Dynamic Sub-grid Scale Modeling of Drift Wave Turbulence within Magnetohydrodynamics

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Outline

- Formulation:
  - Wave kinetics and adiabatic theory
  - Mean field equations for large scales
- Specific Implementation
  - Modulational instability and inverse cascades
  - Self-consistent evolution of a tearing mode in the presence of drift wave turbulence
- Summary

Motivation

- D.N.S. of high Reynolds number systems are computationally impractical due to the high level of resolution necessary
- Filtering procedures are often employed as a means of preventing instabilities from developing on the smallest resolved scales
- However, as is well known strong nonlocal interactions (in scale space) can play an essential role in the evolution of many systems
- Furthermore, for systems which exhibit inverse cascades, it is necessary to treat both the large and small scales on an equal footing
- Here we discuss a dynamic sub-grid scale model which self-consistently describes the evolution of both the small (unresolved) scales and the large (resolved) scales

Wave Kinetics (I)

- Consider a generic fluid equation of the form

$$\frac{\partial \phi_q}{\partial t} + i\omega_q \phi_q = \sum_{p+l=q} A(p,l)\, \phi_p \phi_l$$

- Convenient to separate the system into resolved/unresolved variables

$\phi_k^{<} \leftarrow$ large scale (resolved)

$\phi_k^{>} \leftarrow$ small scale (unresolved)

- Mean field equations can be obtained after performing an average over the rapidly evolving scales

$$\frac{\partial \phi_q^{<}}{\partial t} + i\omega_q \phi_q^{<} = \sum_{p+l=q} A(p,l)\, \phi_p^{<} \phi_l^{<} + \sum_{p+l=q} A(p,l) \left\langle \phi_p^{>} \phi_l^{>} \right\rangle$$

- Effect of unresolved scales is to introduce Reynolds stresses into the mean field equation

- Seek equation describing the evolution of stress term

Wave Kinetics (II)

- Unresolved scales see resolved scales as slowly evolving background fields
  - allows description of small scale evolution via ray tracing equations
- Corresponds to description of individual wave packets being advected and refracted by mean fields

Formulate via adiabatic theory:

- In presence of mean fields, unresolved scales can be modeled via a wave kinetic equation:

$$\frac{\partial}{\partial t}N_k + \overbrace{\frac{\partial}{\partial \mathbf{k}}(\omega_k + \delta\omega_k)\cdot\frac{\partial}{\partial \mathbf{x}}N_k}^{\text{Advection}} - \overbrace{\frac{\partial}{\partial \mathbf{x}}(\omega_k + \delta\omega_k)\cdot\frac{\partial}{\partial \mathbf{k}}N_k}^{\text{Shearing}} = 0$$

$$\omega_k \leftarrow \text{linear frequency}, \quad \delta\omega_k \leftarrow \text{nonlinear frequency modulation}$$

- Here, N_k corresponds to adiabatically conserved quality
  - usually (but not always) wave action density $\equiv E_k/\omega_k$

Wave Kinetics (III)

- Note that the above equation is isomorphic to collisionless Boltzmann equation
- Thus, can be understood to describe the evolution of a 'gas' of quasi-particles whose trajectories are described by:

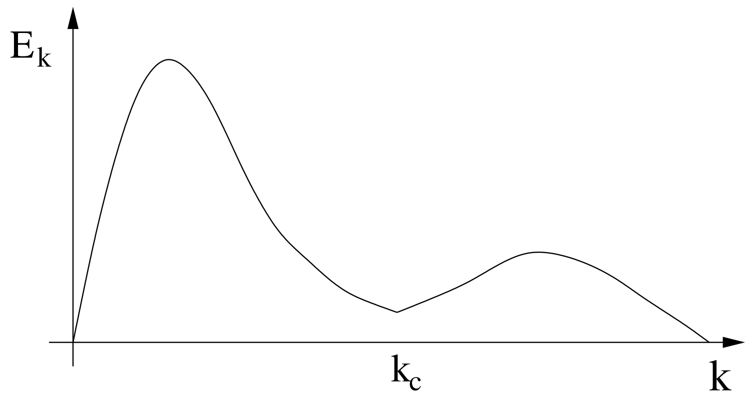
$$\dot{\mathbf{x}} = \frac{\partial}{\partial \mathbf{k}} \left(\omega_k + \delta\omega_k \right), \qquad \dot{\mathbf{k}} = -\frac{\partial}{\partial \mathbf{x}} \left(\omega_k + \delta\omega_k \right)$$

- Hamiltonian structure with, $\omega_k + \delta\omega_k \Leftrightarrow H$, $\mathbf{x} \Leftrightarrow \mathbf{q}$, $\mathbf{k} \Leftrightarrow \mathbf{p}$
- Quasi-particles 'see' mean fields through frequency modulations introduced by the large scale mean fields
- Description well suited to systems exhibiting wave turbulence:
  - Alfvenic turbulence
  - Rossby wave turbulence
  - Langmuir turbulence
  - etc...

Wave Kinetics (IV)

Limitations of wave kinetic formalism:

- Neglects local interactions between unresolved scales in favor of nonlocal interactions with resolved scales
- Requires temporal/spatial scale separation between resolved and unresolved scales, i.e.



- Description especially appropriate for systems which exhibit “inverse cascades”
- However, local interactions can be modeled via the introduction of a “collision” operator on the R.H.S. of the wave kinetic equation

Wave Kinetics (V)

- Closed set of equations given by:

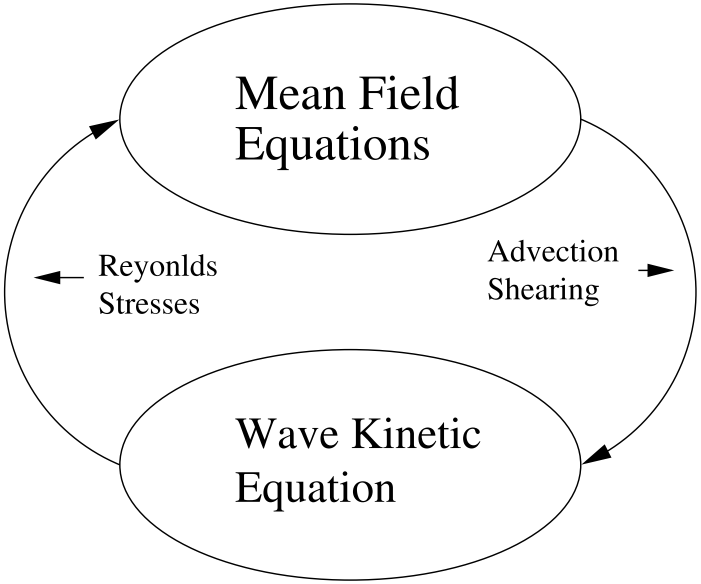
$$\textbf{resolved} \rightarrow \frac{\partial \phi_q^<}{\partial t} + i\omega_q \phi_q^< = \sum_{p+l=q} A(p,l)\,\phi_p^< \phi_l^< + \sum_{p+l=q} A(p,l)\left\langle \phi_p^> \phi_l^> \right\rangle$$

$$\textbf{unresolved} \rightarrow \frac{\partial}{\partial t} N_k + \frac{\partial}{\partial \mathbf{k}}(\omega_k + \delta\omega_k) \cdot \frac{\partial}{\partial \mathbf{x}} N_k - \frac{\partial}{\partial \mathbf{x}}(\omega_k + \delta\omega_k) \cdot \frac{\partial}{\partial \mathbf{k}} N_k = S$$

where

$$\sum_{p+l=q} A(p,l)\left\langle \phi_p^> \phi_l^> \right\rangle \sim \sum_k B(k,q)\, N_k$$

- Small scale 'gas' advected/refracted by mean fields
- React back via stresses on mean fields



Nonlocal Interactions within 2-D Hydrodynamics

- Consider small scale turbulent eddies evolving in the presence of a strong large scale flow
- Evolution of small scale eddies described (Dubrulle and Nazarenko(1997)) by wave kinetic equation

$$\frac{\partial N_k}{\partial t} + \mathbf{v}_0 \cdot \frac{\partial N_k}{\partial \mathbf{x}} - \frac{\partial}{\partial \mathbf{x}} (\mathbf{k} \cdot \mathbf{v}_0) \cdot \frac{\partial N_k}{\partial \mathbf{k}} = 0$$

- where $N_k = k_\perp^4 \left|\phi_k\right|^2 = k_\perp^2 E_k$
- Here, key is that vorticity conserved along fluid trajectories
  - corresponds to conservation of density of vortices/rotons intensity
  - thus, enstrophy density, not wave action density (note $\omega = 0$), conserved along ray trajectories

Modulational Instability (I)

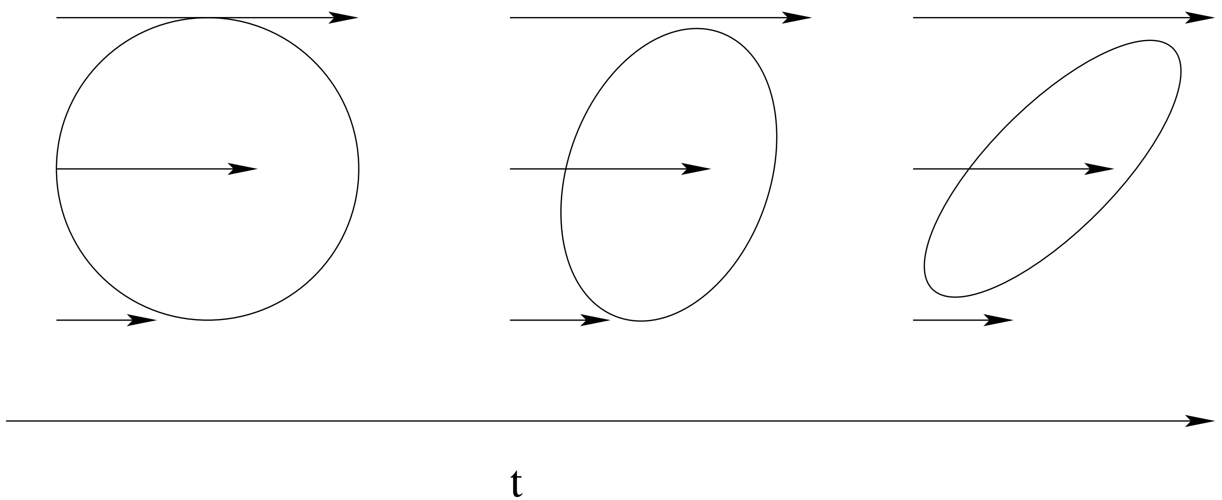
- Interesting to consider characteristics

$$\dot{\mathbf{k}} = -\frac{\partial}{\partial \mathbf{x}} \left(\mathbf{v_0} \cdot \mathbf{k}\right), \quad \dot{\mathbf{x}} = \frac{\partial}{\partial \mathbf{k}} \left(\mathbf{v_0} \cdot \mathbf{k}\right) = \mathbf{v_0}$$

- For simplicity consider a flow directed in the y direction and only varying in the x direction

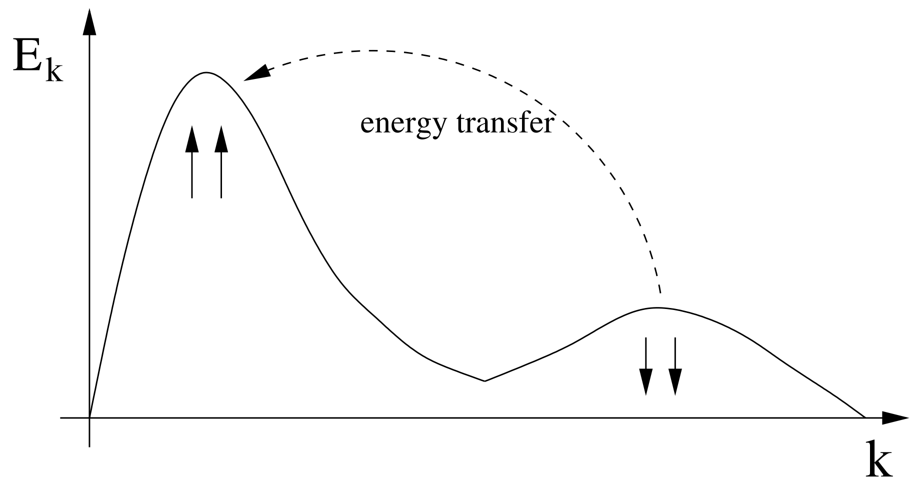
$$\begin{aligned} k_x &\approx k_0 - v_y' k_y t \\ \Rightarrow k_x^2 &\approx \left(v_y' k_y t\right)^2 \end{aligned}$$

- Perpendicular length scale of small scale eddy decreases
- Corresponds to transfer of enstrophy to higher wave number



Modulational Instability (II)

- Thus, since $N_k = k_\perp^2 E_k$ constant along characteristics, E_k must go down
- Energy nonlocally (in scale space) transferred to large scale shear flow



- Note self-consistency essential
  - Strong large scale shear flow generated by inverse cascade
  - Large scales react back via shearing⇒kills small scale drive
- Absence of dynamics on small scales leads to absurd results!

Drift Waves

- As a specific realization of the above model, consider the problem of a tearing mode developing in the presence of drift wave turbulence
- Here, small scale drift wave dynamics described by Charney-Hawegawa-Mima equation

$$0 = \left(\frac{\partial}{\partial t} + \frac{c}{B_0} \left(\hat{\mathbf{z}} \times \nabla \phi^< \right) \cdot \nabla \right) \phi^> + v_e^* \frac{\partial}{\partial y} \phi^> - \rho_s^2 \left(\frac{\partial}{\partial t} + \frac{c}{B_0} \left(\hat{\mathbf{z}} \times \nabla \phi^< \right) \cdot \nabla \right) \nabla_\perp^2 \phi^>$$

- A conservation law for the drift wave enstrophy density can be derived (Smolyakov and Diamond(1999))

$$\frac{\partial}{\partial t} N_k + \frac{\partial}{\partial \mathbf{k}} \left(\omega_k + \mathbf{k} \cdot \mathbf{V}_0 \right) \cdot \frac{\partial}{\partial \mathbf{x}} N_k - \frac{\partial}{\partial \mathbf{x}} \left(\omega_k + \mathbf{k} \cdot \mathbf{V}_0 \right) \cdot \frac{\partial}{\partial \mathbf{k}} N_k = S$$

$$S = \gamma_k N_k - \Delta\omega N_k^2, \quad N_k = \left(1 + \rho_s^2 k_\perp^2 \right)^2 I_k, \quad I_k \left(\mathbf{x}, t \right) \equiv \int d\mathbf{q} e^{i\mathbf{q}\cdot\mathbf{x}} \left\langle \phi_{k+q}^> \phi_{-k}^> \right\rangle$$

Mean Field Equations (I)

- Here we consider mean flow equations, interacting with drift waves
- Note that for electrostatic turbulence, lowest order coupling to micro turbulence is through the polarization nonlinearity

$$0 = \frac{\partial}{\partial t}\psi^{<} + \frac{c}{B_0}\left(\hat{\mathbf{z}} \times \nabla\phi^{<}\right) \cdot \nabla\psi^{<} - v_A\frac{\partial}{\partial z}\phi^{<} - \eta_c\nabla_\perp^2\psi^{<}$$

$$\begin{aligned} 0 \ &= \ \frac{\partial}{\partial t}\nabla_\perp^2\phi^{<} + \frac{c}{B_o}\left(\hat{\mathbf{z}} \times \nabla\phi^{<}\right) \cdot \nabla\nabla_\perp^2\phi^{<} - v_A\frac{\partial}{\partial z}\nabla_\perp^2\psi^{<} \\ &- \ \frac{c}{B_0}\left(\hat{\mathbf{z}} \times \nabla\psi^{<}\right) \cdot \nabla\nabla_\perp^2\psi^{<} + \frac{c}{B_0}\underbrace{\left\langle\left(\hat{\mathbf{z}} \times \nabla\phi^{>}\right) \cdot \nabla\nabla_\perp^2\phi^{>}\right\rangle}_{\text{Coupling to Micro Turbulence}} \end{aligned}$$

- Where the average $\langle ... \rangle$ is over fast spatial and temporal scales.
- To understand response of micro turbulence we calculate response of turbulence spectrum to "seed" asymmetry, symbolically:

$$\left\langle\left(\hat{\mathbf{z}} \times \nabla\phi^{>}\right) \cdot \nabla\nabla_\perp^2\phi^{>}\right\rangle \sim \frac{\partial^2}{\partial x^2}\int d\mathbf{k}M\left(\mathbf{k}\right)\frac{\delta N_k}{\delta\phi^{<}}\phi^{<}$$

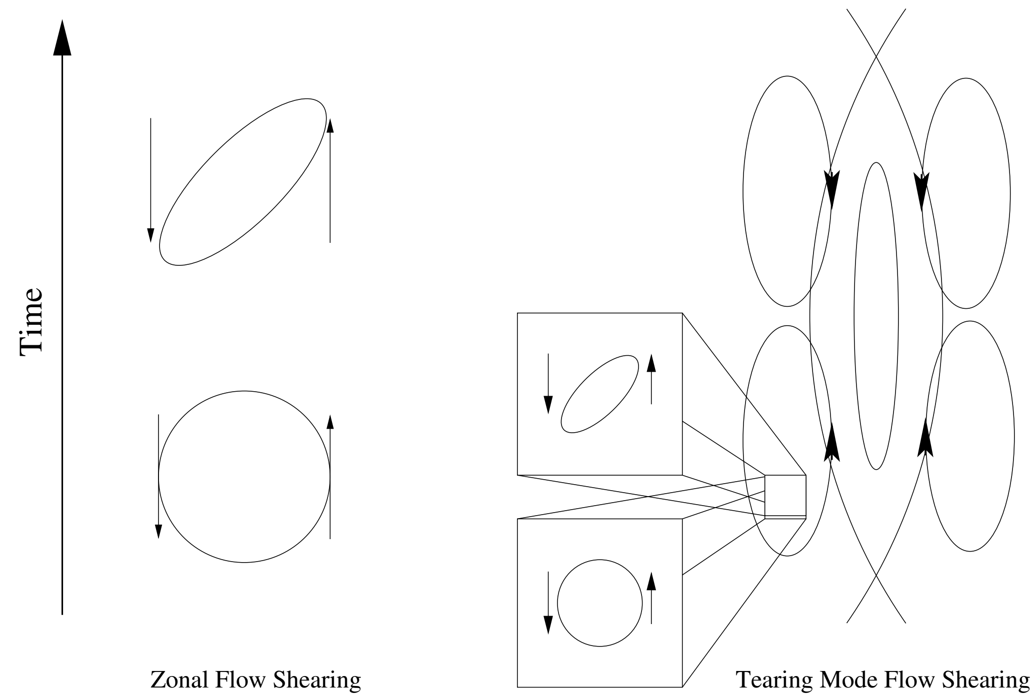
Mean Field Equations (II)

- Response of drift wave turbulence calculated via wave kinetic equation:

$$\left\langle \left(\hat{\mathbf{z}} \times \nabla \phi^{>}\right) \cdot \nabla \nabla_{\perp}^{2} \phi^{>} \right\rangle \approx \underbrace{- c_s^2 \int d\mathbf{k} \frac{\rho_s^2 k_y^2}{\left(1 + \rho_s^2 k_{\perp}^2\right)^2} \frac{\gamma_k}{\left(\gamma_k^2 + \left(\mathbf{q} \cdot \mathbf{v}_{gr}\right)^2\right)} k_x \frac{\partial N_k^0}{\partial k_x}}_{\nu_T} \frac{\partial^4 \phi^{<}}{\partial x^4}$$

  - Thus, for $k_x \frac{dN_k^0}{dk_x} < 0$:

$$\nu_T < 0$$



- Physical mechanism underlying inverse cascade can be seen to be similar to zonal flow excitation

Tearing Mode Equations

- Considering the limit where $\partial/\partial x \gg \partial/\partial y$, the linearized tearing mode equations are given by:

$$\overbrace{\gamma_q \frac{\partial^2 \phi^<}{\partial x^2} = i q_y v_A \frac{x}{L_s} J + \nu_T \frac{\partial^4 \phi^<}{\partial x^4}}^{\text{Vorticity Equation}}, \quad \overbrace{\eta_c J = \gamma_q \psi^< - i q_y v_A \frac{x}{L_s} \phi^<}^{\text{Induction Equation}}$$

Useful simplification:

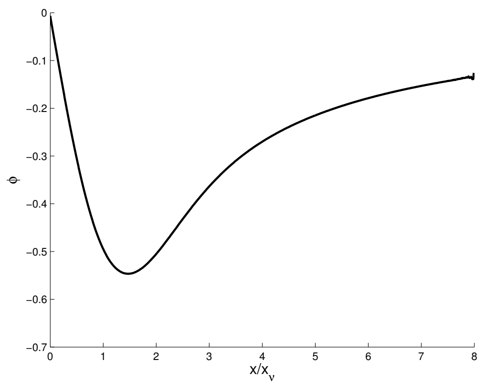
- for $\gamma_T \tau_\eta^{(T)} < 1$, magnetic field is able to diffuse into visco-resistive layer. Thus can approximate $\psi^< \rightarrow \psi_0 = \text{const}$
- Leads to following set of interior equations (in dimensionless units)

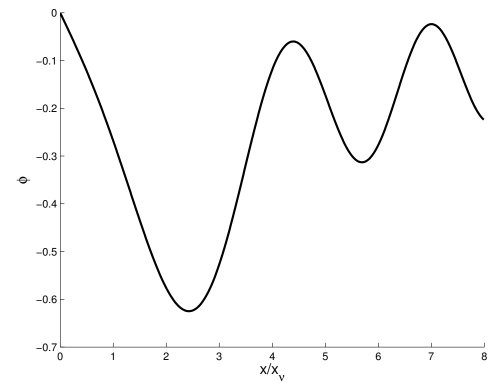
$$0 = -\frac{\partial^4 \Phi}{\partial \sigma^4} - \frac{1}{\alpha}\frac{\partial^2 \Phi}{\partial \sigma^2} + \sigma\left(1 + \sigma\Phi\right) \qquad \Delta' = -\frac{i\omega_q}{\eta_c} x_\nu \int d\sigma \left(1 + \sigma\Phi\right)$$

$$\sigma = x/x_\nu, \quad \alpha = i\left|\nu_T\right| / \left(x_T^2 \omega_q\right) \quad \Delta' = \left(\psi'\left(0^+\right) - \psi'\left(0^-\right)\right)/\psi_0$$

Tearing Mode

- After performing the linear analysis, radial eigenmodes have the following structure

usual F.K.R. $\rightarrow$ 



$\leftarrow$ neg. visc.

- $v_y = \partial \phi / \partial x$, thus oscillations in the radial eigenmode correspond to oscillating shear flows induced near the resonant surface

- Linear growth rate given by:

$$\gamma_q \sim \text{Re}\left(\omega_q\right) \sim \frac{\eta_c}{x_\nu}\Delta' \sim \frac{\eta_c^{5/6}}{\left|\nu_T\right|^{1/6}}\left(\frac{q_y v_A}{L_s}\right)^{1/3}\Delta'$$

- Distortion of flow pattern near resonant surface, leads to slowing down of magnetic reconnection (McDevitt and Diamond(2006))

- Presence of self-induced shear, introduces real frequency

Summary

- Discussion of wave kinetic formalism as sub-grid scale model
- Self-consistent formulation of interaction of a tearing mode with drift wave turbulence
- Identification of the negative turbulent viscosity as the dominant effect on low-m tearing mode
- Calculation of linear growth rate of tearing mode in the presence of negative viscosity

References

B. Dubrulle and S. Nazarenko, Physica D **110**, 123 (1997).

A. Smolyakov and P. Diamond, Phys. Plasmas **6**, 4410 (1999).

C. McDevitt and P. Diamond, Phys. Plasmas **13**, 032302 (2006).