On spectral scaling laws for incompressible anisotropic magnetohydrodynamic turbulence

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Abstract

A heuristic model is given for anisotropic magnetohydrodynamics (MHD) turbulence in the presence of a uniform external magnetic field $B_0 \hat{e}_\parallel$. The model is valid for both moderate and strong $B_0$ and is able to describe both the strong and weak wave turbulence regimes as well as the transition between them. The main ingredient of the model is the assumption of constant ratio at all scales between the linear wave period and the nonlinear turnover timescale. Contrary to the model of critical balance introduced by Goldreich and Sridhar [P. Goldreich and S. Sridhar, ApJ 438, 763 (1995)], it is not assumed in addition that this ratio be equal to unity at all scales. This allows us to make use of the Iroshnikov-Kraichnan phenomenology; it is then possible to recover the widely observed anisotropic scaling law $k_\parallel \propto k_\perp^{2/3}$ between parallel and perpendicular wavenumbers (with reference to $B_0 \hat{e}_\parallel$) and to obtain for the total energy spectrum $E(k_\perp, k_\parallel) \sim k_\perp^{-\alpha} k_\parallel^{-\beta}$ the universal prediction, $3\alpha + 2\beta = 7$. In particular, with such a prediction, the weak Alfvén wave turbulence constant-flux solution is recovered and, for the first time, a possible explanation to its precursor found numerically by Galtier et al. [S. Galtier et al., J. Plasma Phys. 63, 447 (2000)] is given.

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I. INTRODUCTION

Turbulent flows are often studied under the assumptions of homogeneity and isotropy (see, for a review, Ref. [1, 2]). Such assumptions are convenient for theoretical studies but are not always justified physically. For example, it is well known that stratification or rotation applied to neutral flows lead to anisotropic turbulence (see e.g. Ref. [3, 4]). Isotropy is even more difficult to justify in astrophysics where a magnetic field is almost always present at the largest scale of the system. The magnetohydrodynamics (MHD) approximation has proved to be quite successful in the study of a variety of space plasmas. During the last quarter of century many studies have been devoted to incompressible MHD turbulence in the presence of a uniform external magnetic field $B_0 \hat{e}_\parallel$ (see Ref. [5–19]). One of the most clearly established results is that the presence of $B_0$ leads to a bi-dimensionalization of an initial isotropic energy spectrum: the turbulent cascade transfers energy preferentially to perpendicular wavenumbers, i.e. in the direction transverse to $B_0 \hat{e}_\parallel$.

Constant-flux spectra are known to occur in many instances in turbulent flows, the best example of which being the Kolmogorov energy spectrum following a $E(k) \propto k^{-5/3}$ law for three-dimensional Navier-Stokes turbulence (see e.g. Ref. [20]). Power law spectra are also measured in turbulent MHD flows but the value of the scaling index is still hardly discussed in the community. The first prediction in MHD was given independently by Iroshnikov and Kraichnan in Ref. [21, 22] (hereafter IK). They argued that the destruction of phase coherence by Alfvén waves traveling in opposite directions along the local large scale magnetic field introduces a new timescale and a slowing down of energy transfer to small scales. Assuming isotropy, the dimensional analysis for three wave interactions leads to a $E(k) \propto k^{-3/2}$ spectrum for the total energy (see also Ref. [23]). Many direct numerical simulations of strong turbulence in isotropic ($B_0 = 0$) MHD have been made during the last years (see e.g. Ref. [15, 17, 24]) but a definitive conclusion about the spectrum index is still not achieved mainly because the Kolmogorov and IK predictions are very close; furthermore, such scaling laws may be slightly altered by intermittency effects, and the numerical resolution is barely sufficient to determine such spectral indices. Goldreich and Shridar in Ref. [11] proposed in 1995 a heuristic model of strong turbulence for anisotropic (moderate $B_0$) MHD where the distinction between the perpendicular ($k_\perp$) and parallel ($k_\parallel$) wavenumbers is made. This model is based on a critical balance between linear wave periods
\(\tau_A\) and nonlinear turnover timescales \(\tau_{NL}\) for which the equality \(\tau_A = \tau_{NL}\) is assumed to hold at all scales in the inertial range. (Only the symmetric case, for which the Alfvén waves traveling in opposite directions carry equal energy fluxes, is considered here and in the remainder of the paper.) Then the one dimensional perpendicular power spectrum for the total energy scales as \(E(k_\perp) \propto k_\perp^{-5/3}\) whereas the parallel and perpendicular spatial sizes of eddies are correlated according to the scaling law \(k_\parallel \propto k_\perp^{2/3}\). The latter prediction is rather new (see however Ref. [25]) and seems well observed in recent direct numerical simulations for moderate \(B_0\) (see Ref. [15, 17]).

As is well known, in the limit of large \(B_0\), MHD turbulence becomes strongly anisotropic and mainly weak in the sense that, in Fourier space, the domain of applicability of strong turbulence is confined to a region localized around the \(k_\parallel = 0\) plane. The formalism of weak Alfvén wave turbulence developed by Galtier et al. in Ref. [16, 19] is well adapted to this situation. It leads to the so-called wave kinetic equations for which the exact power law solution for the total energy is \(E(k_\perp, k_\parallel) \propto k_\perp^{-2}f(k_\parallel)\). The function \(f(k_\parallel)\) is undetermined because of the dynamical decoupling of parallel planes in Fourier space (this is the signature of the absence of energy transfer along the \(B_0\) direction); it thus represents a shadow of the initial conditions. Numerical simulations of the wave kinetic equations show clearly such a constant-flux spectrum but it also reveals the existence of an earlier transient spectrum during the front propagation towards small scales with a steeper power-law in \(k_\perp^{-7/3}\), the dynamics of which is not yet clarified (see e.g. Ref. [26, 27]). The discovery of such transient spectra in wave turbulence and the possible existence of a family of solutions that are not caught by the usual technique of conformal transform (given e.g. in Ref. [28]) constitute a new exciting topic of research where some progress is currently being made (see, for example, Ref. [29, 30]). When using a shell model of strong turbulence, it has also been found in Ref. [31] that, when considering the decay of energy in time as a power law, initial transients occur that also follow power-laws to leading order and that precede the final power-law decay of the energy; the origin of such transients in time may well be a transient in the Fourier energy spectrum preceding the establishment of a Kolmogorov-like spectrum, although this point has not been documented yet.

In this paper, we propose a heuristic model that describes anisotropic MHD flows for the regimes of strong turbulence (at moderate \(B_0\)) and of weak wave turbulence (for strong \(B_0\)),
as well as the transition between them. As a result of our analysis, a family of solutions is found for the anisotropic total energy spectrum from which the transient spectrum described above is a particular solution. We show that the model supports the same anisotropic scaling law between the parallel and perpendicular wavenumbers as the one found in the context of critical balance but it is more general in the sense that here we do not impose equality between linear wave periods and nonlinear turnover timescales which allows us to use the IK phenomenology. We finally propose to extend the model to two other types of fluids that become anisotropic under the influence of imposed external agents.

II. ANISOTROPIC MHD MODEL

The presence of the mean magnetic field $B_0$ leads to anisotropy and thus to different variations with wavenumber directions, leading us to distinguish between $k_\perp$ and $k_\parallel$. More precisely we will assume that $k_\perp \gg k_\parallel$, i.e. that under the external agent (here, $B_0$), the turbulent flow develops principally in the direction perpendicular to this agent. We will consider the symmetric case for which, in particular, the r.m.s. value at any scale in the inertial range of the fluctuating velocity field $v$ and magnetic field $b$ have the same order of magnitude; note that $b$ is taken in velocity units since we are restricting the analysis to the incompressible MHD case (the density is constant and can be assumed to be equal to unity).

In the classical Kolmogorov phenomenology (hereafter, K41), the fluctuations are distributed isotropically and there is only one timescale, the nonlinear time or eddy turnover time $\tau_{NL}$, which is also the transfer time of the energy to small scales within the system, i.e. $\tau_{tr} = \tau_{NL}$. The rate of energy transfer per unit mass writes $\mathcal{E}_{K41} \sim E/\tau_{tr}$, where $E$ is the total (kinetic plus magnetic) energy at a given scale. In the very same spirit, we can develop the IK phenomenology for anisotropic MHD turbulence using explicitly the fact that the eddy turnover time is $\tau_{NL} \sim (vk)^{-1} \sim (vk_\perp)^{-1}$ and the Alfvén wave period is $\tau_A \sim 1/(k_\parallel B_0)$. We assume that these two timescales are not equal which allows us to use the IK phenomenology. The rate of energy transfer per unit mass now writes

$$\mathcal{E}_{IK} \sim \frac{E}{\tau_{tr}} \sim \frac{v^2}{\tau_{tr}}, \quad (1)$$
where the transfer time will be assumed to scale as
\[
\tau_{tr} = \tau_{NL} \frac{T_{NL}}{\tau_A}, \tag{2}
\]
as is known to be the case for three-wave interaction processes (see e.g. Ref. [16, 21, 22]). The subscript “a” in $IK_a$ stands for the anisotropic version of IK. Including relation (2) into (1), one obtains
\[
\mathcal{E}_{IK_a} \sim \frac{\nu^4 k_{\perp}^2}{B_0 k_{\parallel}}. \tag{3}
\]
We now define the anisotropic energy spectrum, assuming self-similarity in both ($\perp, \parallel$) directions:
\[
E(k_{\perp}, k_{\parallel}) \sim k_{\perp}^{-\alpha} k_{\parallel}^{-\beta}, \tag{4}
\]
where $\alpha$ and $\beta$ are unknown. It is a 2D energy spectrum. In other words, the total energy of the system is recovered by directly integrating the spectrum along $k_{\perp}$ and $k_{\parallel}$, i.e. $E_{sys} = \int \int E(k_{\perp}, k_{\parallel}) dk_{\perp} dk_{\parallel}$ (see e.g. Ref. [16], for a rigorous definition). It is important to note at this stage a difference with the classical phenomenology where the energy spectrum is not introduced with unknown indices but rather deduced by the analysis. The phenomenology proposed to describe MHD turbulence is, of course, not unique; one can find recent approaches in e.g. Ref. [32, 33]. As is well known, the presence of an external magnetic field $B_0$ leads to a reduction of nonlinear transfers along its direction which, in turns, leads to a reduction in size in Fourier space of the inertial range. Numerical simulations, for the most part at moderate resolutions, and/or using the Reduced MHD approximation (see e.g. Ref. [34]) show indeed that a strong $B_0$ leads to an absence of inertial range in $k_{\parallel}$ with a spectrum mainly exponential (see e.g. Ref. [13, 35–37]) as it is usually observed in turbulence at large wavenumbers. Since heuristic models deal with power laws, they are mainly able to describe what happens inside the inertial range, without saying much about the strength of the nonlinear transfers and thus about the size of the inertial range. We are assuming, as noted before (see equation (4)) that power-law develops in $k_{\parallel}$ as well, i.e. that the Reynolds number is extremely large, as found in astrophysical flows; in particular, it may need to be substantially higher than in the fully isotropic case since transfer in the $k_{\parallel}$ direction is strongly impeded in the presence of a uniform magnetic field. Therefore, relation (4) introduced above gives a description of the possible inertial ranges in both $k_{\perp}$ and $k_{\parallel}$ directions; additionally, it offers an opportunity to describe continuously the phase
transition between the strong and weak turbulence regimes. We introduce relation (4) into (3), and by noticing that $v^2 \sim E \sim E(k_\perp, k_\parallel) k_\perp k_\parallel$, we obtain:

$$B_0 \mathcal{E}_{IKa} \sim k_\perp^{1-2\alpha} k_\parallel^{1-2\beta},$$

which can also be rewritten in terms of a $(k_\perp, k_\parallel)$ relationship as:

$$k_\parallel \sim (B_0 \mathcal{E}_{IKa})^{1/2} \frac{4-2\alpha}{2-\beta} k_\perp^{\frac{1}{2-\beta}}.$$

This is our first anisotropic relation. In order to proceed further, we are now seeking a second relationship between the two scaling indices $\alpha$ and $\beta$. One way to obtain a unique relation between them is to use the assumption of constant ratio between $\tau_{NL}$ and $\tau_A$ at all scales in the inertial range. In other words, we will assume that

$$\chi = \frac{\tau_A}{\tau_{NL}}$$

is a constant at all scales but not necessarily equal to one as it is in the critical balance model in Ref. [11]. Condition (7) can be understood as a formal balance between the linear and nonlinear terms in the ideal MHD equations. It is important to note that the precise value of $\chi$ is not important in itself since it is just a numerical factor that does not change the power law scaling. However if $\chi$ is smaller than one we are allowed to use the IK phenomenology instead of the K41 one for $\chi = 1$. A ratio of one seems to be very restrictive and does not correspond to some of the results stemming from direct numerical simulations where $\chi$ can be smaller than unity, as observed for example in Ref. [38], a two-dimensional geometrical case where local anisotropy is possible, or in solar wind in situ measurements (see Ref. [39]) where $\chi$ seems to be smaller than unity. It is also in apparent contradiction with wave turbulence theory for which we have $\chi \ll 1$. We will see that the assumption of constant ratio at all scales in the inertial range is sufficient to achieve a unified model that describes both strong and weak MHD turbulence. With the definition of the timescales given above, we obtain

$$\chi \sim \frac{v k_\perp}{B_0 k_\parallel}.$$

Including expression (8) into (3) gives:

$$k_\parallel \sim \frac{\mathcal{E}_{IKa}^{1/3}}{\chi^{1/3} B_0} \frac{k_\perp^{2/3}}{k_\parallel^{1/3}}.$$
If we use explicitly the fact that both $\chi$ and the rates of energy transfer per unit mass do not depend on the scale (see e.g. Ref. [20–22]), we finally obtain
\[
 k_\parallel \sim \frac{k_\perp^{2/3}}{B_0} .
\]
(10)

This leads us to the first main conclusion of this paper: the same scaling in wavenumbers as the critical balance model is obtained when $\chi$ is not equal to unity, i.e. when the IK phenomenology is used. It thus seems to be a general law for incompressible MHD turbulence, when either the Kolmogorov phenomenology for energy transfer or the $IK_a$ three-wave formulation of energy transfer is utilized, as long as a critical balance (generalized to any ratio differing from unity) is assumed between stretching by velocity gradients and wave motions in the presence of a uniform field $B_0$. Condition (10) may be seen as a path in the $(k_\perp-k_\parallel)$ Fourier space followed naturally by the dynamics for different values of the (imposed) uniform magnetic field. In other words, it means that excitations are concentrated on this curve: the curve does not define a boundary between regions where wave or strong turbulence dominates. An illustration is given in Figure 1 for two different values of $B_0$. It is interesting to note that condition (10) is not incompatible with the weak wave turbulence prediction since in the limit of infinite $B_0$ the curve identifies with the $k_\perp$-axis, and therefore no transfers are possible along the parallel direction.

Another important and somewhat unexpected consequence of this analysis has to deal with the scaling of the energy spectrum. In the context of a critical balance of unity, it is claimed that the energy spectrum derived with an anisotropic Kolmogorov phenomenology scales like $E(k_\perp) \sim k_\perp^{-5/3}$. Here we show that it is in fact possible to find a multitude of spectral indices $\alpha$ and $\beta$ that satisfy the assumption of constant ratio $\chi$ at all scales in the inertial range. Such indices follow in fact a general linear relationship. To find this relation we need to equalize the power-law behaviors found in relations (6) and (10) between the parallel and perpendicular distribution of modal energy in Fourier space. If we use explicitly the fact that the rates of energy transfer per unit mass do not depend on the scale (see e.g. Ref. [20–22]), we finally obtain
\[
 3\alpha + 2\beta = 7 .
\]
(11)

Relationship (11) is general and can be used for strong turbulence as well as for weak wave turbulence. Note that relationship (11) is compatible with the anisotropic Kolmogorov
FIG. 1: Path, in Fourier space, followed by the MHD turbulence dynamics for two different values of the external magnetic field $B_0$. A higher value of $B_0$ leads to weakening of parallel transfers with a curve closer to the $k_\perp$-axis. When $B_0$ is infinite the path identifies with the $k_\perp$-axis and no transfers are possible along the parallel direction. Note that the curves are not extended down to the origin since the anisotropic law is only valid in the inertial range.

spectrum (with $\alpha = 5/3$ and $\beta = 1$) as advocated by Goldreich and Sridhar in Ref. [11], and with the anisotropic IK-like spectrum corresponding to three-wave interactions for weak MHD turbulence ($\alpha = 2$ and $\beta = 1/2$) (see Ref. [16, 40, 41]). We conjecture that it is compatible with the physics of the transitional regime between weak and strong MHD turbulence as well. In other words the law (11) we just derived shows that the $2/3$ scaling between a dependence of $k_\perp$ and $k_\parallel$ is not a unique signature of the Kolmogorov spectrum but rather a signature of the rate at which energy is transferred to small scales where it is dissipated, i.e. a trace of the decay of energy, and as such an unavoidable dimensional law assuming equation (4) holds and that $\chi$ is independent of wave numbers in the inertial range.

III. DISCUSSION

The preceding remark may be linked to the heretofore unexplained following finding: using direct numerical simulations in two space dimensions, it was shown in Ref. [42] that the structure functions of order $p$ based on the energy flux to small scales (as expressed in terms of the exact laws for MHD turbulence derived in Ref. [43, 44]) have a self-similar scaling in the inertial range which is compatible with the scaling for the velocity structure
functions in fluid turbulence, whereas the structure functions of the basic fields (velocity and magnetic fields) are more intermittent insofar as they depart more significantly from a linear scaling with $p$. What this paper shows is that both models (K41 and IK) can be seen in a unifying way, illustrated by the law (11). Note that in the case of the advection of a passive tracer such as temperature, it also seems that the scaling of the structure functions based on the flux of the tracer is close to the fluid scaling laws (see Ref. [45, 46]), whereas the tracer itself is well-known to be strongly intermittent.

More surprisingly perhaps, the choice of $\alpha = 7/3$ and $\beta = 0$ also satisfies the law (11) derived in this paper. This $k^{-7/3}$ spectrum was found in Ref. [16] as a precursor to the constant flux solution of the Alfvén wave kinetic equations which establishes itself later in time; indeed, in that paper, the $7/3$ spectrum is found numerically for the case $k_\parallel = 0$, i.e. without any dependence in $k_\parallel$ (corresponding to $\beta = 0$). The simple model proposed here thus sheds some light, albeit heuristically, on two intriguing facts that have emerged recently concerning weak wave turbulence: (i) the fact that, preceding in time the constant flux solution, a power-law spectrum (called the precursor) establishes itself the origin of which was unrelated to anything known about turbulence spectra (see Ref. [16, 29]) until now, and (ii) the fact that in some cases (see e.g. Ref. [30]) in the weak wave turbulence regime, a wealth of power-law solutions in the $(\alpha, \beta)$ plane can be found numerically as stationary solutions to the wave kinetic equations, although it is not clear whether such solutions are attractive, nor whether they are stable. The link between the dimensional argument given here which allows to recover the $\alpha = 7/3$ precursor spectrum, and the self-similar argument given in Ref. [16], which is compatible with the $\alpha = 7/3$ spectrum, remains to be clarified. Indeed, in Ref. [16] one recovers the numerically observed law of self-similar decay of the energy spectrum, assuming that the energy spectrum scales in the precursor phase as $k^{-7/3}_\perp$, whereas in this paper we give a heuristic justification to that same $-7/3$ law in terms of dimensional arguments compatible with three-wave interactions and with the assumption that there is no $k_\parallel$ dependency in the precursor solution. We show furthermore that this law stems from the same type of unified approach that also gives the other known spectra for MHD turbulence. In order to ascertain the validity of the model derived here, numerical computations with the full three-dimensional MHD equations can be envisaged but being able to distinguish between different power laws which are close in their respective spectral indices may prove difficult. This point is currently investigated and
will be presented elsewhere.

The argumentation delineated here can be used for other types of wave turbulence, like in the case of whistler waves (see Ref. [47]), inertial waves (see Ref. [4]) or gravity waves (see Ref. [48]). For whistler waves that can be encountered in Hall MHD or electron MHD, the characteristic wave period scales as $\tau_W \sim [k_\perp k_\parallel]^{-1}$ (see e.g. Ref. [49]). One finds, following the same analysis as presented before in this paper, that $k_\parallel \sim k_\perp^{1/3}$ (see also Ref. [50]); hence, the prediction for scaling exponents as defined in equation (4) for the whistler turbulence energy spectrum becomes: $3\alpha + \beta = 8$. (Note that here, the eddy turnover time is based on the magnetic field, i.e. $\tau_{NL} = [v k_\perp]^{-1} = [b k_\perp^2]^{-1}$, since $v \propto \nabla \times b$.) The known constant–flux solution, $\alpha = 5/2$ and $\beta = 1/2$, to the wave kinetic equations derived in Ref. [47] is recovered again as well as the strong turbulence prediction, $\alpha = 7/3$ and $\beta = 1$ (see e.g. Ref. [51]).

Finally, when considering inertial waves in rotating turbulence for a Navier-Stokes fluid, the wave period scales as $\tau_I \sim k_\perp/k_\parallel$. The same analysis as before leads to the following relationship between scaling exponents: $3\alpha + 5\beta = 10$. The known constant–flux solution to the wave kinetic equations written in Ref. [4] for wave turbulence corresponds to $\alpha = 5/2$ and $\beta = 1/2$, which does fulfill the above relationship and we recover the Kolmogorov prediction for strong turbulence as well, viz. $\alpha = 5/3$ and $\beta = 1$ for a weak rotation rate (see Appendix). This prediction again could be tested using direct numerical simulations, and the possible existence of precursors could also be studied and checked against the compatibility with the above relationship between the scaling exponents $\alpha$ and $\beta$ in the rotating case.

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Appendix

We assume that the characteristic wave period scales in general as

$$\tau_w \sim k_w^\gamma k_w^\delta; \quad (12)$$

recalling now that the energy spectrum is taken to be of the form $E(k_\perp, k_\parallel) \sim k_\perp^{-\alpha} k_\parallel^{-\beta}$, one finds as the fundamental relationship between parallel and perpendicular wavenumbers of the problem in terms of its four scaling exponents:

$$k_\parallel \sim k_\perp^{-(\gamma-2\alpha+4)/(\delta-2\beta+2)} \sim k_\perp^{-(3\gamma+2)/3\delta}. \quad (13)$$

Note that the last expression comes from the constant ratio assumed between the nonlinear time and the wave period. Hence, the generalization of equation (11) for waves with the above dispersion relation and with the turnover time expressed as

$$\tau_{NL} = \ell / u_\ell \quad (14)$$

is found to be:

$$(3\gamma + 2)(1 - \beta) = \delta(5 - 3\alpha). \quad (15)$$

This indeed recovers equation (11) for Alfvén waves and the relation for inertial waves as well (note that in the case of whistler waves, as already mentioned above, the turnover time is expressed in terms of the magnetic field).

\[ k_{||} \sim k_{\perp}^{2/3} / B_0 \]