HYDRODYNAMIC AND MAGNETOHYDRODYNAMIC TURBULENCE: A RAPID OVERVIEW

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1. Abstract
This text aims at being a soft introduction to hard turbulence. A rapid trip through some of the key concepts in turbulence today is given, with their possible consequences for applications to a few selected topics, in particular the solar environment (intertemittency in the solar wind, and heating of the solar corona), and the role of turbulence in the interstellar medium. Several one-dimensional models are discussed, and the MHD case is particularly emphasized.

2. Introduction
Some of the key concepts relevant to turbulence, together with a few examples taken from MHD as can be encountered in astrophysical flows are illustrated in this short review on fluid and MHD turbulence. What is turbulence made of? Since the times of Kolmogorov and in fact before, we see it as being a superposition of three kinds of structures: (i) large-scale ($L_0$) energy-containing eddies which are not universal but reflect both the instability mechanisms that give rise to them and the boundary conditions that contain them (although nonlinearities can build excitation at scales larger than $L_0$, see §5.4); (ii) inertial–range structures that transport the energy in a loss–less fashion; and (iii) small-scale dissipative eddies. The
actual mechanisms whereby a large-scale blob of vorticity transforms into a dissipative scale are largely unknown, except for the initial exponential phase, although some ideas are emerging (more on that later).

This constant flux engine (to within intermittency corrections, see §3.1 and §3.2) leads to a power–law distribution of energy among modes $E(\kappa) \sim \kappa^{-n}$, with $n$ depending presumably on the problem at hand; for example $n = 5/3$ for the classical three-dimensional Navier–Stokes (NS) incompressible turbulence (Kolmogorov, 1941), and $n = 3/2$ for MHD turbulence and waves (Iroshnikov, 1963; Kraichnan, 1965; hereafter referred to as the IK phenomenology; see e.g. Pouquet (1996) for a recent review). This engine to produce small scales stems from the nonlinearity of the primitive equations: Fourier transforming them leads to a convolution term, i.e. mode coupling. The $5/3$ law for incompressible flows can be recovered from simple dimensional analysis by stating that the mean flux of energy to small scale $\tau = dE^V/dt \sim v^3/l$ is constant (here, the density is constant and can be taken equal to unity), and evaluating $v^3$ through the relationship $v^3 \sim kE^V(\kappa)$. The boundary between weak and strong turbulence on the one hand, and between cascading and shock formation on the other hand is somewhat fuzzy, in part because power–law solutions for the energy spectra obtain i.e. because in all cases a multitude of modes are actually excited, and also because of the presence of coherent structures in both cases (either shocks or vorticity filaments).

One dimensional models of nonlinear behavior are treated in Section 2 (with an application to the problem of the mono–sidedness of proto–stellar jets), the two-dimensional case in Section 4 (with an application to the heating of the solar corona), and the three–dimensional case in Section 5, where the dynamo problem (generation of both small–scale and large–scale magnetic fields) is also briefly mentioned. Section 3 deals mostly with intermittency (i.e. the scarcity of strong small-scale structures, and the departures from Gaussian probability distribution functions, as observed in the solar wind, in solar magnetic fields and in the interstellar medium), and Section 6 describes succinctly a plausible model for the interstellar medium at the scale of the kiloparsec. Section 7 is a brief conclusion.

3. The one–dimensional case

3.1. A FLUID MODEL

There are basically three cartoons that can be drawn for nonlinear mode coupling: a velocity discontinuity due to nonlinear steepening; a shock smoothed by dissipation; and a balance between steepening and dispersive effects, leading to soliton–like behavior. In order to see this, let us write the following *ad hoc* one–dimensional equation:
\[
\frac{\partial u}{\partial t} + A_1 u \frac{\partial u}{\partial x} = A_2 \nu \frac{\partial^2 u}{\partial x^2} + A_3 \chi \frac{\partial^3 u}{\partial x^3} + A_4 \mu \frac{\partial^4 u}{\partial x^4},
\]
and let us consider various cases according to what the \( A_i \) parameters are. Of course, the left-hand side of the above equation, with \( A_1 = 1 \), is the Lagrangian derivative (following the motion of the fluid particle), which is thus the ‘natural’ choice for this parameter. However, we shall see below that for one particular case (the Korteweg de Vries equation), another choice may be convenient for setting the equation in its “classical” form. As for the other \( A_i \ (i \neq 1) \) parameters, there are either 1 or 0, depending on what equation we want to examine in the following.

When \( A_i \equiv 0 \ \forall i \neq 1 \), and \( A_1 = 1 \), the temporal evolution of the velocity starting from smooth initial conditions leads to the development of a shock (velocity discontinuity) forming in a time \( T_0/U_0 \) where \( T_0 \) and \( U_0 \) are respectively the characteristic scale and velocity difference in the system. Of course this discontinuity will be prevented either by viscosity (\( A_2 \nu \neq 0 \)) or by dispersion (\( A_3 \chi \neq 0 \)). For example with \( A_3 = A_4 = 0 \) and \( A_1 = A_2 = 1 \), one has Burgers’ equation. Its solution can be found by changing it to a linear equation, using the Hopf–Cole transformation; writing that the velocity derives from a scalar potential:

\[
u(x, t) = \partial_x s(x, t)
\]
leads to \( s_t + u^2/2 = \nu u_x + f(t) \). Choosing now for the potential \( s(x, t) \):

\[
s(x, t) = -2\nu \ln \Psi(x, t)
\]
we obtain:

\[
\Psi_t = \nu \Psi_{xx} - \Psi f(t)/(2\nu).
\]
In order to arrive at the desired heat equation, we have to eliminate the last term in the above equation; to that effect, we use the gauge freedom of the transformation between \( u(x, t) \) and its potential. Indeed, the fields \( \hat{s}(x, t) = -2\nu \hat{\Psi}(x, t) \) with \( \hat{\Psi}(x, t) = \Psi(x, t)G(t) \), and \( s(x, t) \) yield the same physical velocity field \( u(x, t) \). We now have

\[
\Psi_t = \nu \Psi_{xx} - \Psi [\hat{G}(t)/G(t) + f(t)/(2\nu)]
\]
The heat equation obtains with the choice of gauge \( \hat{G}(t)/G(t) = -f(t)/(2\nu) \).

We thus see that in the dissipative case, an internal boundary layer forms of thickness \( \sqrt{\nu} \) (the solution in the vicinity of the smoothed shock is locally proportional to \( \tanh(x/2\nu) \)). We conclude that the actual ratio of excited length scales in the problem varies as \( R_{1/2} \) (and not \( R_V \)) where

\[
R_V = U_0 T_0/\nu
\]
is the (kinetic) Reynolds number, with $\nu$ the kinematic viscosity. Other power laws in this scaling can arise. For example, Kolmogorov scaling gives $R_{\text{v}}^{-3/4}$, obtained by equating at the dissipative scale $\ell = \ell_D$ the viscous time $\ell^2/\nu$ and the eddy turn-over time $t_e \sim \ell/\nu$, hence the Kolmogorov length $\ell^{(K)}_D \sim R_{\text{v}}^{-3/4}$ (in units of $L_0$). In MHD, the length analogous to the Kolmogorov scale within the framework of the IK phenomenology (that is, with an energy spectrum $E(k) \sim k^{-3/2}$) leads to a $R_{\text{v}}^{+2/3}$ scaling.

On the other hand, returning to equation (1) and retaining only nonlinearity and dispersion ($A_1 = 6$, $A_2\nu = A_4\mu = 0$, $A_3\chi = -1$), i.e. writing the Korteweg de Vries equation or in short KdV, a different solution emerges, namely $u = \text{sech}^2(\xi/2)$. Here, the change of variable into a traveling wave solution

$$u(x,t) = \frac{1}{2} c_0 f(\xi)$$

with

$$\xi = \frac{1}{c_0}(x - c_0 t)$$

is useful (we have anticipated that the amplitude of the solitary wave is a function of its speed). You can check that this is a solution by integrating once, multiplying by $f'$, and integrating again; by supposing that for $\xi \to \infty$, the function and its derivative vanish, you arrive at the relationship $[f''] = f^2(-f + 1)$, hence the $1/cosh^2$ solution. This corresponds to a soliton – propagating the faster the stronger – in which there is exact balance between dispersion and nonlinear steepening (see e.g. Drazin and Johnson (1989) for an introduction). We can simply remark that soliton equations have an infinite number of invariants. Following Kruskal (1974), we write

$$I_n = \int T_n(x)dx$$

where $I_n$ is the $n^{th}$ invariant of flux $X_n$, viz.:

$$\partial_t T_n + \partial_x X_n = 0$$

The first invariant $T_1 = \langle u \rangle$ is the momentum and the second invariant $T_2 = \langle u^2/2 \rangle$ the energy; $T_3 = \langle u^3/3 - 6\partial_x u^2 \rangle$. These invariants can be generated by the Miura transformation for even powers of $e$ (the odd powers do not lead to independent invariants):

$$\partial_t w + \partial_x \left( w^2/2 + (\epsilon^2/18) w^3 + 6(\partial^2 w/\partial x^2) \right) = 0$$

with $u = w + i\sqrt{6} e \partial_x w + (\epsilon^2/6) w^2$. Finally, note that the solitonic solutions to the KdV equation can be built by searching the extremum of $I_n$ subject
to the constraints that all $I_p$ are constant $\forall p < n$ (recall that, from the theorem of Noether (1918), conservation equations can be deduced from

the invariance of a Lagrangian system). Note that, for the KdV equation

with a friction term $\nu'u$, self-organization at large scale of solitons take

place, a phenomenon that does not occur when the usual Laplacian term

is used (Hasegawa, 1985).

With $A_{3\chi} = 0, A_1 = 1$ and $A_2 = A_4 = -1$, one has the Kuramoto–

Sivashinsky equation where the $\nu$ term is a forcing term (involving the

velocity field itself) and the highest derivation term is dissipative.

All such one-dimensional equations have been studied in their own right,

in part because they emerge naturally in a variety of physical (and other)

problems — using e.g. asymptotic techniques. They can be viewed as well as

prototypes for competing effects in non-linear phenomena; see for example

the large body of literature in the context of magnetospheric physics (e.g.

Spangler, 1990; 1994 and references therein) for the Derivative NonLinear

Schrodinger equation or DNLS, as applied e.g. to the Earth’s bow shock;

in the context of the interstellar medium, see Adams, Fatuzzo & Watkins

(1994); and for the thermal instability and the dynamics of fronts, see


In MHD, the following equations were proposed by Thomas (1968, 1970)

with $\delta = 1$ as a model for MHD interactions in the spirit of the Burgers’

model, and by Burgers (with $\delta = -1$) as an extension of its first equation

to compressible gas:

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \delta b \frac{\partial b}{\partial x} + \nu \frac{\partial^2 u}{\partial x^2} \tag{2}
\]

\[
\frac{\partial b}{\partial t} + u \frac{\partial b}{\partial x} = \delta b \frac{\partial u}{\partial x} + \eta \frac{\partial^2 b}{\partial x^2} \tag{3}
\]

With $\nu \equiv 0$, $\eta \equiv 0$, these equations conserve for $\delta = 1$ a “pseudo” energy

(namely $< u^2 + b^2/3 >$), and for $\delta = -1$, the cross-correlation between

the velocity and the magnetic field $< ub >$. They show a dynamo effect,

they lead to shock-like as well as cusp-like solutions (Passot, 1987; Ponty,

1990) and follow a Painlevé integrability property in some cases (Passot &

Pouquet, 1986). When $\delta = 1$ (resp. $-1$), one in fact deals with an isentropic

$\gamma$-law, with $\gamma = -1$ (resp. 3). In the former case, it may correspond to

complex behavior of astrophysical flows when heating and cooling are added

to the energy equation, as occurs in the interstellar medium (see §6.3).

3.2. A NUMERICAL MODEL

Because, for the Burgers equation, the thickness of the viscous layer scales

as $\sqrt{\nu}$, this means that a large fraction of the interacting modes are in fact in
the dissipation range. In three dimensions, using the ansatz $L_0/(2\Delta x) \sim R_V^a$
(where $\Delta x$ is the grid spacing) and with $a = 1$ to simplify the argument, one can show that the fraction of modes pertaining to the dissipative range
varies like $(1-R_V^{-3/2})$ which tends to unity for $R_V \to \infty$. Already for $R_V \sim 10^3$ (i.e. on a grid of $N = 10^9$ points, with the scaling of $a$ chosen here), only of the order of $10^6$ modes are in the inertial (and energy-containing) ranges. Clearly some sort of grid refining or adaptation is needed here. But for a turbulent flow, where many shock-like or filament-like structures develop, such a grid may become too cumbersome to handle. On the other hand, in order to obtain a gain by a factor of two in resolution (i.e. taking a grid with $\Delta x' = \Delta x/2$), the CPU cost is 16 times heavier (for an explicit temporal scheme), and the memory requirement 8 times more.

One can rely on other methods, for example by trying to reduce the number of modes in the dissipative range. Many codes dealing with compressible flows use the Euler (conservative) fluid – and MHD – equations, with some sort of numerical viscosity in shocks. Another such method consists in setting in equation (1) $A_2 = 0$ and $A_4 = -1$, i.e. using for the multi-dimensional problem a bi-Laplacian (in such “hypeviscosity” methods as they are called, powers of $k^2$ or $\partial^2$ higher than 2 can be used). One may also decimate modes in the dissipation range, retaining only one mode out of two in some interval at the onset of the dissipation range, one out of four a bit further in wavenumber, and so on. Such methods have been tested and reproduce well the global energetics of the flow, but not necessarily the detailed structures at small scale (see e.g. Meneguzzi, Politano, Pouquet and Zolfer (1996) and references therein).

A method vastly different in its conception has been developed recently in the context of ocean modeling (Eby and Holloway, 1994). It consists in introducing a small tendency (through a Laplacian term with a negative viscosity) for the computed oceanic circulation to evolve towards its preferred state evaluated from statistical mechanics (although the flow is both driven and damped), taking into account the invariants of the inviscid equations, a state which depends on the bottom topography of the oceanic floor because of the induced vorticity due to the rotation of the Earth. The reason why it apparently works so well may be due to the fact that the ocean has a slow response time (typically of the order of 1,000 days) to the quick meteorological (say, Eolian) excitation that drives it. It is tempting to remark here that, similarly, the response time of the interstellar medium at large scale is at least a factor 100 longer than the small-scale thermal driving (heating and cooling linked with star formation, see §6).
3.3. THE HADA EQUATION AND THE ASYMMETRIC STABILITY OF PROTO-STEMAR JETS

The compressible MHD equations, when including the Hall term in a generalization Ohm’s law, can be written as:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

$$\rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla \left( P + \frac{\rho |\mathbf{B}|^2}{2} \right) + \mathbf{B} \cdot \nabla \mathbf{B}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left( \mathbf{u} \times \mathbf{B} \right) - \frac{1}{\xi} \nabla \times \left( \frac{1}{\rho} \left( \nabla \times \mathbf{B} \right) \times \mathbf{B} \right)$$

$$\nabla \cdot \mathbf{B} = 0 \ ,$$

where \( \mathbf{B} \) has been normalized to a velocity. The \( \xi \) factor, in front of the dispersive Hall term is the ionization fraction of the gas (provided it is not too low, see Bacciotti, Chiuderi and Pouquet (1996), since in that case it would be dominated by the ambipolar drift term.

In the context of magnetospheric physics, Hada (1993) has recently proposed a set of coupled equations stemming from the one-dimensional version of the MHD equations above, but with several simplifying assumptions; the Hada system is in fact very close to the system of Thomas (see equations (3)) and thus can provide an a posteriori justification for them. Now, \( \mathbf{b} = b_y + i b_z \) is a complex divergence-free magnetic field (transverse to a uniform \( B_0 \equiv 1 \) field in the \( x \)-direction):

$$\frac{\partial \mathbf{u}}{\partial T} = -\frac{\partial}{\partial \mathbf{X}} \left( \frac{2}{\Gamma u^2 + \Delta u + \rho b^2}{2} \right)$$

$$\frac{\partial \mathbf{b}}{\partial T} = -\frac{\partial}{\partial \mathbf{X}} \left( \frac{u \mathbf{b}}{\xi \mathbf{b}^2} \right)$$

$$\frac{d}{dt} \left( P/\rho^2 \right) = 0$$

with \( \Gamma = \frac{\gamma - 1}{\gamma} \). Note that in Hada (1993), the medium is assumed to be fully ionized (corresponding to \( \xi \equiv 1 \). In this complex-variable formulation, one has \( \mathbf{b} = i \partial_x \mathbf{a} \) where \( \mathbf{a} = a_y + i a_z \).

These equations are conservative (although dissipative terms could be added, e.g. by making \( \xi \) complex) with invariants \( < u^2 + b^2 > \) (energy) and \( < (ab^* + ba^*)/2 > = < a_z \partial_x a_y - a_y \partial_x a_z \) the magnetic helicity. These equations stem from an asymptotic expansion around a uniform magnetic field in the \( x \) direction, uniform density and zero velocity as the basic state. A scaling on time and space is taken so as to conserve the linear dispersion.
relation \( \omega \sim -k^2 \) stemming from the Hall term and the adiabatic equation
\( d_t(P/\rho^m) \) is transformed using the further constraint that within the plasma,
the ratio of gas to magnetic pressure is very close to unity (without that
extra assumption, one gets the DNLS equation). The Hada equations can
be taken as an extension of the equations of Burgers and of Thomas to a
case where dispersive effects are included.

Hada showed (using a modulational instability) that there is stability
(resp. instability) according to the polarization of the propagating wave,
a result that extends to the case \( \xi \neq 1 \) (Bacciotti et al., 1996). In fact
this asymmetry in the instability diagram leading to the formation of weak
shocks can be used to explain the asymmetry in the emission knots of
protostellar jets (Bacciotti et al., op. cit.), which are observed to be mono-
sided in many instances, as for example HH34 (Reipurth & Heathcote,
1992) and HH111 (Reipurth, Raga and Heathcote, 1992; see also T. Wray,
this Volume, as well as several other Lectures on the topic of jets).

These Herbig–Haro knots can also be interpreted as fragments – or
bullets (Stone and Norman, 1994; Stone, Xu & Mundy, 1995) emanating
from stellar winds subject to hydrodynamical (or MHD) instabilities (i.e.
due to the interaction of the shell produced by the stellar wind, and the
jet itself), as may also occur in supernovae remnants (Franco, Ferrara,
Rozyczka, Tenorio–Tagle and Cox, 1993), and in fact in planetary nebulae
as well (MacLow, 1995).

4. Diagnostics

4.1. STRUCTURE FUNCTIONS

In the absence of a systematic theory of turbulent flows, one relies on mod-
els, on phenomenology, on experiments (including numerical experiments)
and observations. Astrophysical flows have high Reynolds numbers and are
often magnetized. The diagnostics one can perform on such flows are two-
fold. On the one hand, because of the strong noise of the signal, one deals
with statistics. On the other hand, one analyzes the physical structures
that develop (such as intense localized vortices, as those that develop in
the atmosphere of the Earth). The energy spectrum (e.g. the two-point
one time Fourier transform of the velocity correlation function) is the most
common statistical diagnostic, but one also can consider higher moments.
Defining the structure functions as usual as

\[ \delta v_\ell = u(x + \ell) - u(x) , \]

where \( u(x) \) is the longitudinal component of the velocity (or magnetic field)
and assuming self-similar behavior in a range of (inertial) scales, one writes:
\[ <\delta v_t^p > \sim \ell^{\zeta_p} , \]

where \( \zeta_p \) are the unknown scaling exponents to be determined.

Energy is being transferred at an average rate \( \bar{\tau} = U_0^3 / L_0 \). With the further assumption that the energy dissipation in a sphere of radius \( \ell \) scales as well as \( \epsilon_\ell = u_\ell^2 / \ell \) (Kolmogorov refined similarity hypothesis), and defining \( \tau_p \) as the anomalous scaling exponent of transfer – and dissipation – (viz. \( < \epsilon_\ell > \sim \ell^{\tau_p} \)), one obtains:

\[ \zeta_p = p/3 + \tau_p/3 . \]

When no fluctuations of \( \epsilon \) are allowed, the Kolmogorov (1941) scaling is recovered, namely \( \zeta_p = p/3 \). When fluctuations in the energy transfer rate are allowed leading to intermittency, a different scaling obtains. Several models have been proposed to explain the observed departure of scaling exponents of high-order structure functions from a linear \( p/3 \) law, some of which have also been extended to MHD (see Carbone (1994), and for an introduction, Biskamp (1994)). Such models are restricted by exact relations for some \( \zeta_p \) exponents for example, for incompressible homogeneous isotropic and stationary fluids, \( \zeta_3 \equiv 1 \), stemming from the conservation of energy (see for example in Frisch (1995) for details). Similarly, for a passive scalar \( \theta \) such as temperature, one has \( <\delta \theta_\ell^2 \delta \epsilon_\ell > \sim \ell \) (Yaglom, 1949); this result can be extended to two-dimensional MHD, replacing the scalar \( \theta \) by the (scalar) magnetic potential (Caillol, 1995; Caillol, Politano and Pouquet, 1995), although it is an active scalar – it acts on the velocity field through the Lorentz force. This relationship is well verified numerically (although the inverse cascade of magnetic potential leads to a non-stationary behavior).

4.2. THE SHE-LEVÊQUE MODEL

Recently, a model has been proposed (She and Levêque, 1993, referred hereafter as SL) to determine the \( \zeta_p \) exponents, and which agrees to better than \( 1\% \) with experimental data (using the Extended Self-Similarity, or ESS, of Benzi, Ciliberto, Tripiccione, Baudet, Massaioli and Succi, 1993). The SL model relies on equation (9) and furthermore supposes that (i) the characteristic time of the most intermittent structure scales as \( t_\ell \sim \ell^{2/3} \); (ii) the co-dimension of dissipative structures is two (i.e. they are filaments, as indicated – clearly for vorticity, not so clearly for dissipation – by several numerical simulations for three-dimensional incompressible flows, see e.g. Vincent and Meneguzzi, 1991) and (iii) there is an underlying log-Poisson statistics of energy transfer (a multiplicative process leading to a log-distribution, and Poisson statistics because of the rarity of events); note
that richer statistics can obtain for such flows (Dubrulle, 1994; Schertzer, Lovejoy and Schmidt, 1995). The latter condition specifically reads, following Dubrulle (1994):

\[ < \Pi_t^p > = B_p < \Pi_t > \frac{1 - \beta_p}{1 - \beta} \]  

(10)

with \( \beta \neq 1 \) and

\[ < \Pi_t > = \epsilon_t / \epsilon^{(\infty)} \]  

(11)

where \( \epsilon^{(\infty)} \) is the maximum energy available to dissipate in the flow. This relationship has been verified in the laboratory (Ruiz–Chavarría, Baudet and Ciliberto, 1995), at Taylor Reynolds numbers \( R_T \) ranging from 100 to 800 (where \( R_T \) is based on the Taylor scale \( \lambda_T \) constructed from the energy spectrum \( E(k) \) as \( \lambda_T \sim \left[ \int k^2 E(k) dk / \int E(k) dk \right]^{-1/2} \), in other words weighing the small scales that develop in the process of forming turbulent flows). Note that the assumption that \( \epsilon \sim (\partial U / \partial x)^2 \) is made (which holds for strictly homogeneous and isotropic turbulence) in this experimental verification.

In the SL model, \( \epsilon^{(\infty)} \) is evaluated as

\[ \epsilon^{(\infty)} = U_0^2 / t_t \]

and is assumed to be finite (which seems to be verified experimentally up to \( R_T \sim 2,000 \), in so far as the probability distribution functions of the velocity gradients are not power laws in the wings, but remain stretched exponentials. Finally, \( \beta^{SL} \) is set at \( \beta = 2/3 \) (corresponding to the Kolmogorov scaling for \( t_t \)); this leads to \( \zeta^{SL} = \frac{2}{3} + 2(1 - (2/3)^{3/3}) \).

The model has been extended to MHD in the context of the IK phenomenology with the assumption that dissipative structures are now sheets instead of filaments (Grauer, Krug and Marliani, 1994); this leads (with \( t_t \sim \ell^p \) to \( x = 1/2 \) and \( \beta = 1/2 \) (see below, equation (13)); it implies \( \zeta^B = 1 \), a relationship for which there is no proof as yet.

These models can be generalized further to a variable scaling time and variable co-dimension, both for fluids (Dubrulle, 1994) and in MHD (Politoano and Pouquet, 1995) with in the latter case, using the IK phenomenology:

\[ \zeta^B_p = \frac{p}{4} (1 - x_B) + C_0^B (1 - \beta_B^{p/4}) \]  

(12)

together with

\[ C_0^B = \frac{x_B}{1 - \beta_B} . \]  

(13)

Comparing with the observational data of Burlaga (1990) obtained with the Voyager spacecraft, the law fits well the data with \( x_B \sim 1/2 \) and \( \beta_B \sim 1/2 \), although the error bars are too large in order to permit to discriminate
among various fluid (Kolmogorov like) and MHD (IK like) models, although
a recent analysis (Carbone, Velti and Bruno, 1995) tends to show a dif-
ferent behavior at high order for fluid and for MHD flows. Temporally
well-resolved data sets will be very useful in this instance (as for example
with the Ulysses spacecraft).

Although the linear scaling laws for the anomalous exponents of struc-
ture functions may differ (i.e. \( p/3 \) for Kolmogorov fluids, and \( p/4 \) for IK
magneto-fluids) due to different characteristic times for the basic inter-
actions, the intermittency corrections may fall into one (or a few) universal-
ity classes. Indeed, using normalized exponents \( \tilde{\zeta}_p^U = \zeta_p^U / \zeta_3^U \) for the \( U \) variable,
the same approximate values obtain experimentally (Ciliberto, private com-
munication) for Navier–Stokes turbulence, convection and MHD numerical
simulations in two space dimensions. In that light, it should be remarked
that when computing the predicted theoretical values for the She-Leveque
model of fluid turbulence and for the SL–MHD model, normalizing the lat-
ter by \( \zeta_3^B \), one finds values that are very close; for example, \( \zeta_2^{V,SL} \approx 0.696 \)
vs. \( \zeta_2^B \approx 0.692 \), \( \zeta_6^{V,SL} \approx 1.78 \) vs. \( \zeta_6^B \approx 1.769 \) and \( \zeta_{10}^{V,SL} \approx 2.59 \) vs. \( \zeta_{10}^B \approx 2.66 \).
Only accurate data will allow to compute high-order structure functions,
and thus to discriminate between models; this means ideally data sets of
\( \sim 10^{10} \) or more points. Hence the need, in solar wind data, to take the
longest records possible with a good signal–to–noise ratio.

5. The two–dimensional case

5.1. ARE NONLINEAR INTERACTIONS SELF–DEFEATING IN MHD?

We saw in §2.1 that nonlinear interactions between modes can lead, in sim-
ple cases at least, to singularities corresponding in the language of struc-
tures to the formation of shocks (for compressible flows) and presumably
strong velocity gradients. But if there was a way to damp these nonlinear
interactions, they would not be so efficient and the flow could evolve in a
quasi–linear manner. For example, writing first the MHD equations in the
incompressible case as:

\[
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho_0} \nabla P + \nu \nabla^2 \mathbf{u} + \mathbf{j} \times \mathbf{b} ,
\]

\[
\frac{\partial \mathbf{b}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{b}) + \eta \nabla^2 \mathbf{b} ,
\]

\[\nabla \cdot \mathbf{u} = 0 \quad ; \quad \nabla \cdot \mathbf{b} = 0 ,\]

with \( \mathbf{b} = B / \sqrt{\mu_0 \rho_0} \) the Alfvén velocity, \( \mu_0 \) is the permeability, \( \eta \) is the
magnetic diffusivity and \( P \) the pressure, we see immediately that \( \mathbf{v} = \pm \mathbf{b} \)
is an exact solution of the conservative equations (\( \nu \equiv 0, \eta \equiv 0 \)). If such
a flow is attractive in the space of solutions, no further evolution will take place on the dynamical time-scale (of course some dissipation will take place on the slow diffusive time scale when small amounts of viscosity and magnetic diffusivity are reintroduced).

It has been shown that the correlation coefficient

$$\rho_C = 2 < \mathbf{v} \cdot \mathbf{b} > / < v^2 + b^2 >$$

grows with time (see Pouquet (1993) and references therein). So the flow does evolve towards a \( \mathbf{v} = \pm \mathbf{b} \) state! In fact, even when \( \rho_C \sim 0 \) globally (recall that it is a volume integral of a non positive definite quantity), there may be (and in fact, there are, as observed for example in numerical simulations, see Passot, Politano, Pouquet and Sulem, 1990) within the flow regions where locally \( |\rho_C| \sim 1 \) \( i.e. \) rather static regions at the border of which all action takes place in thin dissipative layers. Closure models of MHD turbulence do indicate that \( \rho_C \) is close to zero only around the dissipative scale. This image of MHD turbulence could be backed up by new numerical data at high Reynolds number, at least in two dimensions. In three-dimensional incompressible fluid turbulence, a depletion of non-linearities (from what is expected for a purely random Gaussian velocity field) has been observed on numerical data (Kraichnan and Panda, 1988). It could be attributed to a local quasi bi-dimensionalization of the swirling flows that are induced (through the Biot–Savart law) by the strong vortex filaments that develop in such 3D fluids (see §5.2). The equivalent analysis has not been done for MHD flows.

5.2. DOES RECONNECTION OCCUR AT A FINITE RATE FOR \( R \to \infty \)?

Because the Reynolds numbers for the sun are so large, it is often assumed that dissipation mechanisms are negligible. But there is in fact the \textit{a priori} possibility that dissipation can occur at a finite rate for \( R \to \infty \), provided strong enough velocity gradients develop. This may be of importance for astrophysical flows and in particular in the framework of heating the solar corona, a problem for which recent observations (those of YOHKOH in particular, see Tsuneta, this Volume) clearly indicate that magnetic reconnection plays a fundamental role in shaping the dynamical behavior of the corona (\textit{i.e.}, here we do not choose the static framework considered elsewhere in this Volume, see \textit{e.g.} Low; and Parker).

This question, for a flow that develops small scales in time, amounts to: is there a finite limit, as the Reynolds number \( R \to \infty \), for the dissipation \( \mathcal{D} \)? For an incompressible MHD fluid, the dissipation is given by

$$\mathcal{D} \sim \nu \langle \omega^2 \rangle + \eta \langle j^2 \rangle ,$$
where $\langle \omega^2 \rangle$ and $\langle j^2 \rangle$ are the kinetic and magnetic enstrophies, or in other words the total squared vorticity and current density. The numerical or analytical determination of a limit for $\mathcal{D}$ remains an open challenge in three dimensions (and for incompressible neutral fluids as well) unless a mean magnetic field $B_0$ is present, strong enough to suppress neutral points of the random field.

In two dimensions, there is clear numerical evidence that at early time the flow does not develop any singularity: the growth of small scales is only exponential. However, for later times, it has been shown using both the primitive equations (Politano, Pouquet and Sulem, 1989) and hyper-viscosity codes (Biskamp and Welter, 1989; Passot, Politano, Pouquet and Sulem, 1990) that $\mathcal{D} \rightarrow \mathcal{D}_b$, a finite value. This phenomenon is probably linked to the tearing mode instability which leads to the breaking of long thin current sheets into small islands, thus producing more dissipation (catastrophically). Is it a numerical artefact, due to either too long a time of computation, or (more probable) too low a Reynolds number? Or is it the case that indeed $\mathcal{D}_b \neq 0$?

5.3. A PLAUSIBLE MODEL FOR HEATING THE SOLAR CORONA

As just discussed, the formation and disruption of current layers may be responsible for the heating of the solar corona. It is well known that there is a whole spectrum (both in intensity and in duration) of solar flares, from the nano-flare to the major eruption; this spectrum in fact follows a power-law and the model of Lu, Hamilton, McTierman and Bromund (1993) is very successful in advocating a connection between such observations and the phenomenon of avalanches (and sand piles), linked to self-organized criticality. This model has been extended (see for example Vlahos (1993)) to more complex cases but the basic mechanism remains that, when a gradient becomes too large, the system relaxes to a lower gradient configuration by re-organizing the neighboring points, a reorganization which may propagate in some cases to the whole volume, leading then to major disruptions. The question remains of what is the physical mechanism responsible for the MHD equivalent to the “avalanches”. A recent calculation of two-dimensional MHD incompressible fluids with a forcing term that mimics the random motions of foot points of solar flux tubes shows that the distribution of peaks in the dissipation is random, with numerous small events and rare big events (Einaudi, Velli, Politano and Pouquet, 1996). Similar computations in three-dimensional MHD are performed by Galsgaard and Nordlund (this Volume). This phenomenon is linked to the intermittency of current layers, both spatial and temporal. However, it is not clear whether the major flares can also be produced in such a fashion.
6. The three–dimensional case

6.1. THE BETCHOV RELATION

A simple relation derived by Betchov (1956) links the development of small scales in turbulence (and hence the growth of enstrophy) to the formation of vortex sheets. Starting from the Navier–Stokes equations (4) with \( \mathbf{B} \equiv 0 \), and omitting for brevity to write pressure and viscous terms, one can derive the equation for the time derivative of the velocity gradient tensor; dotting it with \( \partial_t u_j \) and averaging over the whole flow using the hypothesis of isotropy, homogeneity and incompressibility, Betchov showed that:

\[
\frac{D < \omega^2 >}{Dt} \sim -s < a_s b_s c_s > ,
\]

where \( s \) is positive and where \( a_s, b_s \) and \( c_s \) are the three eigenvalues of the symmetrized \( \partial_t u_j \) matrix. Since \( a_s + b_s + c_s = 0 \) because of incompressibility, the sign in the equation above is linked to that of \( b_s \), i.e., the intermediate eigenvalue. In other words, small scales are produced with on average two positive eigenvalues of the rate of strain tensor: an initial roundish blob of vorticity is being stretched in two directions and squeezed in the third direction. Hence the production of vorticity sheets.

6.2. THE STRUCTURING OF VORTICITY INTO FILAMENTS

Numerical data – and now experiments in the laboratory – both indicate that the vorticity, when strong, is structured into filaments, except at early time where numerical data clearly indicates that sheets are formed (see e.g. Brachet, Meneguzzi, Vincent, Politano and Sulem, 1991). The reason is simple: these sheets are unstable (either to Kelvin–Helmholtz instability (Neu, 1984), or through a self–focusing mechanism (Passot, Politano, Sulem, Angilela and Meneguzzi, 1995) and filaments form that are both very strong and long–lived, although in fact they contain little vorticity (roughly 1 %); the background vorticity may still be organized into sheets. Similar structures are found in the compressible case (Porter, Pouquet and Woodward, 1995). Finally, let us remark that because of the analogy between the vorticity equation in the incompressible case (and neglecting diffusion coefficients)

\[
D_t \omega = \omega \cdot \nabla \mathbf{v}
\]

and the induction equation \( D_t \mathbf{b} = \mathbf{b} \cdot \nabla \mathbf{v} \), it has been conjectured that the magnetic field will develop in a similar fashion to vorticity, and indeed the small–scale dynamo favors flux tubes (Galloway and Frisch, 1986), at least in the kinematic phase. The nonlinear problem is poorly explored till
now, because of the high cost of computing in three dimensions (but see
Brandenburg, Jennings, Nordlund, Rieutord, Stein & Tuominen, 1995).

6.3. TRANSPORT COEFFICIENTS AND THE EXCESS OF MAGNETIC
ENERGY IN THE SMALL SCALES

Because we do not know how to solve nonlinear equations in general, we
cheat and simplify the problem. The concept of eddy viscosity arises from
the idea that, by coupling modes nonlinearly and transferring energy to
the small scales where it is ultimately dissipated, such an effect can be
modeled by an enhanced dissipation. A formalization of such a concept can
be achieved through the development of statistical theories — the pioneer
of which is the DIA (Kraichnan, 1959) — which lead to a set of integro-
differential equations for the energy spectrum.

Many such transport coefficients may be thus computed in a variety of
physical contexts. Both small-scale velocity and magnetic fields will con-
tribute to the dissipation of the velocity field; but because the induction
equation is linear in $\mathbf{B}$, in fact only small kinetic scale — to first order —
participate to its dissipation. This has shown in the context of clo-
sures of turbulence (see e.g. Pouquet, Frisch and Léorat, 1976) in three
dimensions. In other words, we can model in the framework of that approx-
imation the temporal evolution of the kinetic and magnetic energy spectra
as diffusion equations with an extra term for the kinetic energy. On that
basis, we can postulate that slightly more magnetic energy will be present
in the small scales, a phenomenon observed in many instances. A global
excess of magnetic energy can in fact be derived from statistical ensemble
of MHD turbulence (Stribling and Matthaeus, 1990). Note that if either
ambipolar drift or the Hall term are added to a generalized Ohm’s law,
leading to terms nonlinear in $\mathbf{B}$ in the induction equation, this result may
not hold any longer (the existence or not of inverse magnetic cascades in
these cases would be interesting to study). Different expressions emerge in
two dimensions (see Pouquet, 1978) because of the inverse cascade of mag-
netic potential linked to a negative magnetic diffusivity, involving small
magnetic scales as well.

6.4. TRANSPORT COEFFICIENTS AND THE DYNAMO PROBLEM

In fact in three dimensions, small-scale (kinetic and magnetic) when hel-
ical, lead to the destabilization of a large-scale magnetic field through
the well-known $\alpha$-effect (Steenbeck, Krause and Rädler, 1966) in the kin-
ematic regime (i.e. with a given velocity field since, at small amplitude, the
Lorenz force is negligible). The efficiency of this mechanism has been put
to doubt recently because even at low levels (well below equi-partition be-
tween kinetic and magnetic energy) a large–scale magnetic field seems to have a strong effect on the underlying Lagrangian chaos of the velocity, presumably quenching the $\alpha$–effect to quasi non–existence.

However, in the turbulent context, the nonlinear dynamo is viewed simply as the manifestation – exemplified with the help of transport coefficients – of the inverse cascade of magnetic helicity postulated on the basis of statistical equilibria of a truncated system of Fourier modes for which the quadratic invariants are conserved (cross correlation, magnetic helicity and total energy). Such inverse cascades are rather well documented (Hasegawa, 1985; see also Pouquet, 1993) in different contexts, e.g. two–dimensional Navier–Stokes and MHD incompressible turbulence, three–dimensional MHD, the Strauss equations (i.e. MHD in the presence of a strong uniform magnetic field), as well as for the Hasegawa–Mima equations for drift–wave turbulence – and Rossby wave turbulence (although the two may differ, see Hasegawa, 1985; and Kukharin, Orszag and Yakhot, 1995).

The important point is that such a mechanism leads to a non–stationary evolution: there is no real saturation of the growth of magnetic energy. It can in fact be shown in the framework of a simple model (Krishnam, 1979) confirmed by numerical experiments – albeit at low resolution (Meneguzzi, Frisch and Pouquet, 1981) – that magnetic helicity $H^M$ grows linearly with time, and because of the Schwartz inequality $E^M(k) > H^M(k)/k$, magnetic energy continues to grow as well. Of course, when the magnetic excitation reaches the size of the overall system, boundary effects come into play and this mechanism will be altered.

6.5. THE DYNAMICAL DEVELOPMENT OF SINGULARITIES

When assuming that the skewness of the velocity field

$$S_k \sim \langle \partial^3_{xxx} u \rangle \bigg/ \langle \partial^2_{xx} u \rangle^{3/2}$$

is time–independent, Betchov in fact showed that $\frac{D <\omega^2>}{Dt} \sim <\omega^2>$, leading to a singularity in a finite time. Experimentally and numerically, it is known that the skewness becomes constant after the enstrophy has reached its maximum, i.e. when dissipation sets in. Other models have been devised over the years which lead to singularity when (arbitrarily) neglecting in the primitive equations some (important?) term, or when setting for all times that some specific symmetry be enforced.

Following a theorem by Beale, Kato and Majda (1989), a good criterion for singularity is to follow in time the development of the peak vorticity $\omega_p$ (or the peak strain), because the theorem states that if a singularity occurs, then the supremum of vorticity must blow–up. There is numerical evidence that such singularities might occur (Kerr, 1985; 1993; 1995; Pelz
and Boratav, 1995) when using specific initial conditions. In the analysis of Bhattacharjee, Ng & Wang (1995), a singularity might occur depending on the sign of \( \partial_x^2 \mathcal{P} \) (where \( \mathcal{P} \) is the pressure) for an initial velocity field satisfying \( u_x = f(y), \ u_y = f(z) \) and \( u_z = f(x) \) (see also Bhattacharjee & Wang, 1992); furthermore, this model agrees with the analysis of Lundgren (1982) as to yielding a \( E^V(k) \sim k^{-5/3} \) spectrum.

7. Turbulence in the interstellar medium

7.1. Evidence of Turbulence Through Line Widths

The interstellar medium (or ISM) is a complex superposition of structures (as for example seen with IRAS and now ISO) encompassing a wide variety of scales, densities and temperatures. Several authors have advocated that it is turbulent (see for example the reviews by Scalo (1985) and by Falgarone (1995)). Evidence for such turbulence is to be seen in the power-law behavior when scaling velocity dispersion against scale of cloud (as first emphasized by Larson in 1981), in the presence of self-similar structures over four order of magnitude in scales, in the computation of fractal dimensions for the boundaries of clouds, and in the non-gaussian wings of velocity spectra, as for example studied in the High Latitude Cloud Ursa-HLC in the \( ^{12}CO(2-1) \) transition (Falgarone, Lis, Phillips, Porter, Pouquet and Woodard, 1994). The fact that with a superposition of modes with a given (Kolmogorov) spectrum but with random phases, one can reproduce almost as well (Dubinsky, Narayan and Philippa, 1995) the observational data (except for large excursions in the vorticity) shows that numerical simulations in three dimensions are still largely under-resolved as far as high Reynolds number turbulence is concerned ... Intermittency may also affect the chemistry balance in the medium, allowing for local high temperatures which may give rise to emission lines unexplained otherwise (Falgarone and Puget, 1995).

7.2. A Model of the ISM at the Kiloparsec Scale

Given the wide range of scales (from the kiloparsec down to the dense cores of 0.01 pc), densities (from roughly 1 particle per \( cm^3 \), up to \( 10^5 \)), and temperatures (from \( 10^6 K \) in the coronal phase, down to \( 10 K \) in dense cores), a direct numerical model of the medium is difficult, even if one resorts to adaptative grids. One can take a different approach, namely resolving scales down to a certain size below which (or above, or both) the physics will not be computed exactly but modeled. For example, radiative processes leading to heating because of ionization winds can be introduced by hand in the energy budget (this stellar heating is due to the eventual collapse of con-
densations as they form); cooling can be introduced as well following the classical temperature-dependent model of Dalgarno and McCray (1972). Several models have been developed in the literature (see e.g. Rosen, Bregman and Norman, 1993; Rosen and Bregman, 1994; Stone and Norman, 1994). In one such model (Vazquez, Passot and Pouquet, 1995a), the resulting computation on a two-dimensional grid, and for fiducial values of the parameters, leads to a realistic description of the medium. In particular, all ingredients (self-gravitation, heating and cooling, and hydrodynamical phenomena) are necessary to obtain an ISM that resembles our own, organized in three phases (hot tenuous pervading gas, cold dense clouds and expanding shells within which self-propagating enhanced stellar formation occurs), and for which there is an energetic balance between all actors. Furthermore, the flow is subsonic with little vorticity in most of the medium in the hot phase, and is mostly supersonic with strong local vorticity in the regions of star formation. Note that these expanding shells are linked to the ionization winds of newly-formed OB stars (see for a discussion of winds of massive young stars, MacLow, 1995).

7.3. AN EFFECTIVE THERMODYNAMICAL LAW

The radiative (heating and cooling) processes occur on a typical time-scale significantly shorter than the hydrodynamical time scale, by a factor of roughly one hundred. Hence, a pressure and temperature equilibrium obtains rapidly, and the observed condensations (associated with Giant Molecular Clouds) must be held by turbulent (num) pressure. Assuming a constant heating law $\Gamma$, and a cooling law $\Lambda$ that is both density- and temperature-dependent (namely $\Lambda = \rho T^n$, with $n$ varying in different temperature intervals, as in the Dalgarno–McCray (1972) model), one can easily show, following Elmegreen (1991), that the ensuing equilibrium pressure writes $P_{\text{eq}} \sim \rho^{\gamma_{\text{eff}}}$ with $\gamma_{\text{eff}} = 1 - 1/n$ (in fact in Elmegreen (1991), a more general law is written with a more general assumed behavior of heating and cooling). For most of the medium which is in the temperature range $2,000 < T < 8,000$, this leads to an effective gamma–law gas with $\gamma_{\text{eff}} = 1/3$; indeed, $P - \rho$ scatter plots obtained from numerical simulations in two space dimensions agree with such an estimate (Vazquez–Semadeni et al., op. cit.) with, in the MHD case, a shallower effective law.

A more realistic model can be built along several lines. On the one hand, one can take into account the self-shielding of clouds to UV radiation, leading to a density-dependent diffuse heating; this leads to more realistic values for the temperature of the cold clouds, and to an enhanced density contrast ($\rho_{\text{max}}/\rho_{\text{min}} \sim 1,000$) (Vazquez, Passot and Pouquet, 1995b). On the other hand, both rotation and magnetic fields should be included.
7.4. THE MHD CASE

In the presence of a strong uniform magnetic field, Alfvén waves propagate in the ISM (see e.g. Falgarone and Puget, 1986; Padritz and Gómez de Castro, 1991). But such waves may steepen, and the approach encompassing turbulence as described in the preceding Section may then be followed again. Furthermore, both rotation and magnetic fields are present in the ISM, with a dynamical role (Mouschovias, 1976ab; and this Volume). That role may not be the same at the kiloparsec scale and at the dense core scale where it is thought that proto-stellar collapse occurs quasi-statically at first, because the tension in magnetic field lines is an efficient braking mechanism. A linear analysis by Elmegreen (1994) – in the context of the galactic spiral arms – shows that rotation and magnetic fields have opposite effects on the gravitational instability (similar in fact to the convective case). In the linear context, in the absence of rotation, the magnetic field simply produces an enhanced pressure (stabilizing the Jeans’ length) for the radial modes. Rotation without magnetic field is stabilizing; but a magnetic field reduces the stabilizing of rotation for the azimuthal modes. In the nonlinear context that can be studied with numerical simulations, it is also shown that the magnetic field can have a “pressure cooker” effect (also envisioned, in the somewhat different context of (one dimensional) spherical symmetry, in Slavin and Cox, 1992, 1993): a closed loop produces an inward Lorenz force that confines the plasma (as in tokamaks), preventing shell expansion and thus sweeping of new material, hence a global reduction in the star formation rate, but within the loop leading to strong turbulence.

In order to reach a more realistic description of the ISM, other effects must be included and other scales (the molecular cloud, the dense core) must be studied. One natural expansion of the previous calculations is to investigate whether the inclusion of supernovae in the heating mechanism will lead to a coronal phase (at a temperature of $10^6K$), and how pervasive is that phase (what is its filling factor). Furthermore, the evolution of proto-stellar outflows and the formation of density shells within them may be somewhat similar, as already mentioned, to the evolution of supernova shells (note that rapid cooling is essential in the process, see Chevalier, Blondin, and Emmering (1992), and Stone et al. (1995) and references therein). At smaller scales, chemistry and ambipolar drift should be included as well.

8. Conclusion

It is of course impossible to introduce key concepts in turbulence research today in so brief a Lecture. The domain is very active, in part (a personal bias) because new experiments and observations are better at pinpointing
the insufficiencies of models. Clearly in Astrophysics this is the case, as observers reach a greater range of scales (such as with the solar THEMIS instrument, or with the radio-wavelengths array at Bures or the millimeter BIMA array for studying the interstellar medium for ground-based instruments, as well as numerous spacecrafts Ulysses, CLUSTER, ISO). Our taking into account of a multitude of scales in complex interactions already proves useful, as this text attempted to show through but a few examples.

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