A Systematic Multi-Scale Framework for Meteorological Modelling I:

Dry Basics

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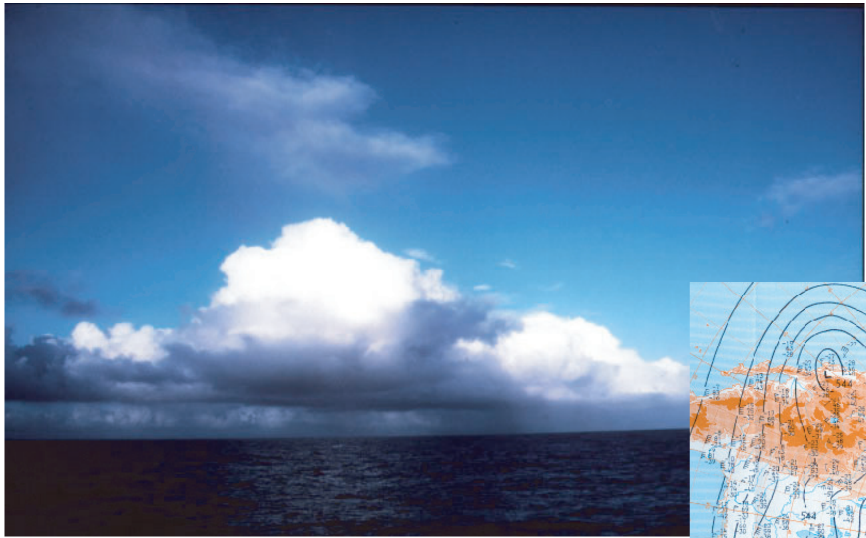
Thanks to ...

Motivation

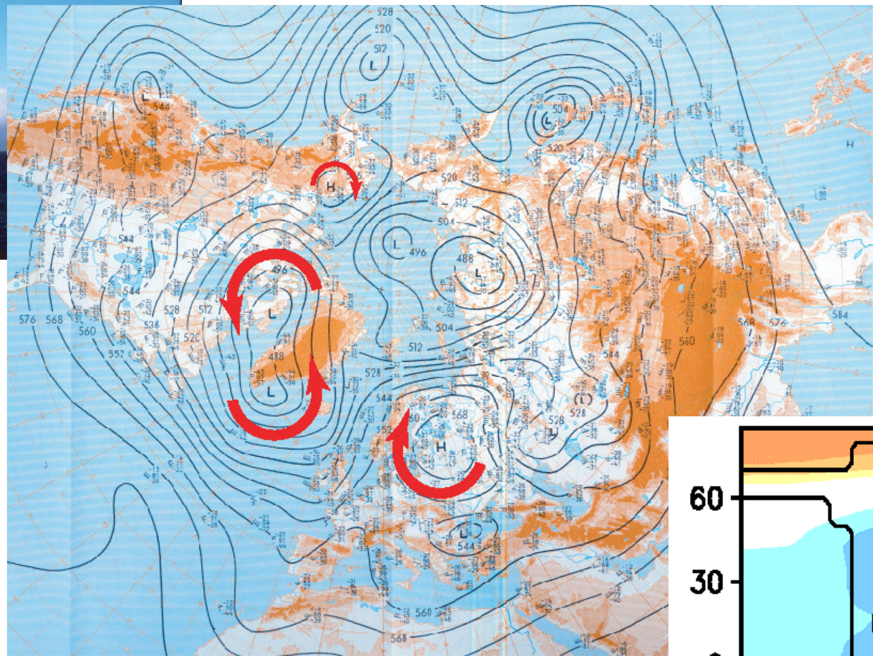
Unified approach via formal asymptotics

Anelastic limits

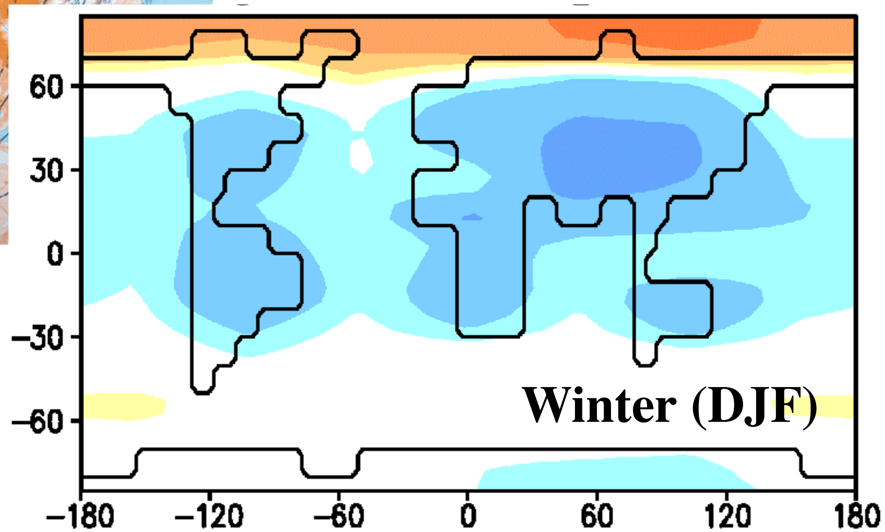
Mesoscale–convective interaction



10 km / 20 min



1000 km / 2 days



10000 km / 1 season

Scales in the Atmosphere

$$
\begin{aligned}
\boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + w\boldsymbol{u}_z + \nabla\pi &= \boldsymbol{S}_{\boldsymbol{u}} \\
w_t + \boldsymbol{u}\cdot\nabla w + w w_z + \pi_z &= -\theta' + S_w \\
\theta'_t + \boldsymbol{u}\cdot\nabla\theta' + w\theta'_z &= S'_\theta \\
\nabla\cdot(\rho_0\boldsymbol{u}) + (\rho_0 w)_z &= 0
\end{aligned}
$$

$$
\theta = 1 + \varepsilon^4\theta'(\boldsymbol{x}, z, t) + o(\varepsilon^4)
$$

Anelastic Boussinesque Model

10 km / 20 min

$$
\left(\partial_\mathcal{T} + \boldsymbol{u}^{(0)}\cdot\nabla\right) q = 0
$$

$$
q = \zeta^{(0)} + \Omega_0\beta\eta + \frac{\Omega_0}{\rho^{(0)}}\frac{\partial}{\partial z}\left(\frac{\rho^{(0)}}{d\Theta/dz}\theta^{(3)}\right)
$$

$$
\zeta^{(0)} = \nabla^2\pi^{(3)}, \qquad \theta^{(3)} = -\frac{\partial\pi^{(3)}}{\partial z}, \qquad \boldsymbol{u}^{(0)} = \frac{1}{\Omega}\boldsymbol{k}\times\nabla\pi^{(3)}
$$

Quasi-geostrophic theory

1000 km / 2 days

$$
\begin{aligned}
\frac{\partial Q_T}{\partial t} + \nabla\cdot\boldsymbol{F}_T &= S_T \\
\frac{\partial Q_q}{\partial t} + \nabla\cdot\boldsymbol{F}_q &= S_q
\end{aligned}
$$

$$
Q_\varphi = \int_{z_s}^{H_a}\rho\,\varphi\,dz\,, \qquad \boldsymbol{F}_\varphi = \int_{z_s}^{H_a}\rho\left(\boldsymbol{u}\,\varphi + (\widehat{\boldsymbol{u}'\varphi'}) + \boldsymbol{D}^\varphi\right)dz\,, \qquad \left(\varphi\in\{T,q\}\right)
$$

$$
T = T_s(t,\boldsymbol{x}) + \Gamma(t,\boldsymbol{x})\left(\min(z,H_T) - z_s\right), \qquad q = q_s(t,\boldsymbol{x})\exp\left(-\frac{z-z_s}{H_q}\right)
$$

$$
\rho = \rho_*\exp\left(-\frac{z}{h_{\mathrm{sc}}}\right), \qquad p = p_*\exp\left(-\frac{\gamma z}{h_{\mathrm{sc}}}\right) + p_0(t,\boldsymbol{x}) + g\rho_*\int_0^z\frac{T}{T_*}\,dz'
$$

$$
\boldsymbol{u} = \boldsymbol{u}_g + \boldsymbol{u}_a\,, \qquad f\rho_*\boldsymbol{k}\times\boldsymbol{u}_g = -\nabla_{\boldsymbol{x}}p \qquad \boldsymbol{u}_\alpha = \alpha\nabla p_0
$$

V. Petoukhov et al., *CLIMBER-2 ...*, Climate Dynamics, 16, (2000)

EMIC - equations (CLIMBER-2)

10000 km / 1 season

Scales in the Atmosphere

Three-dimensional compressible flow equations

$$
\begin{aligned}
\rho_t &+ \nabla \cdot (\rho \boldsymbol{v}) &= 0 \\
(\rho \boldsymbol{v})_t &+ \nabla \cdot (\rho \boldsymbol{v} \circ \boldsymbol{v}) + \nabla p + \boldsymbol{\Omega} \times \rho \boldsymbol{v} &= \boldsymbol{S}_{\rho \boldsymbol{v}} - \rho g \, \boldsymbol{k} \\
(\rho e)_t &+ \nabla \cdot (\boldsymbol{v} \, [\rho e + p]) &= S_{\rho e} \\
(\rho Y_j)_t &+ \nabla \cdot (\rho Y_j \boldsymbol{v}) &= S_{\rho Y_j}
\end{aligned}
$$

$$
(\rho e) = \frac{p}{\gamma - 1} + \frac{1}{2} \rho \boldsymbol{v}^2 + \rho \sum_{j=1}^{N} Q_j Y_j
$$

How are various reduced models related to this system ?

The "Truth"

Motivation

Unified approach via formal asymptotics

Anelastic limits

Mesoscale–convective interaction

Desirables

- **Allways start from 3D compressible flow equations**
- **Asymptotic parameters independent of specific scales**
- **Scale selection motivated by small parameters**
  (not vice versa)

Key ingredients

1. **Identification of**

   - **uniformly valid system scales**
   - **non-dimensional parameters**
   - **distinguished limits**

2. **Specializations of a multiple scales ansatz**

$$
\begin{aligned}
a &= 6 \cdot 10^6 \ \ \mathrm{m} \\
\Omega &= 10^{-4} \ \ 1/\,\mathrm{s} \\
g &= 9.81 \ \ \mathrm{m/s^2} \\
\\
p_{\mathrm{ref}} &= 10^5 \ \ \mathrm{kg/ms^2} \\
\rho_{\mathrm{ref}} &= 1.25 \ \ \mathrm{kg/m^3} \\
u_{\mathrm{ref}} &= 10 \ \ \mathrm{m/s}
\end{aligned}
$$

Key ingredients

1. **Identification of**

   - **uniformly valid system scales**
   - **non-dimensional parameters**
   - **distinguished limits**

2. **Specializations of a multiple scales ansatz**

$$\frac{c_{\mathrm{ref}}}{\Omega\, a} \sim \quad 0.5$$

$$\frac{u_{\mathrm{ref}}}{c_{\mathrm{ref}}} \sim 3 \cdot 10^{-2}$$

$$\frac{a\,\Omega^2}{g} \sim 6 \cdot 10^{-3}$$

$$\left(c_{\mathrm{ref}} = \sqrt{p_{\mathrm{ref}}/\rho_{\mathrm{ref}}}\right)$$

Unified approach via formal asymptotics

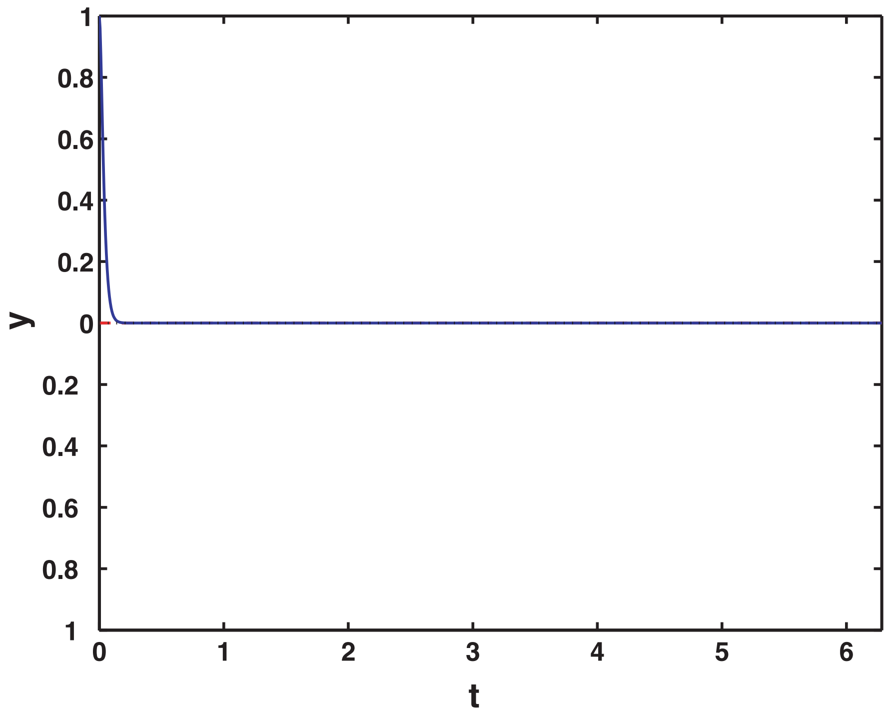
An alternative route

1. Identification of

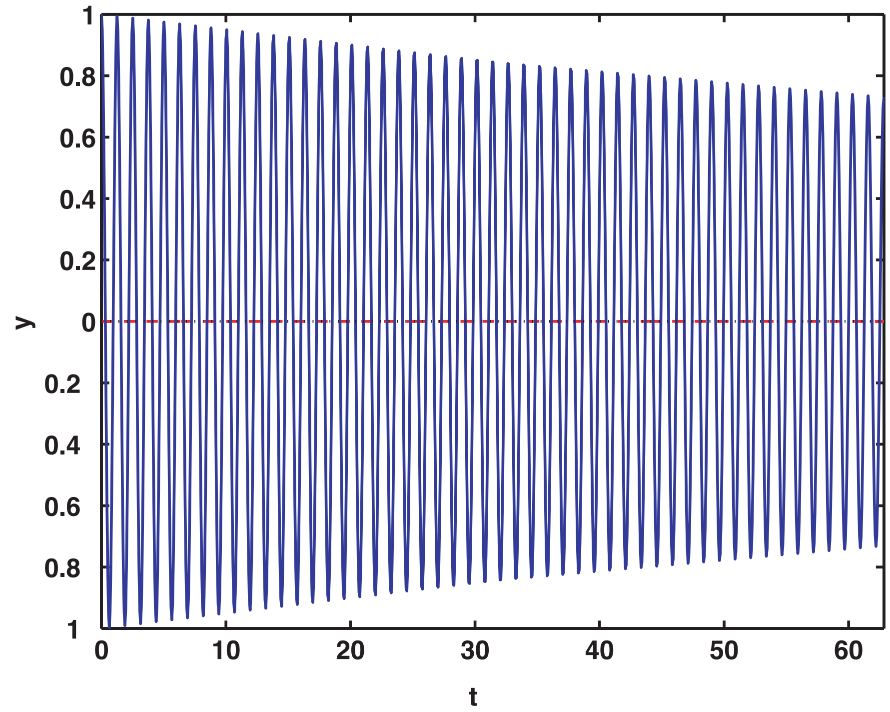
   - uniformly valid system scales
   - non-dimensional parameters
   - distinguished limits

2. Specializations of a multiple scales ansatz

$$\varepsilon\, \ddot{y} + \delta\, \dot{y} + y = 0$$

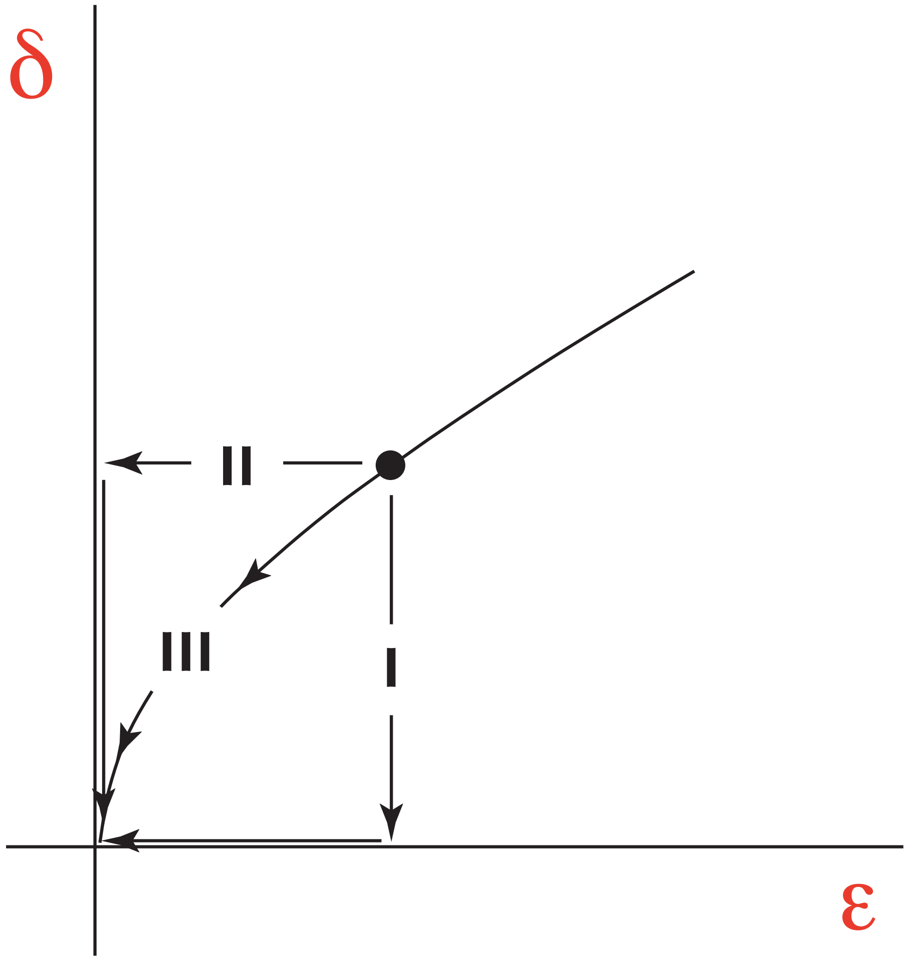


$\varepsilon \ll \delta \ll 1$



$\delta \ll \varepsilon \ll 1$

Limit solutions for $\varepsilon, \delta \to 0$ are path-dependent!!



Distinguished Limits

Key ingredients

1. **Identification of**

   - **uniformly valid system scales**
   - **non-dimensional parameters**
   - **distinguished limits**

$$\frac{c_{\mathrm{ref}}}{\Omega\, a} = O(1)$$

$$\frac{u_{\mathrm{ref}}}{c_{\mathrm{ref}}} = O(\varepsilon^2)$$

$$\frac{a\,\Omega^2}{g} = O(\varepsilon^3)$$

$$(\varepsilon \longrightarrow 0)$$

2. **Specializations of a multiple scales ansatz**

Unified approach via formal asymptotics

Key ingredients

1. **Identification of**

   - **uniformly valid system scales**
   - **non-dimensional parameters**
   - **distinguished limits**

$$
\begin{aligned}
\mathrm{Fr} &\sim \varepsilon \\
\overline{\mathrm{Fr}} &\sim \varepsilon^2 \\
\mathrm{M} &\sim \varepsilon^2 \\
\mathrm{Ro}_{h_{\mathrm{sc}}} &\sim \varepsilon^{-1} \\
\mathrm{Ro}_{L} &\sim \varepsilon
\end{aligned}
$$

etc.

2. **Specializations of a multiple scales ansatz**

Key ingredients

1. Identification of

   - uniformly valid system scales
   - non-dimensional parameters
   - distinguished limits

2. Specializations of a multiple scales ansatz

$$\mathrm{U}(\boldsymbol{x}, z, t; \varepsilon) = \sum_i \varepsilon^i \, \mathrm{U}^{(i)}(\boldsymbol{x}, \varepsilon \boldsymbol{x}, \varepsilon^2 \boldsymbol{x}, \ldots, z, \varepsilon z, \varepsilon^2 z, \ldots, t, \varepsilon t, \varepsilon^2 t, \ldots)$$

Unified approach via formal asymptotics

$\mathbf{U}^{(i)} = \mathbf{U}^{(i)}(t, \boldsymbol{x}, z)$	**Anelastic & pseudo-incompressible models**
$\mathbf{U}^{(i)} = \mathbf{U}^{(i)}(t, \varepsilon\boldsymbol{x}, z)$	**Linear large scale internal gravity waves**
$\mathbf{U}^{(i)} = \mathbf{U}^{(i)}(\frac{t}{\varepsilon}, \boldsymbol{x}, \frac{z}{\varepsilon})$	**Linear small scale internal gravity waves**
$\mathbf{U}^{(i)} = \mathbf{U}^{(i)}(\varepsilon^2 t, \varepsilon^2\boldsymbol{x}, z)$	**Mid-latitude Quasi-Geostrophic Flow**
$\mathbf{U}^{(i)} = \mathbf{U}^{(i)}(\varepsilon^2 t, \varepsilon^2\boldsymbol{x}, z)$	**Equatorial Weak Temperature Gradients**
$\mathbf{U}^{(i)} = \mathbf{U}^{(i)}(\varepsilon^2 t, \varepsilon^{-1}\xi(\varepsilon^2\boldsymbol{x}), z)$	**Semi-geostrophic flow**
$\mathbf{U}^{(i)} = \mathbf{U}^{(i)}(\varepsilon^3 t, \varepsilon^3 x, \varepsilon^2 y, z)$	**Equatorial Kelvin, Yanai & Rossby Waves**

Recovered classical reduced models

Multiple Scales Asymptotics

Numerics in Conservation Form

Realistic estimate:

$$\varepsilon \sim \frac{1}{8} \dots \frac{1}{6}$$

Interacting length scales:

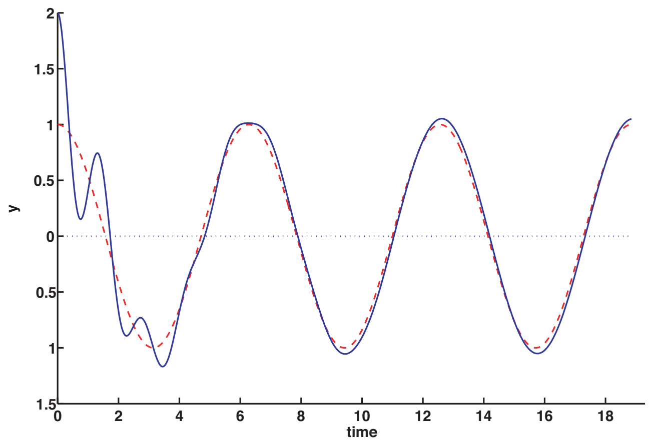
$$10 \dots 70 \dots 500 \dots 3500 \dots \text{ km}$$

Interacting time scales:

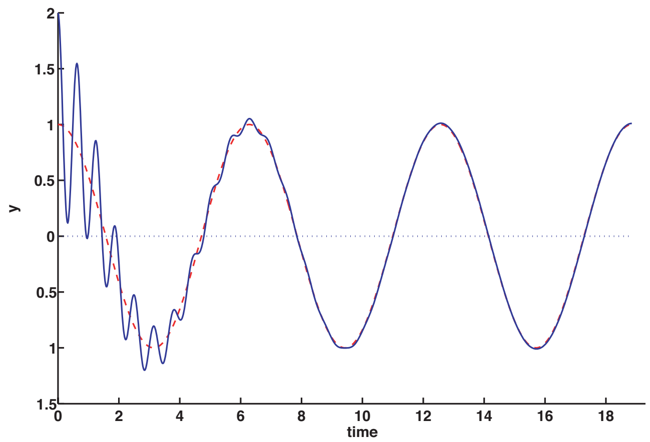
$$3 \dots 20 \text{ min} \dots 3 \text{ h} \dots 1 \text{ day} \dots 1 \text{ week}$$

$$\varepsilon\ddot{y} + \delta\dot{y} + y = \cos(t) \qquad y(0) = 2, \quad \dot{y}(0) = 0$$

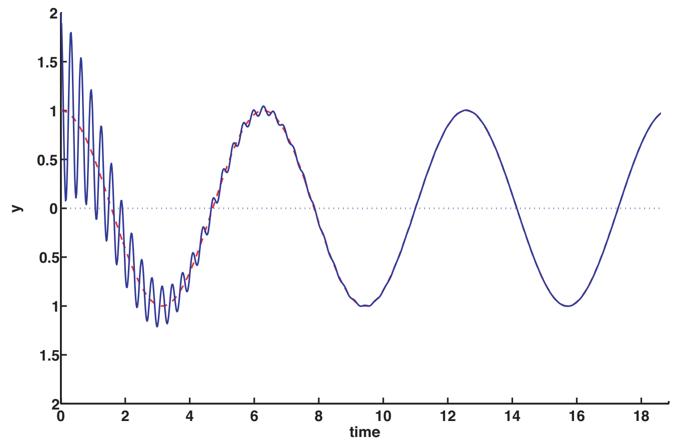
Distinguished limit: $\varepsilon \sim \delta$ $\quad (\delta = \kappa\varepsilon$ as $\varepsilon \to 0)$



$\delta = 0.05\,, \kappa = 1$



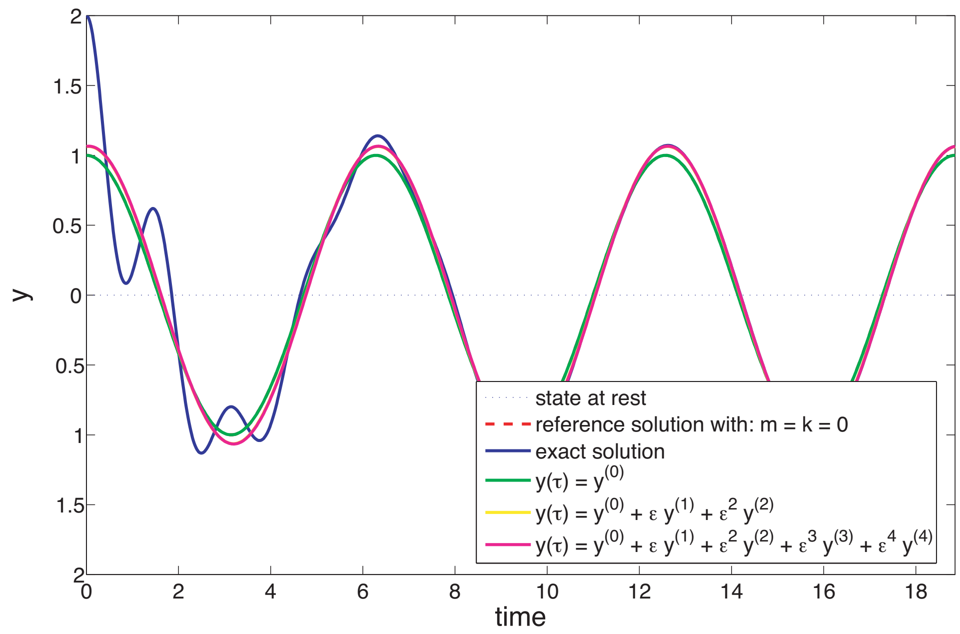
$\delta = 0.01\,, \kappa = 1$



$\delta = 0.0025\,, \kappa = 1$

Slow time expansion:

$$y(t;\varepsilon) = y^{(0)}(t) + \varepsilon\, y^{(1)}(t) + \ldots$$

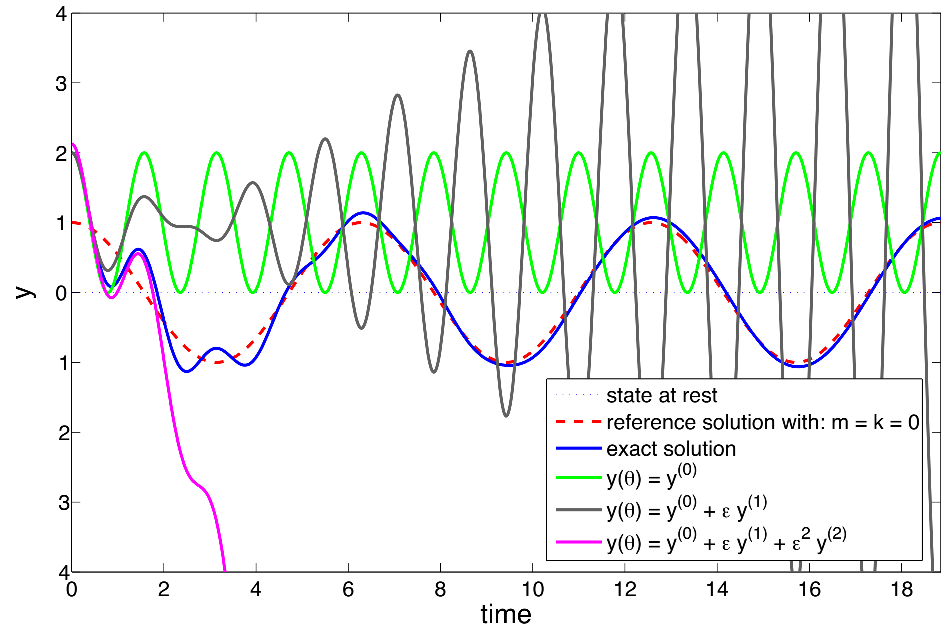


Fast time expansion:

$$y(t;\varepsilon) = Y^{(0)}(\tau) + \varepsilon\, Y^{(1)}(\tau) + \ldots$$

where

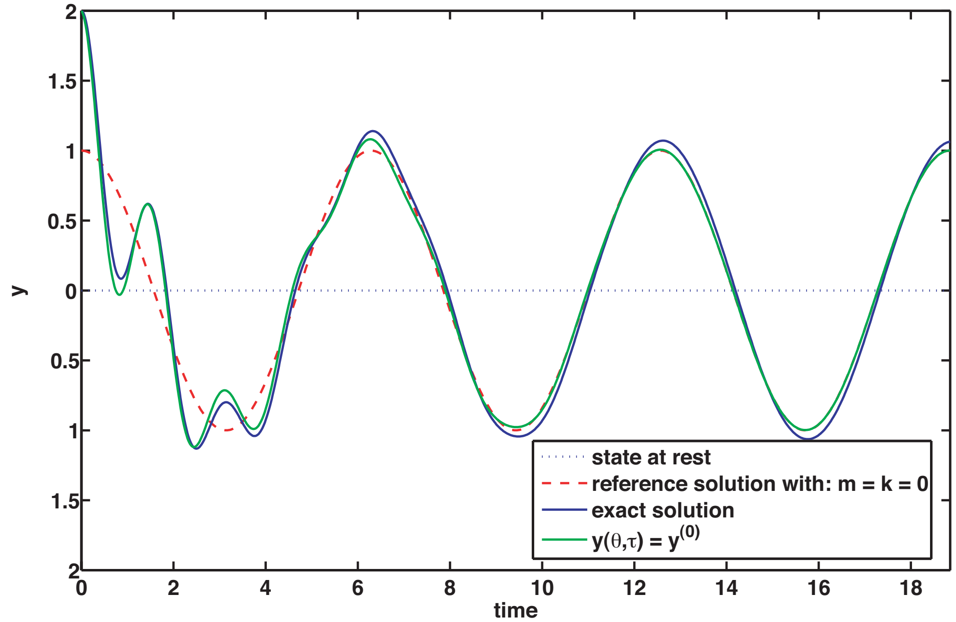
$$\tau = \frac{t}{\sqrt{\varepsilon}}$$



Linear Oscillator / single-scale expansions

Multiple time scale expansion:

$$y(t;\varepsilon) = y^{(0)}(\frac{t}{\sqrt{\varepsilon}}, t) + \sqrt{\varepsilon}\, y^{(1)}(\frac{t}{\sqrt{\varepsilon}}, t) + \ldots$$



Multi-scale regime / multiple scales expansions

Realistic estimate:

$$\varepsilon \sim \frac{1}{8} \dots \frac{1}{6}$$

Interacting length scales:

$$10 \dots 70 \dots 500 \dots 3500 \dots \text{ km}$$

Interacting time scales:

$$3 \dots 20 \text{ min} \dots 3 \text{ h} \dots 1 \,\text{day} \dots 1 \,\text{week}$$

Motivation

Unified approach via formal asymptotics

Anelastic limits

Mesoscale–convective interaction

Regimes:

1. Inkompressible flow (no gravity)

2. Ogura-Phillips-Anelastic flow ($\theta = \mathbf{1} + \varepsilon^4 \boldsymbol{\theta}'$)

3. Buoyancy-controlled flow ($\theta = \boldsymbol{\Theta_0(z)} + \varepsilon^4 \boldsymbol{\theta}'$)

4. Realistic stratification ($\theta = \mathbf{1} + \varepsilon^2 \boldsymbol{\Theta_2(z)} + \varepsilon^4 \boldsymbol{\theta}'$)

Regime 1: Incompressible flow

Leading order result:

$$p = P_\infty + \varepsilon^4 \boldsymbol{p'}$$

Consequences:

$$\rho e \quad = \quad \frac{p}{\gamma - 1} + \varepsilon^4 \frac{\rho \boldsymbol{v}^2}{2} \quad \longrightarrow \quad \frac{P_\infty}{\gamma - 1} \qquad (\varepsilon \longrightarrow 0)$$

$$(\rho e)_t + \nabla \cdot ([\rho e + p] \boldsymbol{v}) = 0 \qquad \longrightarrow \qquad \nabla \cdot \boldsymbol{v}^{(0)} = 0$$

$$\rho_t + \nabla \cdot (\rho \boldsymbol{v}) = 0 \qquad \longrightarrow \qquad \rho_t^{(0)} + \boldsymbol{v}^{(0)} \cdot \nabla \rho^{(0)} = 0$$

$$(\rho \boldsymbol{v})_t + \nabla \cdot (\rho \boldsymbol{v} \circ \boldsymbol{v}) + \frac{1}{\varepsilon^4} \nabla p = 0 \qquad \longrightarrow \qquad (\rho \boldsymbol{v})_t^{(0)} + \nabla \cdot (\rho \boldsymbol{v} \circ \boldsymbol{v})^{(0)} + \nabla \boldsymbol{p'} = 0$$

Regime 2: Ogura-Phillips anelastic flow $\quad (\theta = 1 + \varepsilon^4 \theta')$

Leading order results:

$$p = P_0(z) + \varepsilon^4 p' \qquad \rho = R_0(z) + \varepsilon^4 \rho' \qquad R_0(z) = P_0(z)^{1/\gamma}$$

Consequences:

$$\rho e \;=\; \frac{p}{\gamma - 1} + \varepsilon^4 \frac{\rho \boldsymbol{v}^2}{2} + \rho g z \;\to\; \frac{P_0(z)}{\gamma - 1} + R_0(z) g z \qquad (\varepsilon \to 0)$$

$$(\rho e)_t + \nabla \cdot ([\rho e + p] \boldsymbol{v}) = 0 \qquad \to \qquad \nabla \cdot \left(\boldsymbol{v}^{(0)} R_0(z) \left[\frac{\gamma}{\gamma - 1} \frac{P_0}{R_0}(z) + g z \right] \right) = 0$$

$$\rho_t + \nabla \cdot (\rho \boldsymbol{v}) = 0 \qquad \to \qquad \nabla \cdot \left(\boldsymbol{v}^{(0)} R_0(z) \right) = 0$$

Fortunately

$$\left[\frac{\gamma}{\gamma - 1} \frac{P_0}{R_0} + g z \right] \equiv \text{const.}$$

Regime 2: Ogura-Phillips anelastic flow $(\theta = \mathbf{1} + \varepsilon^4 \boldsymbol{\theta'})$

Consequences (cont'd):

$$(\rho \boldsymbol{v})_t + \nabla \cdot (\rho \boldsymbol{v} \circ \boldsymbol{v}) + \frac{1}{\varepsilon^4}(\nabla p + \rho g) = 0 \quad \longrightarrow \quad (\rho \boldsymbol{v})_t^{(0)} + \nabla \cdot (\rho \boldsymbol{v} \circ \boldsymbol{v})^{(0)} + (\nabla \boldsymbol{p'} + \boldsymbol{\rho'}) = 0$$

Regime 3: Buoyancy-controlled anelastic flow $(\theta = \boldsymbol{\Theta_0(z)} + \boldsymbol{\varepsilon}^4\boldsymbol{\theta'})$

Leading order results:

$$p = P_0(z) + \boldsymbol{\varepsilon}^4 \boldsymbol{p'} \qquad \rho = R_0(z) + \boldsymbol{\varepsilon}^4 \boldsymbol{\rho'} \qquad R_0(z) = P_0(z)^{1/\gamma} / \boldsymbol{\Theta_0(z)}$$

Consequences:

$$\rho e \;=\; \frac{p}{\gamma - 1} + \boldsymbol{\varepsilon}^4 \frac{\rho \boldsymbol{v}^2}{2} + \rho g z \;\to\; \frac{P_0(z)}{\gamma - 1} + R_0(z) g z \qquad (\boldsymbol{\varepsilon} \to 0)$$

$$(\rho e)_t + \nabla \cdot ([\rho e + p] \boldsymbol{v}) = 0 \qquad \to \qquad \nabla \cdot \left(\boldsymbol{v}^{(0)} R_0(z) \left[\frac{\gamma}{\gamma - 1} \frac{P_0}{R_0}(z) + g z \right] \right) = 0$$

$$\rho_t + \nabla \cdot (\rho \boldsymbol{v}) = 0 \qquad \to \qquad \nabla \cdot \left(\boldsymbol{v}^{(0)} R_0(z) \right) = 0$$

Unfortunately

$$\left[\frac{\gamma}{\gamma - 1} \frac{P_0}{R_0} + g z \right] \equiv \boldsymbol{\Phi(z)} \not\equiv \text{const.} \quad \Rightarrow \quad \nabla_{||} \cdot \boldsymbol{u}^{(0)} = 0, \quad \text{and} \quad w^{(0)} = 0$$

Regime 3: Buoyancy-controlled anelastic flow $(\theta = \Theta_0(z) + \varepsilon^4 \theta')$

Consequences (cont'd):

$$(\rho \boldsymbol{v})_t + \nabla \cdot (\rho \boldsymbol{v} \circ \boldsymbol{v}) + \frac{1}{\varepsilon^4}(\nabla p + \rho g) = 0 \quad \longrightarrow \quad (\rho \boldsymbol{v})_t^{(0)} + \nabla \cdot (\rho \boldsymbol{v} \circ \boldsymbol{v})^{(0)} + (\nabla \boldsymbol{p'} + \boldsymbol{\rho'}) = 0$$

Regime 4: Realistic stratification $(\theta = \mathbf{1} + \varepsilon^2 \mathbf{\Theta_2}(z) + \varepsilon^4 \boldsymbol{\theta}')$

Leading order results:

$$p = P_0(z) + \varepsilon^2 \underline{P_2(z)} + \varepsilon^4 \boldsymbol{p}' \qquad \rho = R_0(z) + \varepsilon^2 \underline{R_2(z)} + \varepsilon^4 \boldsymbol{\rho}' \qquad R_0(z) = P_0(z)^{1/\gamma}$$

$$\frac{dP_2}{dz} = -R_2\, g \qquad R_2 = R_0 \left(\frac{1}{\gamma} \frac{P_2}{P_0} - \mathbf{\Theta_2} \right)$$

Consequences (leading order):

$$(\rho e)_t + \nabla \cdot ([\rho e + p] \boldsymbol{v}) = 0 \qquad \to \qquad \nabla \cdot \left(\boldsymbol{v}^{(0)} R_0(z) \left[\frac{\gamma}{\gamma - 1} \frac{P_0}{R_0}(z) + gz \right] \right) = 0$$

$$\rho_t + \nabla \cdot (\rho \boldsymbol{v}) = 0 \qquad \to \qquad \nabla \cdot \left(\boldsymbol{v}^{(0)} R_0(z) \right) = 0$$

Fortunately again

$$\left[\frac{\gamma}{\gamma - 1} \frac{P_0}{R_0} + gz \right] \equiv \text{const.}$$

Regime 4: Realistic stratification $(\theta = \mathbf{1} + \varepsilon^2 \boldsymbol{\Theta}_2(z) + \varepsilon^4 \boldsymbol{\theta}')$

Consequences (**second** order):

$$\nabla \cdot \left(\boldsymbol{v}^{(2)} \frac{\gamma R_0}{\gamma - 1} \right) + \nabla \cdot \left(\boldsymbol{v}^{(0)} \left[\frac{\gamma}{\gamma - 1} P_2(z) + gz\, R_2(z) \right] \right) = \boldsymbol{S}^{(2)}$$

$$\nabla \cdot \left(\boldsymbol{v}^{(2)} R_0 \right) + \nabla \cdot \left(\boldsymbol{v}^{(0)} R_2(z) \right) = 0$$

By elimination of $\boldsymbol{v}^{(2)}$:

$$w^{(0)} \frac{d\boldsymbol{\Theta}_2}{dz} = \frac{\gamma - 1}{\gamma P_0} \boldsymbol{S}^{(2)} \qquad \Rightarrow \qquad \textbf{WTG}$$

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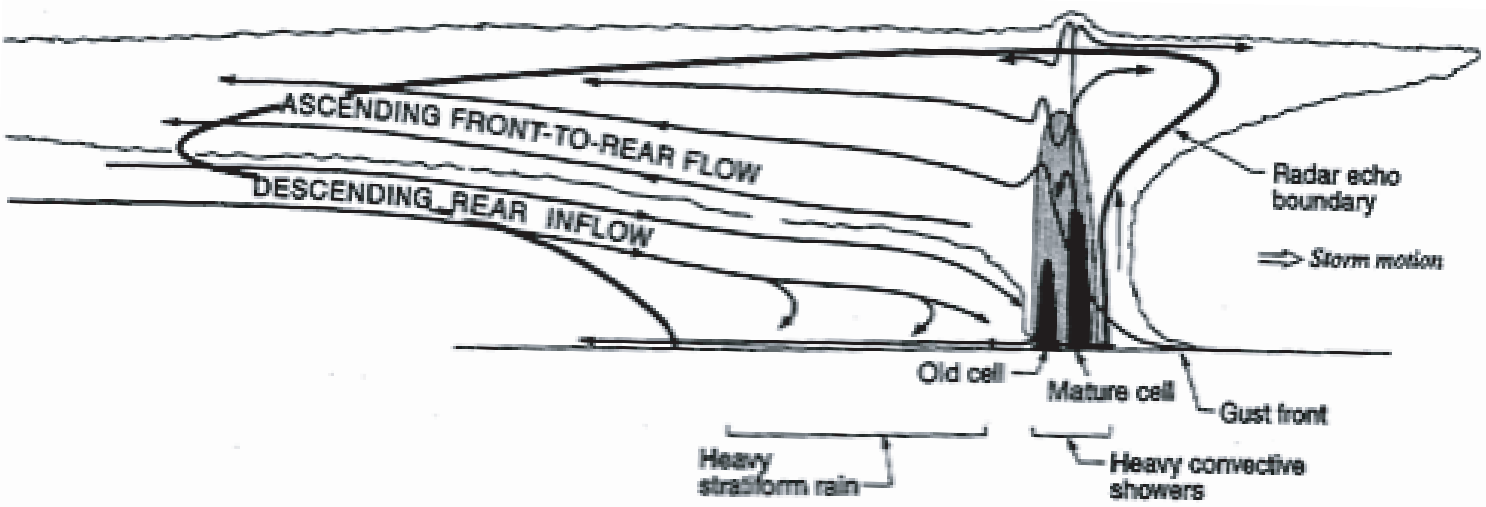

FIG. 1. Conceptual model of a multicell squall line with trailing stratiform precipitation. The storm is viewed in a cross section perpendicular to the convective line (adapted from Houze et al. 1989).

(from: Pandya & Durran (1996))

$$\mathbf{U}(\boldsymbol{x}, z, t; \boldsymbol{\varepsilon}) = \sum_i \boldsymbol{\varepsilon}^i \, \mathbf{U}^{(i)}(\boldsymbol{x}, \boldsymbol{\xi}, z, t),$$

$$\boldsymbol{\xi} = \boldsymbol{\varepsilon} \, \boldsymbol{x}$$

$$\boldsymbol{x} = \frac{\boldsymbol{x}'}{10\,\mathrm{km}}$$

Mesoscale dynamics

$$
\begin{aligned}
\overline{\boldsymbol{u}}_t + \nabla_{\boldsymbol{\xi}}\, \pi' &= -\partial_z \left(\frac{\overline{S_\theta'' \boldsymbol{u}}}{d\Theta_2/dz} \right) \,, \\
\theta_t' + \overline{w'} \frac{d\Theta_2}{dz} &= \overline{S_\theta'}\,, \\
\partial_z \pi' &= \theta'\,, \\
\rho_0 \nabla_{\boldsymbol{\xi}} \cdot \overline{\boldsymbol{u}} + \partial_z \left(\rho_0 \overline{w'} \right) &= 0\,.
\end{aligned}
$$

Convective scale dynamics

Anelastic* for near-moist adiabatic stratification

WTG otherwise $\Rightarrow$ S_θ''

* essentially