Objective Bayesian analysis for Proper Gaussian Markov Random Fields

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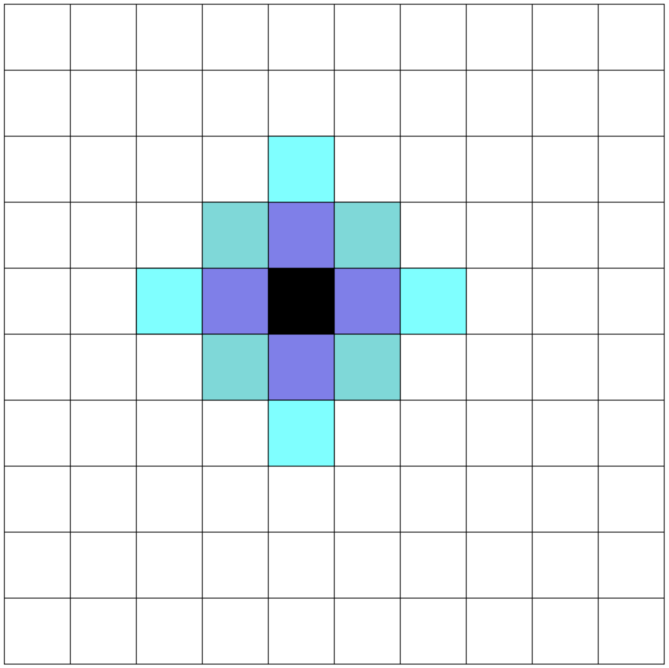
Outline

- Proper Gaussian Markov random fields
- Confounding of parameters
- Reference prior
- Results for some simulated fields
- Frequentist properties
- Discussion

Proper MRF



Proper MRF

x = observed proper Gaussian Markov random field

$p(x|\mu, \alpha, \beta) = N(x|\mu 1_n, \Sigma), \quad \Sigma^{-1} = \beta[Z + \alpha I_n]$

$$\{Z\}_{kl} = \begin{cases} -1, & k \sim l, \\ h_k, & k = l, \\ 0, & \text{otherwise;} \end{cases}$$

h_k = number of neighbors of the block k.

Another look at the likelihood

$$
\begin{aligned}
p(x|\mu,\alpha,\beta) \quad = \quad & (2\pi)^{-0.5n}\beta^{0.5n}\prod_{k=1}^{n}(\lambda_k+\alpha)^{0.5} \\
& \exp\left\{-0.5\beta\left[\sum_{k=1}^{n}(\alpha+h_k)(x_k-\mu)^2\right.\right. \\
& \left.\left.-\sum_{k=1}^{n}\sum_{l\in\partial k}(x_k-\mu)(x_l-\mu)\right]\right\},
\end{aligned}
$$

$\lambda_1 \geq \lambda_2 \geq \ldots, \geq \lambda_{n-1} > \lambda_n = 0$ are the eigenvalues of Z.

Confounding of α and β

When α is big enough:

$$p(x|\mu,\alpha,\beta) \quad \approx \quad (2\pi)^{-0.5n}(\alpha\beta)^{0.5n}\exp\left\{-0.5\alpha\beta\left[\sum_{k=1}^{n}(x_k-\mu)^2\right]\right\},$$

Blocks of x become approximately i.i.d. $N[\mu,(\alpha\beta)^{-1}]$.

Consequence: If we assign independent marginal improper priors for α and β, the marginal posteriors will be improper.

Reference prior

μ is a location parameter, so $\pi^r(\mu|\alpha,\beta) \propto 1$

Integrated likelihood with respect to μ:

$$\begin{aligned}
p(x|\alpha,\beta) &= \int_{-\infty}^{\infty} p(x|\mu,\alpha,\beta)\pi^r(\mu|\alpha,\beta)d\mu \\
&= (2\pi)^{-0.5(n-1)}\beta^{0.5(n-1)}n^{-0.5}\prod_{k=1}^{n-1}(\lambda_k+\alpha)^{0.5} \\
&\quad \exp\left\{-0.5\beta\left[\alpha\sum_{k=1}^{n}(x_k-\bar{x})^2+\sum_{k=1}^{n}h_k x_k^2-\sum_{k=1}^{n}\sum_{l\in\partial k}x_k x_l\right]\right\}
\end{aligned}$$

Reference prior

The reference prior for (α, β) is:

$$\begin{aligned}\pi^r(\alpha,\beta) &\propto \sqrt{J(\alpha,\beta)} \\ &\propto \beta^{-1}\sqrt{(n-1)\sum_{i=1}^{n-1}(\lambda_i+\alpha)^{-2} - \left[\sum_{i=1}^{n-1}(\lambda_i+\alpha)^{-1}\right]^{-2}},\end{aligned}$$

where $J(\alpha, \beta)$ is the determinant of the Fisher information matrix.

α and β are independent a priori.

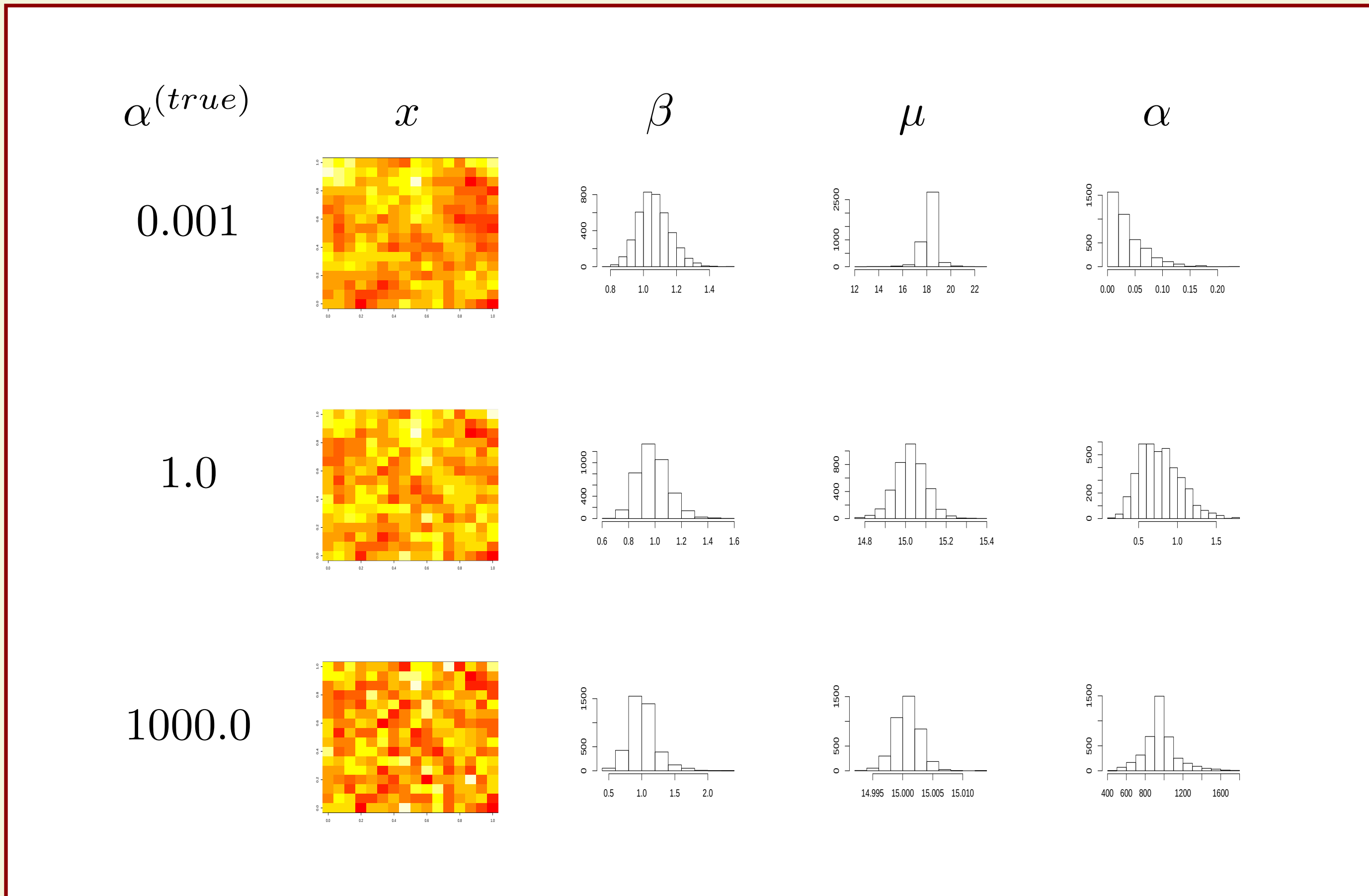
Propriety of prior and posterior

When α goes to zero the prior converges to a constant.
Using first order Taylor expansions around one of x^{-1} and x^{-2}:

$$\begin{aligned}
\pi^r(\alpha) &\propto \alpha^{-1}\sqrt{(n-1)\sum_{k=1}^{n-1}\left(1+\frac{\lambda_k}{\alpha}\right)^{-2} - \left[\sum_{k=1}^{n-1}\left(1+\frac{\lambda_k}{\alpha}\right)^{-1}\right]^2} \\
&\approx \alpha^{-1}\sqrt{(n-1)\sum_{k=1}^{n-1}\left(1-2\frac{\lambda_k}{\alpha}\right) - \left[\sum_{k=1}^{n-1}\left(1-\frac{\lambda_k}{\alpha}\right)\right]^2} \\
&= \alpha^{-2}\left(\sum_{k=1}^{n-1}\lambda_k\right)
\end{aligned}$$

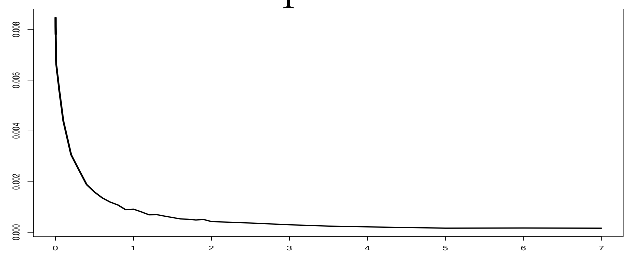
Thus, for large α the prior behaves like α^{-2}.
Therefore, the prior of α is integrable and provides a proper marginal posterior distribution for α.

Results for simulated fields



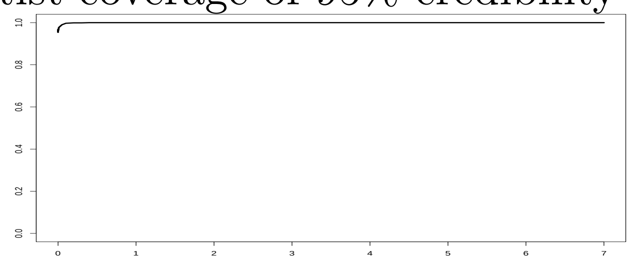
Frequentist properties - Estimation of β

Mean square error



α

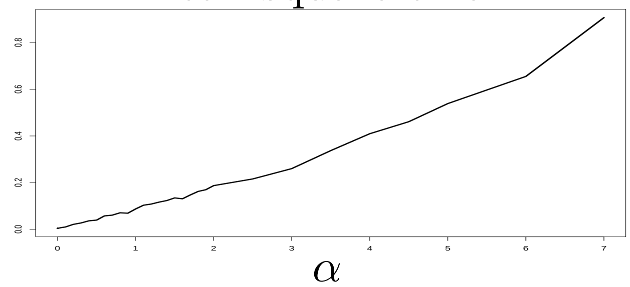
Frequentist coverage of 95% credibility intervals



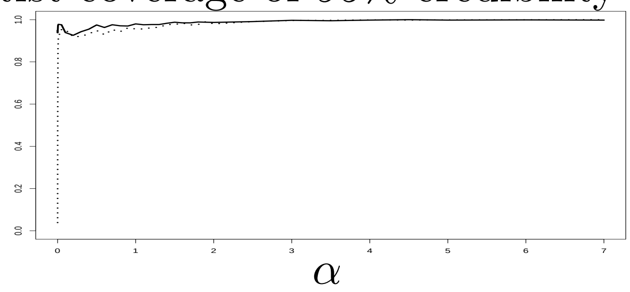
α

Frequentist properties - Estimation of α

Mean square error

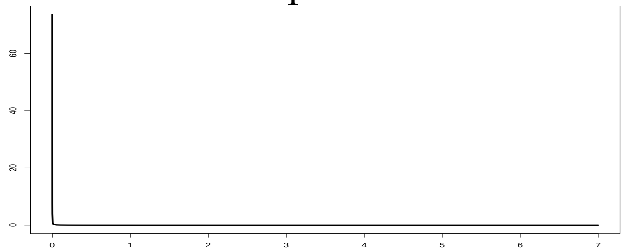


Frequentist coverage of 95% credibility intervals



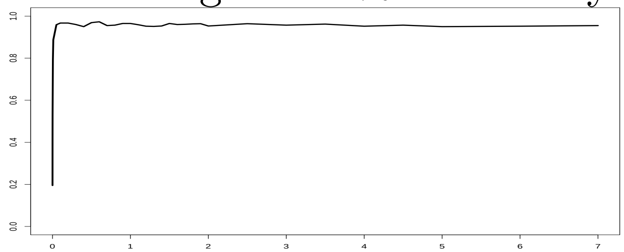
Frequentist properties - Estimation of μ

Mean square error



α

Frequentist coverage of 95% credibility intervals



α

Discussion

- Inference for β is robust with respect to $\alpha^{(true)}$
- HPD credibility interval for α works well for all values of $\alpha^{(true)}$
- Proposed objective Bayesian analysis works very well for $\alpha^{(true)} \geq 0.05$:
  - MSE of the estimators of β, μ and α is small
  - Frequentist coverage of central credibility intervals for β, μ and α is close to nominal value
- When $\alpha^{(true)}$ approaches zero field gets closer to improper:
  - Estimation of μ is more problematic, marginal posterior has heavy tails
  - Frequentist coverage of central credibility intervals for μ and α becomes too small