GTP Workshop on
Modeling MHD Turbulence; Applications to Planetary and Stellar dynamos
at NCAR, 27-29 June, 2006, Boulder, CO, USA

Spatially global & local scale-interaction analyses for nonconforming spectral-element simulations

Aimé Fournier
Institute for Mathematics Applied to Geosciences
National Center for Atmospheric Research, Boulder CO 80307-3000 USA

The classical mathematical tool to analyze scales in spatial functions $u(\vec{x})$ has been the Fourier basis $\mathrm{F}_{\vec{k}}(\vec{x}) := \mathrm{e}^{2\pi\mathrm{i}\vec{k}\cdot\vec{x}}$. The wavevector $\vec{k}$ labels *global* scale content, i.e., if a certain Fourier component $\hat{u}_{\vec{k}} := \langle \mathrm{F}^*_{\vec{k}} u\rangle$ is relatively large then *on average over the spatial domain* the corresponding field $u(\vec{x})$ exhibits relatively significant structure at the corresponding scale $|\vec{k}|^{-1}$. There is information only about scale but not the $\vec{x}$ *location* where the structures occur[1], which can be a serious limitation. Several remedies have been developed to regain that information. Fournier [1, 2, 3], and op. cit. therein, have generalized localized scale interactions (LSI[2]) from wavevector components $\hat{u}_{\vec{k}}$ to wavelet components $\tilde{u}_{\vec{\ell}} := \langle \psi^*_{\vec{\ell}} u\rangle$ using some basis $\psi_{\vec{\ell}}(\vec{x})$. These LSI analyses offer a multiscale analysis tool for which turbulence science has been striving, for a long time and for many purposes [2, for a review].

Another well known approach to multiscale simulation is adaptive mesh refinement (AMR). All AMR codes involve partitioning the problem's spatial domain $\mathbb{D}$ into disjoint elements $\mathbb{D} = \bigcup_{\vec{\ell}\in\mathbb{L}} \mathbb{X}_{\vec{\ell}}$, and most AMR codes use the finite-element method (FEM) or similar discretizations with a small set of values representing the global solution $u(\vec{x}, t)$ locally in each $\mathbb{X}_{\vec{\ell}}$. Thus *most AMR simulations are intrinsically locally low-order* w.r.t. the $\mathbb{X}_{\vec{\ell}}$ size $h_{\vec{\ell}}$. However, a few AMR codes are locally *high-order* w.r.t. a parameter $p_{\vec{\ell}}$ in each $\mathbb{X}_{\vec{\ell}}$; these include adaptive *spectral-element methods* (SEMs, e.g., [6, 7, 8] and op. cit. therein). The combined ***h-p*** analyses built into SEM make it very effective for complicated flows [5, 6, 7, 8]. Using SEM combined with LSI, we can quantitatively model and analyze many important phenomena that involve scale interactions localized in parts of the domain, and that heretofore were mainly only described qualitatively or heuristically.

The fundamental cause of scale interactions is the presence of nonlinearities in the governing dynamics. Nonlinear terms such as $\vec{v}\cdot\vec{\nabla}\vec{v}$ at high Reynolds number can generate significant phenomena, such as coherent vortices, fronts, tubes etc. Historically, important and insightful diagnostic tools for understanding these interactions have been linked to "spectral energetics", e.g., the analysis of Fourier spectra and triad interactions[3]

$$T^{\mathrm{F}}_{a,b,c} := \hat{\vec{u}}^*_{\vec{k}_a}\cdot(\hat{\vec{u}}_{\vec{k}_b}\cdot 2\pi\mathrm{i}\vec{k}_c\delta_{\vec{k}_a,\vec{k}_b+\vec{k}_c})\hat{\vec{u}}_{\vec{k}_c} \tag{1}$$

between modes a, b and c that describe global scale interactions without $\vec{x}$-location information. Using the $\psi_{\vec{\ell}}$ basis, scale resolution of u is degraded,[4] from a sharp wavevector value $\vec{k}$ down to *approximate wavevector elements* $\pm\vec{k} \in \mathbb{K}_{\vec{\ell}} := \operatorname{supp}\hat{\psi}_{\vec{\ell}} \approx \times_{\alpha=1}^{d}[K_\alpha, 2K_\alpha]$ (where $K_\alpha := 2^{\lfloor \log_2 \ell_\alpha\rfloor}$), while location information is augmented, from lack-of-information up to element locations $\vec{X} := \vec{\vec{K}}^{-1}\cdot(\vec{\ell} - \vec{K})$ (where $\vec{\vec{K}} := \operatorname{diag}\vec{K}$). We obtain new energetics diagnostics *describing both scale and location*:

$$T_{a,b,c} := \tilde{\vec{u}}^*_{\vec{\ell}_a}\cdot(\tilde{\vec{u}}_{\vec{\ell}_b}\cdot\langle\psi^*_{\vec{\ell}_a}\psi_{\vec{\ell}_b}\vec{\nabla}\psi_{\vec{\ell}_c}\rangle)\tilde{\vec{u}}_{\vec{\ell}_c}, \tag{2}$$

[1] Location information is dispersed among all $\arg\hat{u}_{\vec{k}}$.

[2] Apologies to *the Shamen*.

[3] In practice, $T_{a,b,c}$ is symmetrized to isolate boundary-flux or divergent-$\vec{v}$ contributions so that "detailed conservation" $T_{a,b,c} + T_{b,c,a} + T_{c,a,b} = 0$ holds.

[4] As required by the Heisenberg uncertainty principle; note that $|\mathbb{K}_{\vec{\ell}}| \gtrsim \prod_{\alpha=1}^{d} K_\alpha = |\mathbb{X}_{\vec{\ell}}|^{-1}$.

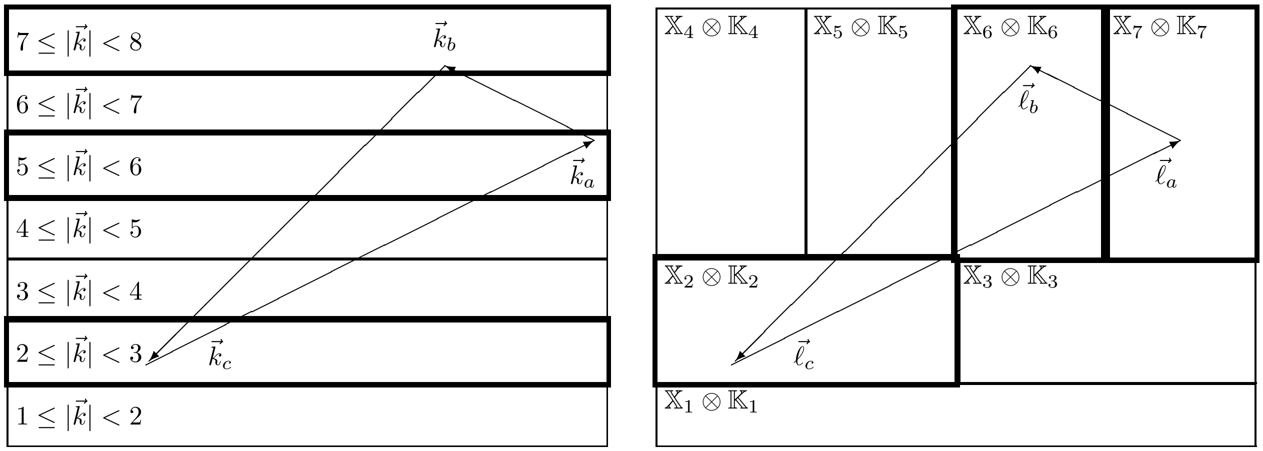

Figure 1: *Schematic illustration of triad interaction in Fourier space* (1)*, left, and wavelet space* (2)*, right.*

the triadic interaction among three structures in $\vec{u}$ that have characteristic scales $\vec{K}_a$, $\vec{K}_b$, $\vec{K}_c$ and locations $\vec{X}_a$, $\vec{X}_b$, $\vec{X}_c$ (Fig. 1). One can see that the triad (2) generalizes the Fourier triad (1). In fact, (2) is even more general, in that the $\psi_{\vec{\ell}}$ can be *any* orthogonal basis. In order to construct LSI we may use a SEM basis $\boldsymbol{\psi}_{\vec{\ell}}$ built up from a one-dimensional single-element basis $\psi_j(\xi)$ that can be either Legendre or interpolation polynomials for $0 \le \xi \le 1$. In the Legendre case the indexes $\vec{\jmath}$ augment the spectral resolution, while in the interpolation case they augment the spatial resolution.[5] Recently it was shown that it is also possible to use a SEM basis to compute (1) to machine precision [4].

References

[1] A. Fournier 1995. *Wavelet zonal spectral analysis of observed geopotential and winds: Scale-orthogonal decomposition of blocking patterns and local kinetic energy and enstrophy transfer between scales*, Eos, Trans. Amer. Geophys. Union **76**, Fall Meet. Suppl. Abstract A41A-02.

[2] A. Fournier 2003. *Atmospheric energetics in the wavelet domain II: Time-averaged observed atmospheric blocking*, J. Atmos. Sci. **60**, 319–338.

[3] A. Fournier 2005. *Instantaneous wavelet energetic transfers between atmospheric blocking and local eddies*, J. Climate **13**, 2151–2171.

[4] A. Fournier 2006. *Exact calculation of Fourier series in nonconforming spectral-element methods*, J. Comp. Phys. **215**, 1–5.

[5] A. Fournier, G. Beylkin and V. Cheruvu 2005. *Multiresolution adaptive space refinement in geophysical fluid dynamics simulation*, Lecture Notes Comp. Sci. Eng. **41**, 161–170.

[6] A. Fournier, M.A. Taylor and J.J. Tribbia 2004. *The spectral element atmosphere model (SEAM): High-resolution parallel computation and localized resolution of regional dynamics*, Mon. Wea. Rev. **132**, 726–748.

[7] D. Rosenberg and A. Fournier 2006. Presented elsewhere in this Workshop.

[8] D. Rosenberg, A. Fournier, P. Fischer and Annick Pouquet 2006. *Geophysical-astrophysical spectral-element adaptive refinement (GASpAR): Object-oriented h-adaptive fluid dynamics simulation*, J. Comp. Phys. **215**, 59–80.

[5] One must also high-pass filter to remove the large-scale polynomial space $\mathbb{P}_{\vec{p}}(\mathbb{X}_{\vec{\ell}})$ from the union $\bigcup_i \mathbb{P}_{\vec{p}}(\mathbb{X}_{i,\vec{\ell}})$ of its subdivided spaces, where $\bigcup_i \mathbb{X}_{i,\vec{\ell}} = \mathbb{X}_{\vec{\ell}}$ and $\mathbb{X}_{i,\vec{\ell}} \bigcap \mathbb{X}_{i' \neq i,\vec{\ell}} = \varnothing$.