very small, except for a very small neighborhood of 0, $|\widehat{K}_{\alpha 2^{-j},\nu}^{(L-\ell)}(\xi)|$ behaves like (up to a constant depending on ℓ) $|\xi|^{-(2\nu+1)-(L-\ell)}$, for $\ell = 0, 1, \ldots, L$. This imposes a restriction on how large ν can be in order that the phenomena of rapid decay of the covariances take place. This is because, by assumption on ψ, the ℓ-th derivative of $|\widehat{\psi}(\xi)\overline{\widehat{\psi}}(2^{j-j'}\xi)|$ behaves like $o(|\xi|^{2L-\ell})$ in a neighborhood of 0. This means that, at least for reasonably small $\alpha > 0$ and large j, the effective range of values of ν for the decay phenomena to be observed is $(0.5, (L-1)/2)$. Looking at it from another angle, one needs to have a wavelet with enough vanishing moments to guarantee that the decay phenomena are preserved for all values of $\alpha > 0$, at all scales and for a large enough range of values of ν.

Remark 3 : At this point it becomes clear that the decay of covariances of wavelet coefficients depend very weakly on the behavior of the autocorrelation function. Properties (a) and (b) are the only significant properties that are desired of the kernel K_j, derived from the original kernel K through $\widehat{K}_j(\cdot) := \widehat{K}(2^j\cdot)$. And, as *Remark 2* shows, even properties (b) can be weakened to something like $|\widehat{K}_j^{(\ell)}(\xi)| = O(|\xi|^{-a})$ as $\xi \to 0$, for some $a > 0$, for all $\ell = 0, 1, \ldots, L$.

6 Mixture of stationary processes

It is important to point out that even the assumption of stationarity of the stochastic process is an overkill. One can work with a somewhat weaker assumption of *local stationarity* and work with basis functions other than wavelets, as is done in Mallat, Papanicolaou and Zhang (1998) who use local cosine basis to study this type of processes. (A good reference for wavelet packets, local cosine basis and similar objects is Mallat (1998)).

There is one possible generalization of the Matérn covariance model for which the results presented here apply immediately. This refers to the class of finite mixture of Matérn class processes. Symbolically, such a process may be denoted by

$$X(t) = \pi_1 X_1(t) + \ldots + \pi_m X_m(t), \qquad t \in \mathbb{R}, \tag{19}$$

where $\pi_1, \ldots, \pi_m > 0$, $\sum_{l=1}^{m} \pi_l = 1$ and $X_1(\cdot), \ldots, X_m(\cdot)$ are independent zero mean stationary processes on $\mathbb{R}$ with $X_l(t)$ having a Matérn autocovariance function with parameter $(\beta_l, \alpha_l, \nu_l)$ for each l. Then the process $X(\cdot)$ is a stationary process on $\mathbb{R}$ with autocovariance function $K(\cdot)$ given by

$$K(t) = \sum_{l=1}^{m} \pi_l \beta_l K_{\alpha_l,\nu_l}(t), \qquad t \in \mathbb{R}. \tag{20}$$

Therefore the results derived in the previous sections imply that if the Meyer wavelet basis is used in the decomposition and if the scaling and wavelet coefficients $\{a_{jk}\}$ and $\{b_{jk}\}$ are defined by (1), where $X(t)$ is as in (19), then the covariances $A_{jk,j'k'}$, $B_{jk,j'k'}$ and $C_{jk,j'k'}$ have similar decay properties as in the case of a single component Matérn model. For example,

$$|B_{jk,jk'}| = |B_{j0,j(k-k')}| \leq c_{\pi,\beta,\alpha 2^{-j},\nu,L} 2^{-j(2\nu_{\min}+1)} |k-k'|^{-L}, \qquad L \geq 0. \tag{21}$$

Here $\nu_{\min} = \min_{1\leq l\leq m} \nu_l$, and the constant $c_{\pi,\beta,\alpha 2^{-j},\nu}$ is given by

$$c_{\pi,\beta,\alpha 2^{-j},\nu,L} = \sum_{l=1}^{m} \pi_l \beta_l c_{\alpha_l 2^{-j},\nu_l,L},$$

where $c_{\alpha_l 2^{-j},\nu_l,L}$ is as in (13) with (α,ν) replaced by (α_l,ν_l).

7 Two dimensional Matérn covariance

For dealing with two dimensional stationary processes, we use a tensor product-based 2-D wavelet basis. For a particular pair (ϕ,ψ) of 1-D scaling function and wavelet, define

$$\Phi_{jk_1k_2} = \phi_{jk_1} \otimes \phi_{jk_2}, \quad \Psi^{LH}_{jk_1k_2} = \phi_{jk_1} \otimes \psi_{jk_2}, \quad \Psi^{HH}_{jk_1k_2} = \psi_{jk_1} \otimes \psi_{jk_2}, \quad \Psi^{HL}_{jk_1k_2} = \psi_{jk_1} \otimes \phi_{jk_2},$$

for $j_1, k_1, k_2 \in \mathbb{Z}$. Here the notations LL, LH, HL and HH should be interpreted as *low-low*, *low-high*, *high-low* and *high-high* signifying the order in which the low-pass and high-pass filters are appearing in the tensor product. A convenient way of indexing these wavelets is to denote the ordered triplet (j, k_1, k_2) by λ. It is well-known (Mallat, 1998) that if the wavelet basis generated in 1-D by ψ is orthonormal, then the functions $\{\Psi^{LH}_\lambda, \Psi^{HH}_\lambda, \Psi^{HL}_\lambda\}_{\lambda\in\mathbb{Z}^3}$ form an orthonormal basis of $\mathbf{L}^2(\mathbb{R}^2)$.

Then the two dimensional Fourier transforms of the scaling function and wavelets are just tensor products of Fourier transforms of one dimensional wavelets. For example,

$$\widehat{\Phi}_{jk_1k_2}(\omega_1,\omega_2) = \widehat{\phi}_{jk_1}(\omega_1)\widehat{\phi}_{jk_2}(\omega_2), \quad \omega_1,\omega_2 \in \mathbb{R}.$$

Let $X(s,t)$ be a stationary process in $\mathbb{R}^2$ with zero mean and autocovariance function belonging to the isotropic Matérn class. To be more specific, suppose that the Fourier transform of the autocovariance function $K_{\alpha,\nu}(\cdot,\cdot)$ is given by

$$\widehat{K}_{\alpha,\nu}(\omega_1,\omega_2) = [\alpha^2 + (\omega_1^2 + \omega_2^2)]^{-\nu-1} \tag{22}$$

where we have taken the scaling constant to be 1.

Define the wavelet and scaling coefficients as

$$a_{jk_1k_2} = \int_{\mathbb{R}^2} \Phi_{jk_1k_2}(s,t)X(s,t)dsdt, \qquad b_{jk_1k_2} = \int_{\mathbb{R}^2} \Psi^{LH}_{jk_1k_2}(s,t)X(s,t)dsdt,$$
$$c_{jk_1k_2} = \int_{\mathbb{R}^2} \Psi^{HL}_{jk_1k_2}(s,t)X(s,t)dsdt, \qquad d_{jk_1k_2} = \int_{\mathbb{R}^2} \Phi^{HH}_{jk_1k_2}(s,t)X(s,t)dsdt.$$

Then define

$$A^{LL}_{jk_1k_2,j'k_1'k_2'} = \text{cov}(a_{jk_1k_2}, a_{j'k_1'k_2'}), \qquad A^{LH}_{jk_1k_2,j'k_1'k_2'} = \text{cov}(a_{jk_1k_2}, b_{j'k_1'k_2'}),$$
$$A^{HL}_{jk_1k_2,j'k_1'k_2'} = \text{cov}(a_{jk_1k_2}, c_{j'k_1'k_2'}), \qquad A^{HH}_{jk_1k_2,j'k_1'k_2'} = \text{cov}(a_{jk_1k_2}, d_{j'k_1'k_2'}).$$

Similarly,

$$B^{LL}_{jk_1k_2,j'k_1'k_2'} = \text{cov}(b_{jk_1k_2}, a_{j'k_1'k_2'}), \qquad B^{LH}_{jk_1k_2,j'k_1'k_2'} = \text{cov}(b_{jk_1k_2}, b_{j'k_1'k_2'}),$$
$$B^{HL}_{jk_1k_2,j'k_1'k_2'} = \text{cov}(b_{jk_1k_2}, c_{j'k_1'k_2'}), \qquad B^{HH}_{jk_1k_2,j'k_1'k_2'} = \text{cov}(b_{jk_1k_2}, d_{j'k_1'k_2'});$$

$$C^{LL}_{jk_1k_2,j'k_1'k_2'} = \text{cov}(c_{jk_1k_2}, a_{j'k_1'k_2'}), \qquad C^{LH}_{jk_1k_2,j'k_1'k_2'} = \text{cov}(c_{jk_1k_2}, b_{j'k_1'k_2'}),$$
$$C^{HL}_{jk_1k_2,j'k_1'k_2'} = \text{cov}(c_{jk_1k_2}, c_{j'k_1'k_2'}), \qquad C^{HH}_{jk_1k_2,j'k_1'k_2'} = \text{cov}(c_{jk_1k_2}, d_{j'k_1'k_2'});$$

and

$$D^{LL}_{jk_1k_2,j'k_1'k_2'} = \text{cov}(d_{jk_1k_2}, a_{j'k_1'k_2'}), \qquad D^{LH}_{jk_1k_2,j'k_1'k_2'} = \text{cov}(d_{jk_1k_2}, b_{j'k_1'k_2'}),$$
$$D^{HL}_{jk_1k_2,j'k_1'k_2'} = \text{cov}(d_{jk_1k_2}, c_{j'k_1'k_2'}), \qquad D^{HH}_{jk_1k_2,j'k_1'k_2'} = \text{cov}(d_{jk_1k_2}, d_{j'k_1'k_2'}).$$

Clearly there are some redundancies in these definitions. The following relations hold.

$$A^{LH}_{jk_1k_2,j'k_1'k_2'} = B^{LL}_{j'k_1'k_2',jk_1k_2}, \; A^{HL}_{jk_1k_2,j'k_1'k_2'} = C^{LL}_{j'k_1'k_2',jk_1k_2}, \; A^{HH}_{jk_1k_2,j'k_1'k_2'} = D^{LL}_{j'k_1'k_2',jk_1k_2},$$
$$B^{HL}_{jk_1k_2,j'k_1'k_2'} = C^{LH}_{j'k_1'k_2',jk_1k_2}, \; B^{HH}_{jk_1k_2,j'k_1'k_2'} = D^{LH}_{j'k_1'k_2',jk_1k_2},$$
$$C^{HH}_{jk_1k_2,j'k_1'k_2'} = D^{HL}_{j'k_1'k_2',jk_1k_2}.$$

Since the analysis of the decay properties of these covariances is necessarily going to be repetitive, we give the results for only $B^{LL}_{jk_1k_2,j'k_1'k_2'}$, $B^{LH}_{jk_1k_2,j'k_1'k_2'}$, $B^{HL}_{jk_1k_2,j'k_1'k_2'}$, $B^{HH}_{jk_1k_2,j'k_1'k_2'}$ and $A^{LL}_{jk_1k_2,j'k_1'k_2'}$ to demonstrate the main features.

Usual Fourier analytic considerations give

$$B^{LL}_{jk_1k_2,j'k_1'k_2'} = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \widehat{K}_{\alpha,\nu}(\omega_1,\omega_2)\widehat{\phi}_{jk_1}(\omega_1)\overline{\widehat{\phi}_{j'k_1'}}(\omega_1)\widehat{\psi}_{jk_2}(\omega_2)\overline{\widehat{\phi}_{j'k_2'}}(\omega_2)d\omega_1 d\omega_2, \quad (23)$$

$$B^{LH}_{jk_1k_2,j'k_1'k_2'} = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \widehat{K}_{\alpha,\nu}(\omega_1,\omega_2)\widehat{\phi}_{jk_1}(\omega_1)\overline{\widehat{\phi}_{j'k_1'}}(\omega_1)\widehat{\psi}_{jk_2}(\omega_2)\overline{\widehat{\psi}_{j'k_2'}}(\omega_2)d\omega_1 d\omega_2, \quad (24)$$

$$B^{HL}_{jk_1k_2,j'k_1'k_2'} = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \widehat{K}_{\alpha,\nu}(\omega_1,\omega_2)\widehat{\phi}_{jk_1}(\omega_1)\overline{\widehat{\psi}_{j'k_1'}}(\omega_1)\widehat{\psi}_{jk_2}(\omega_2)\overline{\widehat{\phi}_{j'k_2'}}(\omega_2)d\omega_1 d\omega_2, \quad (25)$$

$$B^{HH}_{jk_1k_2,j'k_1'k_2'} = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \widehat{K}_{\alpha,\nu}(\omega_1,\omega_2)\widehat{\phi}_{jk_1}(\omega_1)\overline{\widehat{\psi}_{j'k_1'}}(\omega_1)\widehat{\psi}_{jk_2}(\omega_2)\overline{\widehat{\psi}_{j'k_2'}}(\omega_2)d\omega_1 d\omega_2, \quad (26)$$

$$A^{LL}_{jk_1k_2,j'k_1'k_2'} = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \widehat{K}_{\alpha,\nu}(\omega_1,\omega_2)\widehat{\phi}_{jk_1}(\omega_1)\overline{\widehat{\phi}_{j'k_1'}}(\omega_1)\widehat{\phi}_{jk_2}(\omega_2)\overline{\widehat{\phi}_{j'k_2'}}(\omega_2)d\omega_1 d\omega_2. \quad (27)$$

Define $r_1 = r(j, k_1, j', k_1') = k_1 - 2^{j-j'}k_1'$ and $r_2 = r(j, k_2, j', k_2') = k_2 - 2^{j-j'}k_2'$. Also define $\xi_1 = 2^{-j}\omega_1$, $\xi_2 = 2^{-j}\omega_2$. Then it follows that

$$\begin{aligned} B^{LH}_{jk_1k_2,j'k_1'k_2'} &= 2^{-j(2\nu+2)-(j'-j)}\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} K_{\alpha 2^{-j},\nu}(\xi_1,\xi_2)\widehat{\phi}(\xi_1)\overline{\widehat{\phi}}(2^{j-j'}\xi_1)e^{ir_1\xi_1} \\ &\qquad \widehat{\psi}(\xi_2)\overline{\widehat{\psi}}(2^{j-j'}\xi_2)e^{ir_2\xi_2}d\xi_1 d\xi_2. \end{aligned} \tag{28}$$

Let $h^{LH}_{jj'}(\xi_1,\xi_2) = K_{\alpha 2^{-j},\nu}(\xi_1,\xi_2)\widehat{\phi}(\xi_1)\overline{\widehat{\phi}}(2^{j-j'}\xi_1)\widehat{\psi}(\xi_2)\overline{\widehat{\psi}}(2^{j-j'}\xi_2)$. Also, let

$$h^{LH}_{jj';\ell_1\ell_2}(\xi_1,\xi_2) = \frac{\partial^{\ell_1}}{\partial\xi_1^{\ell_1}}\frac{\partial^{\ell_2}}{\partial\xi_2^{\ell_2}}h^{LH}_{jj'}(\xi_1,\xi_2)$$

assuming that the relevant quantities exist. Now suppose that the mother wavelet ψ has $M+1$ vanishing moments and $\widehat{\psi}$ has M continuous, bounded, derivatives. Assume further that $\widehat{\phi}$ has $\overline{M}$ continuous, bounded, derivatives, where $0 \le \overline{M} \le M$. Then we use the expression (28) and the fact that the relevant partial derivatives vanish at $\pm\infty$ independently in each variable, to first integrate by parts $\overline{M}$ times w.r.t. ξ_1, and then (separately) integrate by parts M times w.r.t. ξ_2, and finally use arguments similar to those in Section 3 and 4, to conclude that, for $r_1 \neq 0$ and $r_2 \neq 0$,

$$B^{LH}_{jk_1k_2,j'k_1'k_2'} = i^{M+\overline{M}}r_1^{-\overline{M}}r_2^{-M}2^{-j(2\nu+2)-(j'-j)}\int_{\mathbb{R}^2} h^{LH}_{jj';\overline{M}M}(\xi_1,\xi_2)e^{i(r_1\xi_1+r_2\xi_2)}d\xi_1 d\xi_2. \tag{29}$$

On the other hand, if $r_1 = 0$ and $r_2 \neq 0$ then

$$B^{LH}_{jk_1k_2,j'k_1'k_2'} = i^{M}r_2^{-M}2^{-j(2\nu+2)-(j'-j)}\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} h^{LH}_{jj';0M}(\xi_1,\xi_2)e^{ir_2\xi_2}d\xi_1 d\xi_2. \tag{30}$$

And finally, if $r_1 \neq 0$ and $r_2 = 0$, then

$$B^{LH}_{jk_1k_2,j'k_1'k_2'} = i^{\overline{M}}r_1^{-\overline{M}}2^{-j(2\nu+2)-(j'-j)}\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} h^{LH}_{jj';\overline{M}0}(\xi_1,\xi_2)e^{i(r_1\xi_1+r_2\xi_2)}d\xi_1 d\xi_2. \tag{31}$$

This shows that, with δ_x denoting the indicator of x,

(a) the covariance model is stationary in both the dimensions within each scale j, and

$$\begin{aligned} |B^{LH}_{jk_1k_2;jk_1'k_2'}| &= |B^{LH}_{j00;j(k_1-k_2)(k_1'-k_2')}| \\ &\le c^{B;LH}_{\alpha 2^{-j},\nu,M,\overline{M}}2^{-j(2\nu+2)}|k_1-k_1'|^{-\overline{M}\wedge\delta_{k_1-k_1'}}|k_2-k_2'|^{-M\wedge\delta_{k_2-k_2'}}; \end{aligned} \tag{32}$$

(b) for any two scales j, j' with $j' > j$,

$$|B^{LH}_{jk_1k_2;j'k_1'k_2'}| \le c^{B;LH}_{\alpha 2^{-j},\nu,M,\overline{M}}2^{-j(2\nu+2)-(j'-j)}|k_1-2^{j-j'}k_1'|^{-\overline{M}\wedge\delta_{r_1}}|k_2-2^{j-j'}k_2'|^{-M\wedge\delta_{r_2}}; \tag{33}$$

for some constant $c^{B;LH}_{\alpha 2^{-j},\nu,M,\overline{M}} > 0$. Similar computations yield

$$
\begin{aligned}
|B^{LL}_{jk_1k_2;j'k_1'k_2'}| &\leq c^{B;LL}_{\alpha 2^{-j},\nu,\overline{M}} 2^{-j(2\nu+2)-(j'-j)}|k_1 - 2^{j-j'}k_1'|^{-\overline{M}\wedge\delta_{r_1}}|k_2 - 2^{j-j'}k_2'|^{-\overline{M}\wedge\delta_{r_2}}; \quad (34)\\
|B^{HL}_{jk_1k_2;j'k_1'k_2'}| &\leq c^{B;HL}_{\alpha 2^{-j},\nu,\overline{M}} 2^{-j(2\nu+2)-(j'-j)}|k_1 - 2^{j-j'}k_1'|^{-\overline{M}\wedge\delta_{r_1}}|k_2 - 2^{j-j'}k_2'|^{-\overline{M}\wedge\delta_{r_2}}; \quad (35)\\
|B^{HH}_{jk_1k_2;j'k_1'k_2'}| &\leq c^{B;HH}_{\alpha 2^{-j},\nu,M,\overline{M}} 2^{-j(2\nu+2)-(j'-j)}|k_1 - 2^{j-j'}k_1'|^{-\overline{M}\wedge\delta_{r_1}}|k_2 - 2^{j-j'}k_2'|^{-M\wedge\delta_{r_2}}; \quad (36)\\
|A^{LL}_{jk_1k_2;j'k_1'k_2'}| &\leq c^{A;LL}_{\alpha 2^{-j},\nu,\overline{M}} 2^{-j(2\nu+2)-(j'-j)}|k_1 - 2^{j-j'}k_1'|^{-\overline{M}\wedge\delta_{r_1}}|k_2 - 2^{j-j'}k_2'|^{-\overline{M}\wedge\delta_{r_2}}; \quad (37)
\end{aligned}
$$

for constants $c^{B;LL}_{\alpha 2^{-j},\nu,\overline{M}} > 0$, $c^{B;HL}_{\alpha 2^{-j},\nu,\overline{M}} > 0$, $c^{B;HH}_{\alpha 2^{-j},\nu,M,\overline{M}} > 0$ and $c^{A;LL}_{\alpha 2^{-j},\nu,\overline{M}} > 0$. A note of caution though, about the behavior of these constants. Apropos of *Remark 2*, as the scale increases, the function $\widehat{K}_{\alpha 2^{-j},\nu}$ tends to become more concentrated near zero and tends to behave more like $|\xi_1 + \xi_2|^{-2(2\nu+1)}$. This implies that, unless $\widehat{\phi}^{(\ell)}$ vanishes at 0, the constant appearing in the bound on the covariances will tend to be large for large j except in the cases where both the coefficients correspond to high-pass filters. Indeed the Coiflet family of wavelets has the desirable property that $\widehat{\phi}^{(\ell)}(0) = 0$ for $\ell = 1, \ldots, M$ when the corresponding mother wavelet ψ has $M + 1$ vanishing moments. Of course Meyer wavelets also have this property. However, for a fixed j, the partial derivatives of $\widehat{K}_{\alpha 2^{-j},\nu}(\xi_1, \xi_2)$ tend to become more spiky at 0 as the order of these derivatives increase. Since $\widehat{\phi}(0) = 1$, there is no cancellation at 0. And so the constant $c^{A;LL}_{\alpha 2^{-j},\nu,\overline{M}}$ tends to increase fast for increasing $\overline{M}$. Analogous comments apply to the covariances of coefficients corresponding to basis elements involving at least one low-pass component.

8 A class of nonstationary covariance functions

Following Pintore and Holmes (2006) we consider a class of nonstationary covariance function defined through

$$C_{NS}(s,t) = \int_{\mathbb{R}^N} e^{i\omega^T(s-t)}\sqrt{f(\omega, s; \theta(s))}\sqrt{f(\omega, t; \theta(t))}d\omega. \quad (38)$$

here $f(\omega, s; \theta) = h(s)f_\theta(\omega)$ for some nonnegative functions $h(s)$ and some parametric spectral density $f_\theta(\omega)$ (of a stationary process on $\mathbb{R}^N$). The necessary and sufficient condition for (38) to define a valid covariance function is that

$$\int_{\mathbb{R}^N} |f(\omega, s; \theta(s))|ds < \infty.$$

In the situation when $f_\theta(\omega)$ is the spectral density function of Matérn class covariance with $\theta = (\alpha, \nu)$, the corresponding nonstationary covariance K_{NS} is of the form

$$K_{NS}(s,t) = \overline{h}(s,t)(\alpha \parallel s - t \parallel)^{\nu(s,t)}K_{\nu(s,t)}(\alpha \parallel s - t \parallel), \quad (39)$$

where

$$\nu(s,t) = \frac{\nu(s) + \nu(t)}{2}, \qquad \text{and} \qquad \overline{h}(s,t) = \frac{h(s)h(t)\pi^{N/2}}{2^{\nu(s,t)-1}\Gamma(\nu(s,t) + N/2)}\alpha^{-2\nu(s,t)}.$$

We shall show that when the function $h(s)$ is smooth (meaning analytic) and $\nu(s) \equiv \nu$ (for simplicity), the qualitative behavior of the covariances of the wavelet coefficients from this process remains similar to the case when $h(s) \equiv 1$ (i.e., the stationary Matérn case). One can get analogous results locally (depending on the "flatness" of the function $\nu(\cdot)$ and in higher scales) for the general model (39) as long as the function $\nu(\cdot)$ is smooth and has a bounded range. To simplify the exposition we shall consider the $N = 1$ case (one dimensional process) only. The generalization for higher dimensions is straightforward.

8.1 Covariance of wavelet coefficients

For the ease of exposition we assume that that the scale function h is a polynomial of order Q. Then we can express h in terms of its Taylor series expansion around $2^{-j}k$ as

$$h(s) = \sum_{q=0}^{Q} h^{(q)}(2^{-j}k)\frac{(s-2^{-j}k)^q}{q!} = h_Q(s; 2^{-j}k), \quad \text{say.} \tag{40}$$

Then using $\overline{B}_{jk,j'k'}$ to denote $\int\int \psi_{jk}(s)h_Q(s;2^{-j}k)h_Q(t;2^{-j'}k')K(s-t)\psi_{j'k'}(t)dt$, we have

$$\overline{B}_{jk,j'k'} = \sum_{q=0}^{Q}\sum_{q'=0}^{Q} \frac{h^{(q)}(2^{-j}k)}{q!}\frac{h^{(q')}(2^{-j'}k')}{q'!}\int\int \psi_{jk}(s)(s-2^{-j}k)^q K(s-t)\psi_{j'k'}(t)(t-2^{-j'}k')^{q'}dsdt. \tag{41}$$

Observe that, $\overline{B}_{jk,j'k'}$ is the covariance between the wavelet coefficients corresponding to wavelets ψ_{jk} and $\psi_{j'k'}$ of the process $Y(s) = h(s)X(s)$ where $X(s)$ is the one dimensional stationary Matérn process with covariance kernel $K(\cdot)$. Again, to avoid repetitions of the results described in Section 4, we shall focus our attention to the behavior of $\overline{B}_{jk,j'k'}$ only.

Since

$$\psi_{jk}(s)(s-2^{-j}k)^q = 2^{j/2}\psi(2^j(s-2^{-j}k))(s-2^{-j}k)^q = \psi_{j0}(s-2^{-j}k)(s-2^{-j}k)^q,$$

we have

$$\mathcal{F}(\psi_{jk}(\cdot)(\cdot\; - 2^{-j}k)^q)(\omega) = i^q\widehat{\psi}_{j0}^{(q)}(\omega)e^{i\omega 2^{-j}k}, \tag{42}$$

where $\mathcal{F}(g)(\omega) = \widehat{g}(\omega)$ denotes the Fourier transform of g at $\omega \in \mathbb{R}$. Also, observe that

$$\widehat{\psi}_{j0}(\omega) = \widehat{\psi}(2^j\cdot)(\omega) = \widehat{\psi}(2^{-j}\omega), \quad \text{and hence} \quad \widehat{\psi}_{j0}^{(q)}(\omega) = 2^{-jq}\widehat{\psi}^{(q)}(2^{-j}\omega).$$

Substituting this and (42) in (41), we have

$$\begin{aligned}
\overline{B}_{jk,j'k'} &= \sum_{q=0}^{Q}\sum_{q'=0}^{Q}\frac{h^{(q)}(2^{-j}k)}{q!}\frac{h^{(q')}(2^{-j'}k')}{q'!}i^{q-q'}\frac{1}{2\pi}\int_{\mathbb{R}}\widehat{K}(\omega)\widehat{\psi}^{(q)}_{j0}(\omega)\overline{\widehat{\psi}^{(q')}_{j'0}}(\omega)e^{i(2^{-j}k-2^{-j'}k')}d\omega \\
&= \sum_{q=0}^{Q}\sum_{q'=0}^{Q}\frac{h^{(q)}(2^{-j}k)}{q!}\frac{h^{(q')}(2^{-j'}k')}{q'!}2^{-(jq+j'q')} \\
&\quad \cdot\ 2^{(j-j')/2}i^{q-q'}\frac{1}{2\pi}\int_{\mathbb{R}}\widehat{K}(2^{j}\xi)\widehat{\psi}^{(q)}(\xi)\overline{\widehat{\psi}^{(q')}}(2^{j-j'}\xi)e^{i(k-2^{j-j'}k')\xi}d\xi \\
&= 2^{-j(2\nu+1)}\sum_{q=0}^{Q}\sum_{q'=0}^{Q}(2^{-jq}\frac{h^{(q)}(2^{-j}k)}{q!})(2^{-j'q'}\frac{h^{(q')}(2^{-j'}k')}{q'!}) \\
&\quad \cdot\ i^{q-q'}2^{(j-j')/2}\widetilde{K}_{\alpha 2^{-j},\nu}(q,q';2^{j-j'},k,k'),
\end{aligned} \tag{43}$$

where

$$\widetilde{K}_{\alpha 2^{-j},\nu}(q,q';2^{j-j'},k,k') = \frac{1}{2\pi}\int_{\mathbb{R}}\widehat{K}_{\alpha 2^{-j},\nu}(\xi)\widehat{\psi}^{(q)}(\xi)\overline{\widehat{\psi}^{(q')}}(\xi)d\xi.$$

Using calculations similar to the ones used in Section 4.1, we have the following bound

$$|\widetilde{K}_{\alpha 2^{-j},\nu}(q,q';2^{j-j'},k,k')| \leq \kappa_{\alpha 2^{-j},\nu,L}(q,q')|(k-\frac{1}{2})-2^{j-j'}(k'-\frac{1}{2})|^{-L}, \qquad \text{for} \ \ L\geq 0,$$

where $\kappa_{\alpha 2^{-j},\nu,L}(q,q')>0$ is some constant independent of k,k'. This shows, using (43), that

$$\begin{aligned}
|\overline{B}_{jk,j'k'}| &\leq 2^{-(2\nu+1)j-(j'-j)/2}|(k-\frac{1}{2})-2^{j-j'}(k'-\frac{1}{2})|^{-L} \\
&\quad \cdot \sum_{q=0}^{Q}\sum_{q'=0}^{Q}(2^{-jq}\frac{|h^{(q)}(2^{-j}k)|}{q!})(2^{-j'q'}\frac{|h^{(q')}(2^{-j'}k')|}{q'!})\kappa_{\alpha 2^{-j},\nu,L}(q,q')
\end{aligned}$$

for all $L\geq 0$. Thus, the asymptotic behavior of $|\overline{B}_{jk,j'k'}|$ is very similar to the situation when $h(s)\equiv 1$.

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