NUMERICAL SIMULATIONS OF MHD TURBULENCE

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Abstract

In the context of geo- and astrophysical conductive flows, several issues that may be settled by numerical simulations in both two and three dimensions are mentioned, together with some of the open questions that remain. The useful rôle of the inclusion of models of small-scale dissipation in the implementation of the numerics of the primitive equations, as well as the rôle of Elsässer variables in the analysis of the data is stressed. Discussions are presented concerning in particular the possibly rapid development of current and vorticity sheets which can lead to efficient heating mechanisms, and the origin of both small-scale chaotic and large-scale helical magnetic fields.

I. Introduction

Three-dimensional MHD turbulence occurs in a wide variety of phenomena, in the laboratory eg in the context of fusion, and in many geophysical and astrophysical space plasmas. The motivations behind the various works presented here stem from (i) solar problems in the contexts of the solar wind, the heating of the solar corona, and the development and structuring of small-scale magnetic fields (in particular in view of the development of instrumentation such as the THEMIS project); (ii) the interstellar medium both at large (kiloparsec) and intermediate (pc) scales – although compressibility, gravitation, radiative transfer and chemistry all play a rôle; and (iii) magnetospheric physics, in particular reconnection phenomena, for example in Flux Transfer Events (see \[57\] for a review) and plasmoid ejection and coalescence [27].

The tool of investigation here is numerical, either through direct simulations of the primitive equations or with the inclusion of a model, in particular of the small scales which cannot be included explicitly because of the huge Reynolds numbers involved in such flows. For several reasons – cost and difficulty of visualisation in three space dimensions for example – a comparative approach will be taken: what do we learn from
the two dimensional geometry (Section 2)? And from the dynamo regime, both in the
kinematic and the nonlinear regimes (Section 3)? Are they any analogy with the 3D
non–conducting fluid (Section 4)? What then can we expect in the three–dimensional
MHD case (Section 5)? Finally, a brief conclusion is given in Section 6.

For completeness, let us write here the MHD equations for the velocity \( \mathbf{v} \) and the
magnetic field \( \mathbf{B} \) normalized to an Alfvén velocity \( \mathbf{b} = \mathbf{B}/\sqrt{\mu \rho_0} \) in the incompressible
case:

\[
\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho_0} \nabla P_0 + \nu \nabla^2 \mathbf{v} + \mathbf{b} \cdot \nabla \mathbf{b} \\
\frac{\partial \mathbf{b}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{b}) + \lambda \nabla^2 \mathbf{b} \\
\nabla \cdot \mathbf{v} = 0; \quad \nabla \cdot \mathbf{b} = 0.
\]

(1)

The vorticity is \( \omega = \nabla \times \mathbf{v} \) and the current density \( \mathbf{j} = \nabla \times \mathbf{b} \); finally, \( \nu \) and \( \lambda \) are the
kinematic viscosity and the magnetic diffusivity respectively, and \( P_0 \) is the pressure.

II. The two–dimensional regime

IIa. Dissipation versus Reynolds number

It has been long established that the small–scale structures that develop in 2DMHD are
thin current sheets and vorticity quadrupoles in the vicinity of these current structures
(see [3] for a critical discussion of the internal structure of such sheets in the inviscid
phase, compatible with a local pressure–balanced field \( \mathbf{B} \sim \tanh x \) across the sheet; see also [63] for the three–dimensional case). Note that the functional relationship
that is deduced from such a tanh profile between current \( \mathbf{j} \) and magnetic potential \( \mathbf{A}_M \)
is reminiscent of the \( \omega = f(\psi) \) relationship found in two–dimensional Navier–Stokes fluids
[39] for long times – where \( \mathbf{v} = \nabla \times \psi \) with \( \psi \) the stream–function. In the
large scales, an inverse cascade of squared magnetic potential \( \mathbf{A}_M^2 \) (with \( \mathbf{B} = \nabla \times \mathbf{A}_M \))
may develop, for which there is also good numerical evidence, although the long–time
behavior is still an open problem.

One of the key remaining questions in the 2D case concerns the limit in the high
Reynolds number limit of the rate of energy dissipation \( \mathcal{D} \). Assuming \( \nu = \lambda \) (unit
magnetic Prandtl number), we have:

\[
\mathcal{D} = \frac{dE}{dt} = \nu \Omega^V + \lambda \Omega^M = \nu \Omega^T
\]

(2)

where \( \Omega^M = \langle j^2 \rangle \) and \( \Omega^V = \langle \omega^2 \rangle \) are the magnetic and kinetic enstrophies, i.e.
total square current density and vorticity. Does \( \mathcal{D} \) have a non–zero limit \( \mathcal{D}_* \) when
\( \nu \to 0 \)? What is the time of growth of current and vorticity? Do they develop generic
characteristic structures irrespective of initial conditions or external forcing?

Closure models of turbulence [52] predict that \( \mathcal{D}_* = \mathcal{O}(1) \) at a time \( t_* = 1 \) for characteristic
velocity and length scales of unity. This is in contradiction with numerical
simulations performed in the inviscid case [17] assuming periodic boundary conditions. For example, taking as initial conditions the Orszag–Tang vortex [44] for the stream–function \( \psi \) and the magnetic potential \( A_M \):

\[
\psi = 2(\cos x + \cos y) \\
A_M = \cos x + \cos 2y ,
\]

the first peak in the dissipation occurs at \( t = t_1 \sim 1.4 \) (in units of the large–scale eddy turnover time) and scales with diffusivity as \( \mathcal{D}_{t_1} \sim \lambda^{0.45} \), as sketched on a log–log scale in Figure 1 (broken line) which is a reanalysis of data discussed in [48]). However, once the dissipation structures have formed, they are unstable to tearing for example and the scaling of dissipation in that second phase is much flatter: \( \mathcal{D}_{t_2} \sim \lambda^{-0.88} \), as seen again in Figure 1 for \( t_2 = 3.6 \) (solid line) corresponding to the secondary peak in the enstrophy at the onset of the strong tearing of the central current sheet. The highest resolution at which these computations were performed was on a uniform grid of \( 1024^2 \) points. The same qualitative and quantitative results hold [4] with a variant of the OT vortex and a hyperviscosity model (see below and Section IIa), as well as with random initial conditions and also a hyperviscosity model (see Figure 1 of [45]). By hyperviscosity model is meant that the Laplacian operators \( \nu \Delta \mathbf{v} \) and \( \lambda \Delta \mathbf{b} \) appearing in equations (1) are replaced by effective dissipation \( \nu_n \Delta^n \mathbf{v} \) and \( \lambda_n \Delta^n \mathbf{b} \), which effectively enlarges the range of wavenumbers for which dissipative effects are negligible compared to nonlinear ones. The highest \( n \) one can choose depends on the grid resolution so that the filter be not too abrupt at the cut–off wavenumber \( k_{\text{max}} \) of the computation. Another small–scale model developed for 2DMHD [41] in which the number of modes is drastically reduced in the dissipation range but keeping a standard Laplacian \( (n = 1) \) for dissipation also describes accurately the large–scale evolution of the flow (with computations for resolutions up to \( 512^2 \) grid points): the large–scale dynamics seems insensitive to small–scale interactions, provided enough physics – such as reconnection – is preserved \(^1\). It has been recently shown [60] that the tearing–mode instability proceeds much in the same way for \( n > 1 \) as for \( n = 1 \), but with a modified power–law dependence on the magnetic resistivity of the characteristic time of the linear development of the instability. There are numerous papers investigating the scaling \( \mathcal{D} \sim \lambda^d \) in the solar context of heating the corona (see eg [11] and references therein) – with either a positive, zero or negative index \( d \). Although specific configurations, in particular boundary conditions, and forcing functions may lead to variations in this scaling, the computations of decaying MHD turbulence suggest that this variation can be attributed to the existence of several temporal phases in the development of MHD flows, as shown here.

On the basis of these results it is tempting to suggest that one of the open problems for MHD – namely the dependency (or lack thereof) on the Reynolds number of energy dissipation – may be tackled again numerically using hyperviscous codes: do the scaling

\(^1\)The inclusion of kinetic effects, stemming from example from a two–fluid analysis, is another open problem not dealt with in this paper.
laws given above persist? Of course, this computation can also be done in three dimensions, both with and without hyperviscosity.

IIb. Velocity–magnetic field correlation

Another striking feature of 2DMHD flows is the development of strong correlations between the velocity and magnetic field. It has been shown that their correlation coefficient defined here as

$$\rho = \frac{\langle \mathbf{v} \cdot \mathbf{b} \rangle}{\sqrt{\langle v^2 \rangle \langle b^2 \rangle}}$$

(with $|\rho| \leq 0.5$) grows in time, although more slowly for higher Reynolds number (in the inviscid case, $\rho$ is the ratio of two invariants of the equations, namely $E^C = \langle \mathbf{v} \cdot \mathbf{b} \rangle$ and energy $2E^T = \langle v^2 + b^2 \rangle$). Moreover, spectral (Fourier) information indicates that both the small scales and the large scales become highly correlated, although with opposite signs. In Figures 2 and 3 are shown the correlation coefficient for the Orszag–Tang vortex for a dealiased computation with periodic boundary conditions, using a pseudo–spectral method on a grid of 768² points (see [41]); the size of the flow is $L_0$, and $\nu = \lambda = 0.025$, corresponding to initial kinetic and magnetic Reynolds numbers of 800. At $t = 0$, the correlation coefficient is equal to 0.25, but as can be seen from Figure (2a), it varies smoothly between its extrema. The maximum of the enstrophy is reached near $t \sim 1.4$. At that time, the Reynolds numbers have dropped to 317, and $\rho$ has increased slightly to a value of 0.267. The correlation coefficient is shown for the entire flow at (2a) $t = 0$ and (2b) $t = 1.0$. In Figures (3a) and (3b) are displayed a blow–up of the central region – where the most important current sheet forms – at $t = 1.0$ and $t = 1.4$. From these figures, it can be seen that the moderate value of $\rho$ hides the fact that the flow organises into highly ±–correlated regions which can be seen as forward or backward propagating Alfvén waves on a “mean” turbulent magnetic field at scale $L_0$ or slightly smaller. Only in the zones surrounding the current sheets where most of the dissipation takes place do we have an intermingled complex structure (see also [45], fig. 14) possibly due to an accumulation of Alfvén waves in the vicinity of a zero–point of the magnetic field, with the potentiality of an important energy accumulation – and thus release, for example in flares – near such points (see [13]); similar results hold for random flows. The three–dimensional case is presently under investigation.

We may thus infer that, at least in two dimensions for which we have numerical evidence, even though $\rho$ may initially be moderate globally, and remain so as $R_M \to \infty$, the dynamical evolution of MHD turbulence may be such that the correlation coefficient reaches values close to its ± extrema in many places of the flow since no such conservation constraint exist locally; one should thus also consider the unsigned coefficient $\rho^u = \langle |\mathbf{v} \cdot \mathbf{b}| \rangle / \langle E^T \rangle$ as a measure of true nonlinearities in the flow.

IIc. Invariants
Another important issue in MHD is related to the fact that there is an infinite number of invariants in the non-dissipative case of the form:

\[ I_{H,G} = \int [H(A_M) + \omega G(A_M)] \, dx \, dy \]

where \( A_M \) is the magnetic potential and where \( H \) and \( G \) are arbitrary functions. As pointed out by several authors, these invariants may strongly constrain the dynamical evolution of such flows (see also Gruzinov, this conference), but how precisely? And how does it alter reconnection phenomena? These questions are more general than in the pure MHD (or fluid) case; indeed, it has been shown recently [34] that such geometrical invariants also exist – still in the two-dimensional case – for plasmas with spatial scales going down to the electron skin depth.

Another problem arises when considering truncated systems: a simple Fourier truncation will only preserve the quadratic invariants \( E_T, E_C \) and \( E^A = < A_M^2 >. \) Formalisms allowing for the preservation of \((N-1)\) invariants in a system of \( N^2 \) modes have been developed [68] which involve a non-cartesian volume element of phase space taking into account the curvature of that space because of the integral constraints. The numerical integration of such systems is difficult since an algorithm must be found that accurately conserve these invariants. A related question concerns the problem of geometrical constraints versus dynamical constraints: there are several systems for which the phase space has an identical structure (metric) but with different dynamical operators (Zeitlin, op cit). It remains an open problem whether such systems evolve in identical ways (indicative of strong geometrical constraints) or not (indicative of strong dynamical constraints).

IId. Hyperviscosity

A possible numerical effect when using hyperviscosity can be seen in the following model. We try to evaluate in a phenomenological manner the possible ranges and their extension in wavenumber space in the near-dissipation regime, in two ways.

On the one hand, we know from several numerical investigations both in 3D Navier-Stokes – and 3DMHD in the kinematic dynamo regime – that filaments (of vorticity in the former case, of magnetic field in the latter one) obtain. Let us write the equation for the magnetic field (or equivalently the vorticity) as:

\[ \frac{D\mathbf{b}}{Dt} = \mathbf{b} \cdot \nabla \mathbf{v} + (-1)^{n+1} \mathbf{R}_n^{-1} \nabla^{2n} \mathbf{b} \ . \]

In the kinematic problem, velocity gradients are assumed \( O(1) \) and a steady solution obtains with characteristic width for magnetic or vorticity filaments:

\[ k_{F,n}^{-1} \sim R_n^{-1/2n} \ . \]

However, if an eddy viscosity of the form \( R_{turb}^{-1} \nabla^2 \mathbf{b} \) develops – as for example a renormalization group analysis would indicate (see [16] for the MHD case) – then we recover the well-known relation \( k_F^{-1} \sim R_V^{-1/2} \).
On the other hand, we can estimate the wavenumber $k_D$ at which dissipation sets in by equating the non-linear and dissipative times at that wavenumber:

$$t_{NL}(k_D) = t_{diss}(k_D) \, .$$

With $t_{NL} \sim (k^3 E(k))^{-1/2}$ and $t_{diss} \sim R_n k^{-2m}$ where $E(k) \sim k^{-m}$ is the energy spectrum and $m$ the spectral index, we obtain:

$$k_{D,m,n} \sim R_n^a$$

with

$$a = \frac{2}{4n - 3 + m} \, .$$

For $n = 1$ and $m = 5/3$, we recover the Kolmogorov scaling $k_D \sim R_V^{3/4}$ with $R_V = R_1$ the usual Reynolds number.

Comparing now the ratio of these two wavenumbers as a function of Reynolds number, we have

$$r_{m,n} = \frac{k_{F,n}}{k_{D,m,n}} \sim R_n^a \, , \quad a_r = \frac{m - 3}{2n(4n - 3 + m)} \, .$$

This ratio has a slight $R_n$ variation for $n = 1$, but as soon as $n > 1$, one can consider that $r_{m,n}$ is independent of $R_n$, except of course for $m \sim 3$, which may occur for highly correlated flows [23] [53]. The magnitude in wavenumber of the intermediate range in which such filaments may prevail before sharp dissipation sets in will remain quasi-constant with $R_n$ in a hyperviscous run – unless once again renormalised transport coefficients are invoked with a $k^2$ dependence – whereas it will slowly augment with $R_V$ for $n = 1$.

III. The dynamo problem

It has been known for a long time that a large scale magnetic field can grow when the velocity field at small scale is helical (non zero velocity – vorticity correlations) through the so-called $a$–effect, where the transport coefficient $a$ is proportional to the kinetic helicity $H^V = \langle \mathbf{v} \cdot \mathbf{\omega} \rangle$ of the small–scale flow. Recently, attention has focused rather on small–scale magnetic fields (at scales comparable to that of the velocity and down to the dissipation scale), in particular in the context of the underlying chaos of the velocity field. The best known example is that of the ABC flows:

$$\mathbf{u}_{ABC} = (A_0 \sin k_0 z + C_0 \cos k_0 y, \ B_0 \sin k_0 x + A_0 \cos k_0 z, \ C_0 \sin k_0 y + B_0 \cos k_0 x) \, .$$

The kinematic dynamo (given velocity field) with the $A_0 : B_0 : C_0 = 1 : 1 : 1$ flow with $k_0 = 1$ [1] is probably fast, by which is meant that the growth rate of the magnetic field has a non–zero limit in the limit of infinite magnetic Reynolds number $R_M$. It was shown in [19] that the magnetic field develops at the location of the stagnation points of the velocity – where temporal chaos is also present – in the form of elongated
cigars of thickness $\sim R_{M}^{-1/2}$ (a balance between stretching by $O(1)$ velocity gradients and diffusion). In the absence of stagnation points, either in the linear regime [21] or in the non-linear regime [18], sheets rather than cigars develop.

By setting one of the ABC parameters to zero, the flow becomes two-dimensional and Hamiltonian. Temporal chaos can be reintroduced by applying a temporal sinusoidal perturbation of frequency $\Omega_0$ and amplitude $\delta$ [20] which, when small, allows one to use tools of Hamiltonian chaos. It can then be shown that, as a function of $\Omega_0$, the growth rate of the magnetic field for the (presumably) fast dynamo correlates best with the Melnikov function which measures, at first order in the amplitude of the perturbation, the distance between the stable and unstable manifolds — and to a lesser extent with Lyapunov exponents [50]. The magnetic structures are sheets centered on the unstable manifold of the velocity field. Superimposed to these structures are magnetic eddies in the vicinity of the elliptic stagnation point, whose growth rate diminishes with $R_M$. A more detailed study is in progress [50].

It was shown in the full 3D case that when $k_0 \neq 1$, the growth rates of the $\mathbf{B}$ field is substantially larger [18]. Furthermore, when sufficient scale-separation is present a slower non-oscillatory growth of the magnetic field at the largest scale is observed (presumably corresponding to an equivalent $\alpha$-effect), by opposition to a rapid oscillatory growth of $\mathbf{B}$ at a scale comparable to that of the velocity (corresponding to a fast dynamo). However, in the non-linear regime, the Fourier shell associated with the largest scale in the flow is energetically dominant. Presumably, there are different saturation mechanisms for the fast and $\alpha$ dynamos, a point that deserves further numerical study but which is costly. If the phenomenology deduced from two-point closure computations [55] is valid, then the large-scale $\alpha$ dynamo is in fact associated with an inverse cascade of magnetic helicity $H^M = \langle \mathbf{A}_M \cdot \mathbf{B} \rangle$ towards larger and larger scales, with a quasi-saturation at scale $\ell$ being provided by Alfvén waves in a time $\ell/B_0$, with $B_0$ the mean field (or average large-scale field).

We show in Figure (4a) sketches of the ratio $E = E^M/E^V$ of the magnetic to kinetic energy in the saturated regime as a function of $K = k_0/k_{\text{min}}$ with $k_{\text{min}}$ the minimum wavenumber of the computation, and (4b) the coefficient $Q_\rho = \rho^V$ (top) or $Q_\rho = \rho^M$ (bottom) corresponding to the normalized kinetic and magnetic helicities

$$\rho^V = \int \mathbf{v} \cdot \mathbf{\omega} \, dx / \sqrt{\int v^2 \, dx \int \omega^2 \, dx}$$

and

$$\rho^M = \int \mathbf{A}_M \cdot \mathbf{b} \, dx / \sqrt{\int A_M^2 \, dx \int b^2 \, dx}$$

for a computation with $A_0 : B_0 : C_0 = 1 : 1 : 1$ and $R_V = R_M = 12$ as described in [18]. In Figure (4a), the dotted line corresponds to equipartition, which is seen to be reached as $k_0$ grows; similarly, in (4b), the two $Q_\rho$-factors tend to equalize with $k_0$ increasing, as would be the case following the phenomenology developed in [55].

One challenge for dynamo computations is clearly to study such phenomena at substantially higher Reynolds numbers, for example enforcing symmetries [8]. It should be
noted that in the context of penetrative compressible magnetoconvection, it has been shown [9] that the magnetic field that develops has several features of fast dynamos, in particular its spatial concentration and possibly the low level of saturation of the magnetic energy (although this may be due to the moderate Reynolds number of all such computations in the fully non-linear case). One important point (not only because of computational costs) is to unravel the respective roles of nonlinear dynamics as exemplified in somewhat academic models as the ABC flows (with both $\alpha$-effect and chaos) and of the actual physical conditions of the flow (such as boundaries, convection, rotation and radiative transfer) in the development of the characteristic temporal and spatial structures of dynamo-induced velocity and magnetic fields (see also Section Va).

IV. Three-dimensional Navier–Stokes equations

There is presently a surge of activity concerning the detailed study of small-scale turbulence. It has been known for some time that in the inviscid phase vorticity sheets form, that are unstable and lead to numerous filaments of vorticity (e.g. [31] [66]). The mechanism of formation of these filaments is still debated [43] [37] [47] and can be due to a Kelvin–Helmholtz instability, or to a self–focusing instability of a vorticity sheet embedded in an external large–scale strain [47]. The inertial range (and early dissipation range) that results from these filaments themselves can be either a Kolmogorov spectrum $E(k) \sim k^{-5/3}$ or a flatter spectrum $E(k) \sim k^{-1}$ (see for a recent discussion [58] and references therein; see also [22]). Evidence for the simultaneous presence of two such ranges – although barely resolved – has been obtained in a different context, that of the computation of decaying compressible flows in the subsonic phase [51], although one cannot yet rule out a numerical artifact of the PPM [67] algorithm at small scales. Conditional sampling of filaments in an incompressible flow also leads to a $k^{-1}$ spectrum [30].

Indeed, following [6], the velocity

$$v_x = f(ze^\mu)$$
$$v_y = \gamma y$$
$$v_z = -\gamma z$$

(10)

is an exact solution of the Euler ($\nu = 0$) equations where $v_x$ follows a self–similar law and where $(v_y, v_z)$ is a strain of uniform strength $\gamma$; the vorticity for that flow is in the $y$–direction and grows exponentially in time with growth rate $\gamma$. In the presence of viscosity $\nu$, this layer is in fact of thickness $\sqrt{\nu/\gamma}$ and can be represented by the Burgers’ vorticity sheet; whether such filaments persist at high Reynolds number is questioned. They are unstable to kink, as observed for example in numerical simulations of compressible flows [51].

Another striking feature of three–dimensional fluid flows is the alignment of vorticity with the second eigenvector $e_2$ of the rate of strain tensor $\partial_i v_j$ (made symmetric), as first found in [31]. It should be noted [47] that this alignment is compatible with
the facts that (i) vorticity and velocity gradients grow at the same rate \(^2\) and (ii) vorticity is observed numerically to grow exponentially (as opposed to faster than exponential) in the inviscid phase, the equation for that phase being \(D\omega/Dr = \omega \cdot \nabla \mathbf{v}\). A self-similar model using as a velocity field equations (10) leads explicitly to such an alignment of \(\omega\) and \(\mathbf{e}_2\) [6] in the inviscid phase, showing that at least at intermediate times this alignment cannot be ascribed to the quasi two-dimensional configuration of a vortex filament embedded in its own strain, as suggested in [30] for later times. However, it should be noted that, following the nomenclature of [6], at early times \(\lambda_2\) is the largest eigenvalue, so that at those early times, the vorticity aligns with the eigenvector corresponding to the dominant eigenvalue; the exponential growth of \(\lambda_1\) in time together with the constant value of \(\lambda_2\) in that model, however, then leads to \(\omega // \mathbf{e}_2\); the time at which \(\lambda_1\) and \(\lambda_2\) switch as far as being the largest eigenvalue for the flow may not necessarily coincide with the time at which the self-strain created by the temporally-growing vorticity dominates the original external large-scale strain of strength \(\gamma\), at which (latter) time the \(\omega - \mathbf{e}_2\) alignment becomes a trivial consequence of the local quasi-axisymmetry of vorticity filaments.

Townsend [64] has derived the spectrum due to filaments \(E(k) \sim \varepsilon^{1/2}k^{1/2}k^{-1}\); for an energy injection rate \(\varepsilon\) assumed constant, the amplitude of this spectrum diminishes with Reynolds number in a way compatible with the scaling of the onset of the dissipation range as \(k_F \sim R_v^{-1/2}\). The emerging picture (that of a Kolmogorov spectrum \(\sim k^{-5/3}\) until \(k_F\), followed by a \(k^{-1}\) spectrum until \(k_D\)) may have been observed in recent numerical simulations of decaying supersonic flows [51] using the PPM algorithm, even though PPM has an effective dissipation functional \(\sim k^{2n}\) with \(n\) measured empirically as \(n \sim 4\) or 5. Such an increase (from the \(E(k) \sim k^{-5/3}\) level) in the energy at the end of the inertial range has been recently modeled [15] as a “bottleneck” phenomenon due to strong depletion in the exponentially-decreasing dissipation range of the energy spectrum leading to inefficient triple-wave interactions.

Finally let us remark that if we consistently compute \(k_D\) using now the \(k^{-1}\) spectrum given above, we nevertheless find \(k_D \sim R_v^{3/4}\) because this latter spectrum involves dissipation in its formulation (see above).

The equivalent determination of inertial, near and far-dissipation ranges remains to be done in 3DMHD. We already know that for the Kraichnan phenomenology of Alfvén-wave-dominated transfer [33], velocity–magnetic field correlation may play a role (see [54] for a review) in the development of the inertial index for the fully non-linear regime (note however that in [61] this spectrum is replaced by a steeper law in the weak turbulence regime, due to the possible irrelevance of three-wave interactions). In the dynamo case with weak fields, other inertial spectra may develop \((E(k) \sim k^{-1}\) as in [59], or \(E(k) \sim k^{1/3}\) as in [9]). Moreover, in the large scales, a \(k^{-1}\) magnetic energy spectrum may develop due to an inverse cascade of magnetic helicity [55] – computed for a maximal rate, unless coherent structures alter the flow.

\(^2\)It can be shown that for homogeneous incompressible flows the mean square vorticity and strain are equal – a purely kinematic result; under the same assumptions, using in MHD the Elsässer variables \(z^\pm\) defined in (13), identical relationships obtain for the symmetric (± strain) and anti-symmetric (± vorticity) parts of the \(\partial_t z_j^\pm\) tensors as given in Section Vc, equation (19).
V. Three-dimensional structures in MHD

Va. The weak dynamo

It has been conjectured for a long time that large-scale structures in MHD may be helical, through the basis of statistical mechanics and closure models of MHD turbulence (see e.g. [54] for a review). Numerical evidence for a positive transfer of magnetic helicity to the large scales was obtained for periodic boundary conditions [56] as well as for more realistic boundaries [28] including in the acoustic case [29]. The fully supersonic case, as encountered in the interstellar medium (ISM), has not been investigated yet for such large-scale enhancement. Evidence for such structures are numerous: in coronal loops, in the ISM (the L204 cloud, [25]) and possibly in flux transfer events in the magnetosphere (see for a review [57]). As discussed in the previous Section, the nonlinear cascade of magnetic helicity towards large scales may well give rise to such helical fields.

On the other hand, chaotic dynamo may lead to a highly intermittent small-scale magnetic field. Such an intermittent field was but barely noticeable in early computations of non-helical dynamos [40], although in that computation intermittency may be due to the proximity of the transition \( R_M \sim R_{Ms} \), with dynamo action taking place for \( R_M > R_{Ms} \). On the other hand, using maps instead of flows, numerous examples in that context have been found, of magnetic fields with small-scale cancellation (see also Ruzmaikin, this conference, for an application of such concepts to solar observations).

In dynamo computations in a convective layer [9], it is found that the magnetic field grows mostly at stagnation points of the velocity, in the form of filaments wrapping around strong vortex filaments. Furthermore, these authors showed that there was a tendency for the vorticity to align with \( \mathbf{e}_2 \), the second eigenvector of the rate of strain, when vorticity is strong, whereas the alignment for the magnetic field is debatable. This indicates that the analogy between growth of vorticity and growth of magnetic field is not fully functional. In numerical simulations of decaying MHD turbulence with initial energetically equivalent velocity and magnetic fields, similar results hold: when vorticity is strong, it is aligned with \( \mathbf{e}_2 \) [49].
MHD computations may be tackled from many points of view, among which: (i) as a dynamo problem, both in the linear and saturation phase; (ii) as a perturbation to a non-conducting flow (how are the vortex filaments modified by the growth of a magnetic field? Is there a critical level of the magnetic field intensity for such disturbance to occur or is it progressive?); (iii) as an analogy with the dynamical evolution of vorticity.

Another point of view consists in taking *ab initio* a magnetic field of comparable amplitude to the velocity and ask what kind of features develop in time. Computations with periodic boundary conditions together with a uniform grid of $180^3$ points are presently being analysed in that regime. The magnetic Prandtl number is taken equal to unity, and the initial conditions are either random and centered in the large-scales, or a large-amplitude perturbation in the third direction of the Orszag–Tang vortex [44]. Preliminary results [49] indicate that: (a) there is an exponential decrease of both kinetic and magnetic characteristic scales in time in the early non-dissipative phase; (β) the growth of the current density is both faster and stronger than that of vorticity. Similar computations with a code enforcing symmetries in order to enhance resolution are in preparation [7].

One of the open questions raised by such computations, by analogy with the two-dimensional case, is the following: of the two simple solutions:

(a) $\mathbf{v} = \pm \beta \mathbf{b}$ everywhere – the Alfvén flow when $\beta = 1$; and
(b) $\mathbf{v}$ and $\mathbf{b}$ exist in separate patches – corresponding to a $\mathbf{v} - \mathbf{b}$ “exclusion” – which one will prevail, or are they both realized, at different times and/or locations in the flow?

In each case, a very different dynamics would occur: indeed if the nonlinear interactions develop Alfvén-like solutions, the further evolution will occur on the slow diffusive time as opposed to the eddy turn-over time, after a transient phase. These questions take us back to Section 2 and the problem of finite dissipation in a finite time in the limit of high Reynolds numbers.

Finally, let us note that the presence of zeros of the magnetic field as a necessary ingredient for strong currents to occur is debated in the literature [24] [26] [35]; at $v_A = 0$, it is often argued that a substantial amount of energy due to stagnation of Alfvén waves may accumulate to be released in a current sheet, but otherwise it is recognised that in the magnetosphere a strong non-zero component of the magnetic field is present, at the location of the current sheet.

**Vc. The solar corona**

The heating of the solar corona as a whole remains an open problem (see for example [12] for a review), as well as the observations that there exists a continuous spectrum in intensity and duration of solar flares ([38] and this conference). In two dimensions, a recently developed model [13] that includes a forcing function in the magnetic equation leads to the clear development of current sheets with random ratio of output (current
bursts) to input (modelised flux–tubes foot motions). As already conjectured by several authors, the continuous spectrum of intensity and duration of flares, from the nano–events to the large disruptions – could be a turbulent feature, but the largest flares may require the almost simultaneous lighting–up of the whole corona, in a semi–coherent fashion [12]. In terms of turbulent eddies, this may mean the triggering of tearing instabilities. Three dimensional computations taking into account the anchoring of magnetic foot points in the solar photosphere as well as their motions due to convection [42] [62] show the clear development of current sheets; their further evolution – and in particular whether they destabilize into filaments or tend to burst – is not analysed in these papers.

\textit{Vd. Vorticity and currents}

We now write the MHD equations in terms of vorticity and current in the non–dissipative case:

\[
\frac{\partial \omega}{\partial t} + \mathbf{v} \cdot \nabla \omega - \mathbf{b} \cdot \nabla \mathbf{j} = \omega \cdot \nabla \mathbf{v} - \mathbf{j} \cdot \nabla \mathbf{b} \quad (11)
\]
\[
\frac{\partial \mathbf{j}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{j} - \mathbf{b} \cdot \nabla \mathbf{w} = \mathbf{j} \cdot \nabla \mathbf{v} - \omega \cdot \nabla \mathbf{b} - 2 \sum_m \nabla u_m \times \nabla b_m , \quad (12)
\]

where the following identity is of use:

\[
\nabla \times (\mathbf{u}_1 \cdot \nabla \mathbf{u}_2) = \mathbf{u}_1 \cdot \nabla \omega_2 - \omega_2 \cdot \nabla \mathbf{u}_1 + \sum_m \nabla u_{1m} \times \nabla u_{2m}
\]

with $\omega_2 = \nabla \times \mathbf{u}_2$, and where $\nabla \cdot \mathbf{u}_1 = 0$ has been assumed. The terms on the \textit{rhs} of (12) can also be written as:

\[
-\nabla(\mathbf{v} \cdot \nabla) \times \mathbf{b} + \nabla(\mathbf{b} \cdot \nabla) \times \mathbf{v}.
\]

To implement the analogy with stretching of vorticity by velocity gradients, it is useful to write the above equations in a form made symmetric using the Elsässer variables

\[
\mathbf{z}^\pm = \mathbf{v} \pm \mathbf{b} .
\]

We first define the Lagrangian derivative in the $\mathbf{z}^\pm$ variables with

\[
D^\pm / Dt = \partial_t + \mathbf{z}^\pm \cdot \nabla .
\]

We then clearly see that the terms in the \textit{lhs} of equations (11–12) participate to the Lagrangian derivative in the $\mathbf{z}^\pm$ variables – carrying structures along $\mathbf{z}^\pm$–trajectories – and only do the terms on the \textit{rhs} are actively producing vorticity and currents, or equivalently $\omega^\pm = \omega \pm \mathbf{j}$.

The first three sets of equations (again without dissipation) for the fields, their curl or generalised vorticity $\omega^\pm$ and again their curl – noting $\eta^\pm = \nabla \times \omega^\pm = \eta \pm \mu$ with $\eta = \nabla \times \omega$ and $\mu = \nabla \times \mathbf{j}$ – in the non–dissipative case now read (with $\pm$ symmetry not explicitly written in order to ease the deciphering):
\[
\frac{D_{-}z^+}{Dt} = -\nabla P_0
\]
(14)
\[
\frac{D_{-}\omega^+}{Dt} = \omega^+ \cdot \nabla z^- + \sum_m \nabla z^+_m \times \nabla z^-_m
\]
(15)
\[
\frac{D_{-}\eta^+}{Dt} = -\nabla P_2 - \eta^- \cdot \nabla z^+ - z^- \cdot \nabla \eta^+ + \sum_{j,k} (\partial_j z^-) \partial_k \eta^+ \nabla^+ \cdot \nabla \eta^+,
\]
(16)

with \( P_2 = -\nabla^2 P_0 \). Equation (16) can also be written as:
\[
\frac{D_{-}\eta^+}{Dt} = \eta^+ \cdot \nabla z^- + \omega^+ \cdot \nabla \omega^- - \omega^- \cdot \nabla \omega^+ + 2 \sum_m \nabla \omega^+_m \times \nabla z^-_m + \sum_m \nabla \times (\nabla z^+_m \times \nabla z^-_m),
\]
(17)

where the last two terms, once developed, also read:
\[
\sum_m [ (\nabla z^-_m \cdot \nabla) \nabla z^+_m - (\nabla z^+_m \cdot \nabla) \nabla z^-_m + \eta^+_m \nabla z^-_m - \eta^-_m \nabla z^+_m ].
\]

In two dimensions, equation (15) gives
\[
\frac{D_{\pm}\omega^\pm}{Dt} = \pm \sum_m \nabla z^+_m \times \nabla z^-_m,
\]
an equation that can be found in [3] under a slightly different but equivalent form since:
\[
\sum_m \nabla z^+_m \times \nabla z^-_m = 2 \sum_m \nabla \partial_m A_M \times \nabla \partial_m \psi = \sum_m \nabla \partial_m \psi^+ \times \nabla \partial_m \psi^-,
\]
with \( \psi \) and \( A_M \) the stream function and the magnetic potential respectively, and \( \psi^\pm = \psi \pm A_M \) the symmetric potentials. On the other hand, equation (17) gives
\[
\frac{D_{\pm}\eta^\pm}{Dt} = \eta^\pm \cdot \nabla \eta^\mp \pm \mathcal{S}
\]
with the source term
\[
\mathcal{S} = \nabla \times (\sum_m \nabla z^+_m \times \nabla z^-_m),
\]
the fluid version of which can be found in [5], noting that in that case \( \mathcal{S} = 0 \). Indeed, the equations in dimension two in terms of the curl of vorticity \( \eta = \nabla \times \omega \) read:
\[
\partial_t \eta + \mathbf{v} \cdot \nabla \eta = \eta \cdot \nabla \mathbf{v}.
\]
(18)

In the \( \pm \) variables (equations 15), one sees immediately that the growth of \( \pm \) vorticity is due to two terms: the first one is similar to the standard three-dimensional Navier–Stokes case, leading to an exponential growth in the inviscid phase at a rate depending on the magnitude of the \( \pm \) vorticity gradient assumed to be \( \mathcal{O}(1) \); but the second
term, assuming that these vorticity gradients grow themselves exponentially may in fact insure a growth twice as fast, and should not necessarily be neglected, contrary to what is assumed in [32] (see also [3]), at least in the inviscid phase of initial growth.

Vc. The Betchov relation

Following Betchov [2], we write the gradient matrices \( g_{n_{ij}} = \partial_i h_j \) - for a vector field \( h \) which can be indifferently either the velocity, the magnetic field or the \( z^\pm \) fields - as a sum of a strain \( \sigma_{n_{ij}} = \frac{1}{2}(\partial_i h_j + \partial_j h_i) \) symmetric tensor and an anti-symmetric rotation tensor \( \rho_{n_{ij}} = \frac{1}{2}(\partial_i h_j - \partial_j h_i) \). In the principal axis of the strain of each of the \( (v, b, z^\pm) \) fields, we have:

\[
g_{n_{ij}} = \begin{pmatrix}
  s_h^{(1)} & R_h^{(3)} - R_h^{(2)} & 0 \\
  -R_h^{(3)} & s_h^{(2)} & R_h^{(1)} \\
  R_h^{(2)} & R_h^{(1)} & s_h^{(3)}
\end{pmatrix}
\]

with \( \sum_i s_h^{(i)} = 0 \) because of incompressibility; \( \nabla \times h_i = 2R_h^{(i)} \) is the \( i \)-th-component of the pseudo-vector generalised vorticity of the \( h \)-field (the current density for the \( h = b \) magnetic field). We shall in fact also use the notations \( (s_v^{(1)}, s_v^{(2)}, s_v^{(3)}) = (a, b, c) \) and \( (R_v^{(1)}, R_v^{(2)}, R_v^{(3)}) = (A, B, C) \) for the velocity, \( (s_b^{(1)}, s_b^{(2)}, s_b^{(3)}) = (d, e, f) \) and \( (R_b^{(1)}, R_b^{(2)}, R_b^{(3)}) = (D, E, F) \) for the magnetic field, \( (s_z^{+}, s_z^+, s_z^+) = (p_1, p_2, p_3) \) and \( (R_z^{+}, R_z^+, R_z^+) = (P_1, P_2, P_3) \) for the \( z^+ \) field, and finally \( (s_z^-, s_z^-, s_z^-) = (m_1, m_2, m_3) \) and \( (R_z^{1}, R_z^{2}, R_z^{3}) = (M_1, M_2, M_3) \) for the \( z^- \) field.

As already mentioned in the footnote of Section IV, the following relationship holds for each of the divergence-free \( h \)-field:

\[
\sum_i < (s_h^{(i)})^2 >= 2 \sum_i < (R_h^{(i)})^2 > 
\]

(19)

where homogeneity has been assumed, a relationship also valid for cross-terms:

\[
< m_1 p_1 + m_2 p_2 + m_3 p_3 > = 2 < M_1 P_1 + M_2 P_2 + M_3 P_3 > ,
\]

\[
< a d + b e + c f > = 2 < A D + B E + C F > .
\]

Assuming isotropy as well, kinematics again yields:

\[
\sum_i < (s_h^{(i)})^3 >= 3 \sum_i < s_h^{(i)} (R_h^{(i)})^2 > ,
\]

(20)

a result stemming from evaluating the triple sum \( < g_{ij} g_{jk} g_{ki} > = 0 \). Cross-terms are more numerous; for example one has:

\[
< m_1 p_1^2 + m_2 p_2^2 + m_3 p_3^2 > = < m_1 P_1^2 + m_2 P_2^2 + m_3 P_3^2 > ,
\]

leading now in the equations for the \( \pm \) vorticities to the three terms:

\[
\partial_t < (p_1^2 + p_2^2 + p_3^2) > \sim - < (m_1 p_2 p_3 + m_2 p_3 + p_1 p_2 m_3) > ,
\]

(21)
\( (p_1, p_2, p_3) \) and \( (m_1, m_2, m_3) \) being the three eigenvalues of the symmetric \( \partial_t \mathbf{z}^\pm \) tensors as stated above. For \( \mathbf{z}^+ = \mathbf{z}^- \) (the fluid case), we recover the Betchov relation [2] with the lhs of (21) equal to \(-3 < abc >\). According to (21), what would guide the growth or decay of the \( \pm \) enstrophies is the sign of the middle eigenvalues \( p_2 \) and \( m_2 \), as for fluids, but here more cases must be considered, and cross-terms may alter these conclusions. This will be done in a forthcoming paper.

\textit{Vf. The interstellar medium}

As a final point, mention can be made of an ongoing investigation of the interstellar medium (see [69]), and in particular of the role played by turbulence in its dynamical evolution. It was shown, using both numerical simulations in two dimensions and phenomenology, that supersonic flows can stop gravitational collapse of molecular clouds at the scale of a few parsec at all scales, due to the early formation of shocks [36]. At the scale of the kiloparsec, and with the inclusion of heating and cooling terms to mimic for example the role of stellar ionisation winds in stirring the medium at small scales, the ISM organises in three distinct phases – (i) hot, dilute, subsonic, large-scale gas, (ii) cold, dense mostly supersonic giant molecular clouds, and (iii) expanding shells corresponding to HII regions – for which all ingredients (self-gravity, turbulence and heating and cooling) are essential and are part of a self-sustained energetic cycle [65]; the extension of the model to include rotation and magnetic field is presently being worked out. In weakly ionised gas such as the ISM, it might also be useful to deal with dispersive effects not present in MHD with a simple Ohm’s law, for example ambipolar diffusion [10], and Hall effect [46]. Finally, a comparison of data sets stemming from three-dimensional numerical simulations of supersonic flows and spectra obtained in high-latitude clouds, show that turbulent flows may very well be present in these clouds [14].

\textbf{VI. Conclusion}

Several challenges have been identified that can be partially resolved in the next decade, for example the problem of the limit of the rate of energy dissipation when the Reynolds number goes to infinity, possibly using hyperviscosity models, in particular when extending this research to the 3D case. In the dynamo problem, one of the open challenges is to understand the interplay between the small-scale chaotic “fast” magnetic field, and the large scale helical magnetic field. In particular the model derived in [55] in the context of closures of turbulence – and thus ignoring possible effects due to both the geometry of structures, and to chaos – remains unchallenged and yet present-day computers are powerful enough that some of its predictions – albeit in the simplistic context of homogeneous turbulent flows without boundaries – may be confirmed or proven wrong.

The statistical approach to turbulence leads to the development of models, of the concepts of direct and inverse cascades, and to the evaluation of transport coefficients. On the other hand, if we deal with structures, we shall rather work with shear flows,
neutral X–points, sheets, ribbons and filaments, considering their development, their
instability, and their temporal and spatial intermittency. With the help of computers
and visualisation tools, one can identify basic generic configurations, such as vorticity
filaments, and unravel some of their properties, such as the alignment of vorticity with
the second eigenvector of the strain.

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Figure captions

**Figure 1** Energy dissipation $\mathcal{D}$ versus Reynolds number $R$ for two-dimensional MHD flows; the computations are performed on a uniform grid, using a pseudo-spectral method, with periodic boundary conditions and no forcing; log-log coordinates. The magnetic Prandtl number is equal to unity ($\nu = \lambda$), and the initial conditions are those of the Orszag–Tang vortex [44]. Results are taken from a re-analysis of data published in [48]. Dots correspond to the time at which the total enstrophy $\Omega^T = \mathcal{D}/\nu$ (see equation 2) is maximum, and crosses correspond to the secondary peak in the total enstrophy, when strong tearing has destabilised the central current sheet. Equivalent computations with random initial conditions and hyperviscosity [45] give similar results. Note the quasi–independence of dissipation with Reynolds number in the reconnection phase, as opposed to the inviscid phase.

**Figure 2** Contour lines of the correlation coefficient between the velocity and magnetic field given in equation (4) for the Orszag–Tang flow (3) at (a) $t = 0$ and (b) $t = 1$. Note the organisation of the flow in mostly positively–correlated regions at large scales (due to the initial conditions), whereas small scales are mostly negatively–correlated.

**Figure 3** Contour lines of a blow–up of the correlation coefficient $\rho$ between the velocity and magnetic field near the central current sheet for the Orszag–Tang flow at (a) $t = 1.0$ and (b) $t = 1.4$. Most of the dynamics occur in the vicinity of the current sheet, with rapid localised spatial variations of $\rho$. 
Figure 4 As a function of the normalised characteristic wavenumber of the velocity $K$ are drawn – in the non-linearly saturated ABC dynamo regime – (top) the magnetic to kinetic energy ratio and (bottom) the kinetic and magnetic relative helicities $Q_\rho$ (see equations (8 – 9) for definitions). The data is taken from [18]. Note the strong tendency towards equipartition as soon as $K \neq 1$. 