From Quasigeostrophic Turbulence to Stratified Turbulence

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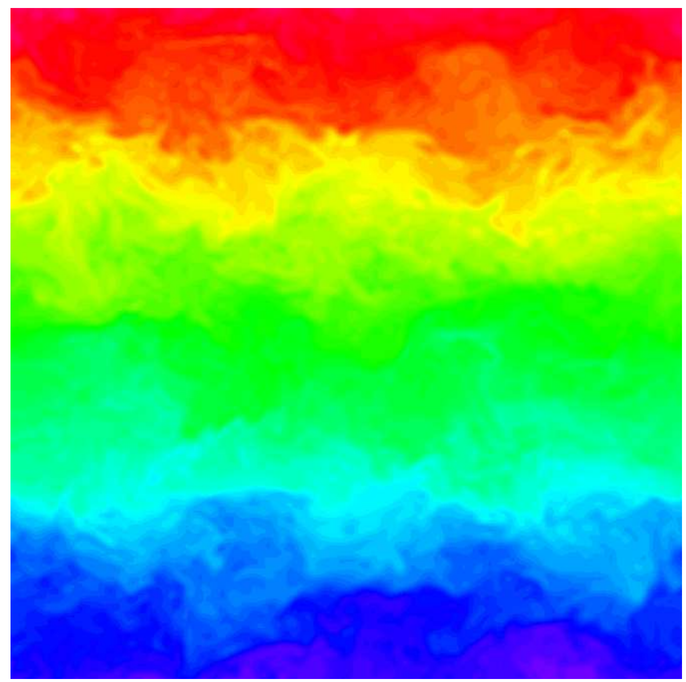


Mesoscale atmospheric spectrum

- Gage (1979) then Lilly (1983) argued that there was an inverse cascade from convective scales. Rotation unimportant.

- Van Zandt (1982) said that it was a “gravity-wave spectrum”.

An early test of the Gage-Lilly idea

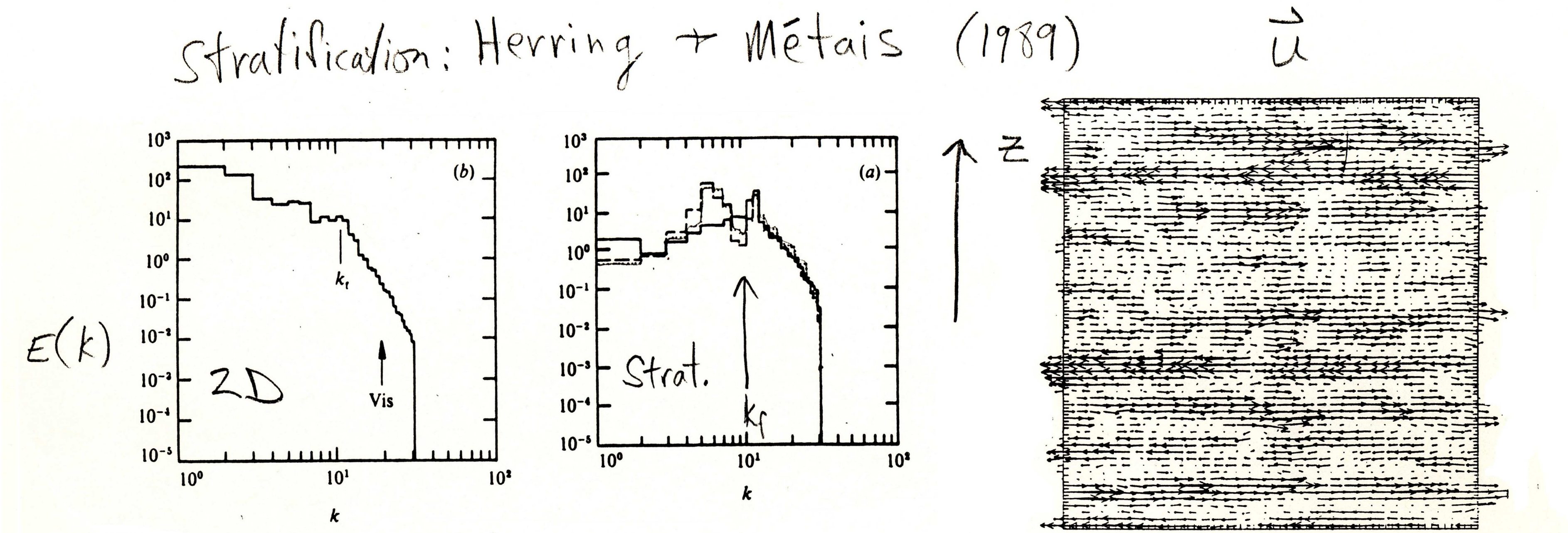


FIGURE 11. Vector plots of $u(x, y, z, t)$. $N = 80\pi$ for (x, z)-slice at the midplanes of the flow. Flow is statistically stationary.

- No inverse cascade with strong stratification and no rotation.
- Flow reached statistical stationarity (except for shear modes). Therefore energy went downscale.
- Later, Métais *et al.* added strong rotation and obtained an inverse cascade.

Boussinesq Normal Mode Decomposition

Let $G_{\mathbf{k}} = B_{\mathbf{k}}^{(0)}$ and $A_{\mathbf{k}}^{(\pm)}(\epsilon t)\mathrm{e}^{\pm i\omega_{\mathbf{k}} t} = B_{\mathbf{k}}^{(\pm)}$, where $\epsilon = \min(Ro, Fr)$.

$$\frac{\partial}{\partial(\epsilon t)}\langle |G_{\mathbf{k}}|^2\rangle = \sum_{\Delta} N_1 \langle G_{\mathbf{k}} G_{\mathbf{p}} G_{\mathbf{q}}\rangle +$$

$$N_2 \langle G_{\mathbf{k}} G_{\mathbf{p}} A_{\mathbf{q}}\rangle \mathrm{e}^{i\omega_{\mathbf{q}} t} + N_3 \langle G_{\mathbf{k}} A_{\mathbf{p}} A_{\mathbf{q}}\rangle \mathrm{e}^{i(\omega_{\mathbf{q}}+\omega_{\mathbf{p}})t}$$

$\to 0$

1. conservation

2. resonance

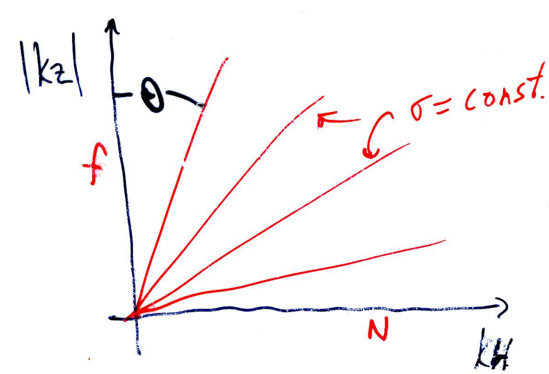
$$\frac{\partial}{\partial(\epsilon t)}\langle |A_{\mathbf{k}}|^2\rangle = \sum_{\Delta} M_1 \langle A_{\mathbf{k}} G_{\mathbf{p}} G_{\mathbf{q}}\rangle \mathrm{e}^{i\omega_{\mathbf{k}} t} + M_2 \langle A_{\mathbf{k}} A_{\mathbf{p}} G_{\mathbf{q}}\rangle \mathrm{e}^{i(\omega_{\mathbf{k}}+\omega_{\mathbf{p}})t}$$

$$+ M_3 \langle A_{\mathbf{k}} A_{\mathbf{p}} A_{\mathbf{q}}\rangle \mathrm{e}^{i(\omega_{\mathbf{k}}+\omega_{\mathbf{p}}+\omega_{\mathbf{q}})t}$$

Energy, $E = \sum_{\mathbf{k}} |G_{\mathbf{k}}|^2 + |A_{\mathbf{k}}|^2 = E_G + E_A$.

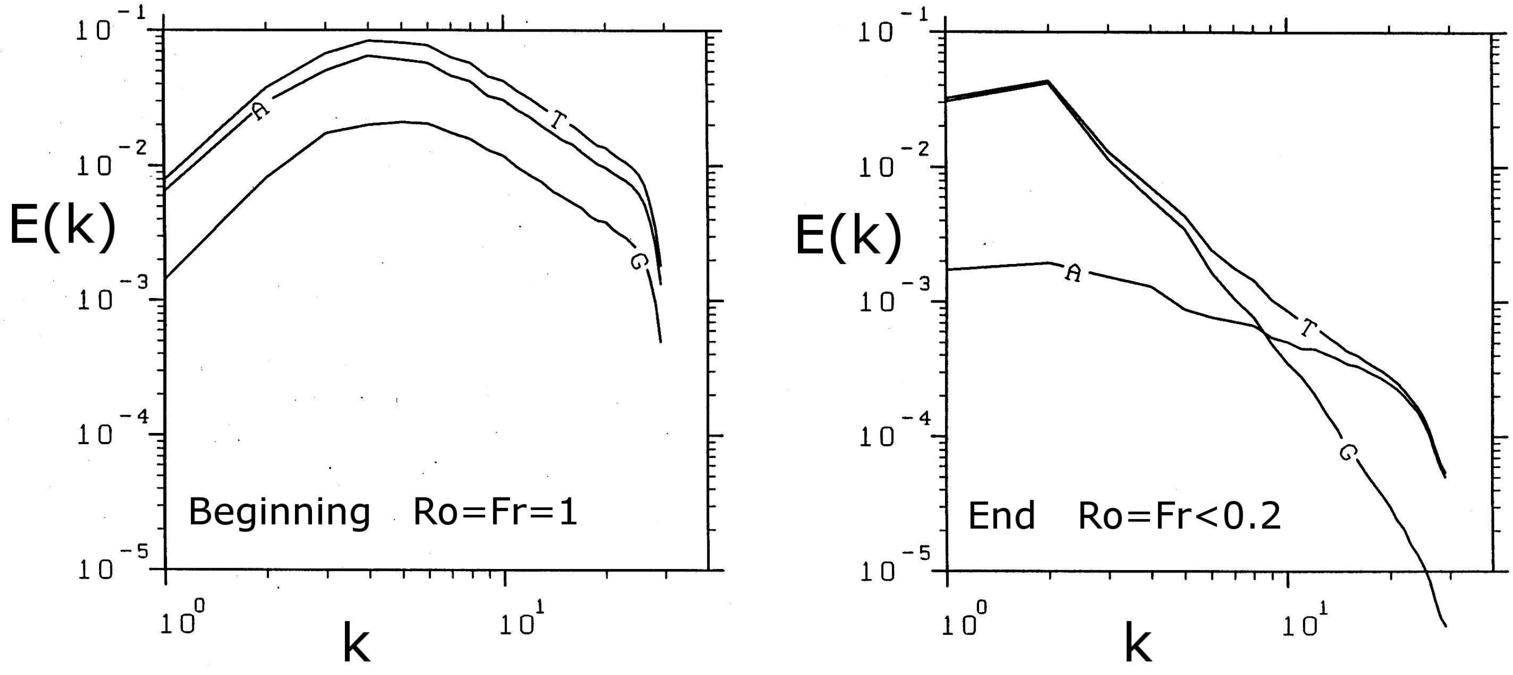
Potential Enstrophy, $V \doteq \sum_{\mathbf{k}} \sigma_{\mathbf{k}}^2 k^2 |G_{\mathbf{k}}|^2$ as

Gravity-wave dispersion relation, $\sigma_{\mathbf{k}} = \Big(f^2$ c



Strong rotation and strong stratification

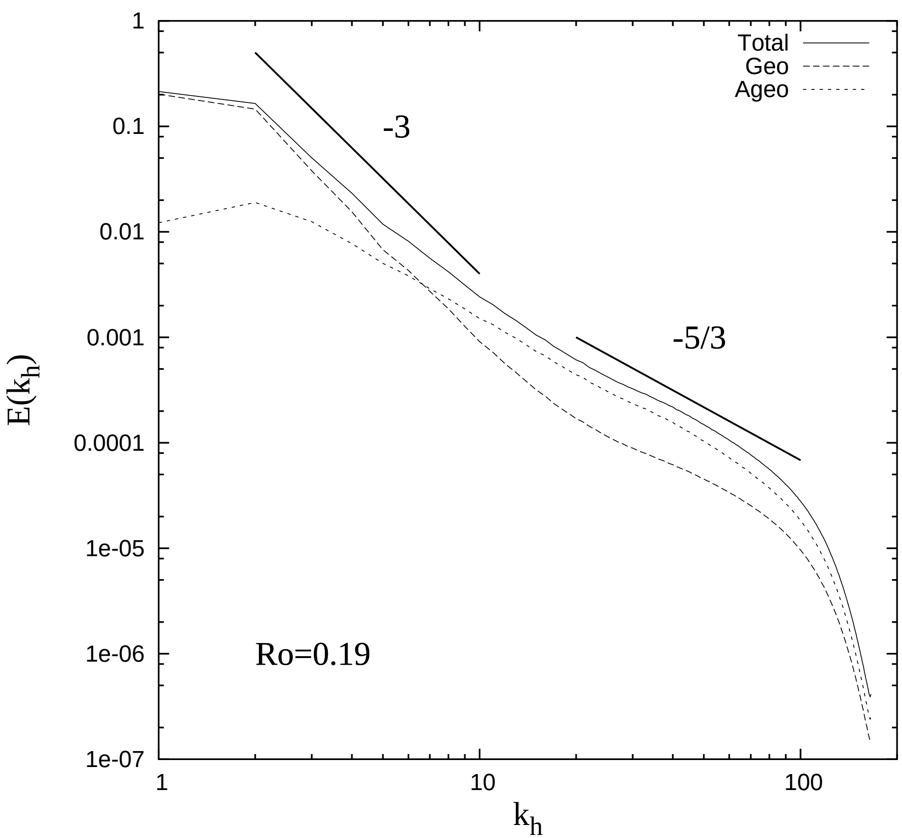
Geostrophic, G and ageostrophic, A spectra



- G modes cascade upscale, A modes downscale $\rightarrow$ balance
- Balance requires dissipation of balanced modes to be less than that of unbalanced ones (e.g. inverse cascades).

The atmosphere?

- Boussinesq with N/f = 10 $+$ Rayleigh damping
- Grid: $500 \times 500 \times 50$ with $\Delta x = \Delta y = \Delta z$
- A set of baroclinic QG modes with $L/H = N/f$ are forced.



Stratification only: Dimensionless equations

- Nondimensionalise using $\boldsymbol{u} \sim U$, x, $y \sim L$, $z \sim H$, etc.
- Wave: $T \sim L/NH$ Vortical: $T \sim L/U$.
- Two Froude numbers:

$$Fr_h = \frac{U}{NL}, \qquad Fr_z = \frac{U}{NH}.$$

Two scaling ideas:

- Integral scale vertical Froude number $\rightarrow O(1)$, i.e. $H_i \sim \sqrt{E}/N$
- Vertical Froude is unity at all vertical scales, i.e. $H(k_z) \sim [k_z E(k_z)]^{1/2}/N$ or equivalently $E(k_z) \sim N^2 k_z^{-3}$

Limit of Strong Stratification ($\Omega = 0$)

- The vertical scale collapses until dissipation scale reached.
- Two limits:
  - 1. $Fr_h \to 0$, Fr_h/Fr_z fixed (RMW 1981)
  - Appropriate if flow is initially isotropic or when vertical scale determined by dissipation
  - Potential enstrophy approximately quadratic

  - 2. $Fr_h \to 0$, Fr_z fixed
  - Implies $H \sim U/N$ set by stratification. Due to
    - i) Thorpe (1977) (introduced characteristic scale for overturning. U/N is an upper bound.)
    - ii) Munk (1981) (identifi ed it as small-scale end of Garret-Munk spectrum)
    - iii) Hines (1996) (transition scale between unsaturated and saturated waves)
    - iv) Billant & Chomaz (2001)

Stratification only

- Vectors and ∇, etc. are horizontal components
- Use vortical time scale $T \sim L/U$:

$$
\begin{aligned}
\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} + Fr_z^2\, w \frac{\partial \boldsymbol{u}}{\partial z} &= -\boldsymbol{\nabla} p, \\
Fr_h^2 \left(\frac{\partial w}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} w + Fr_z^2\, w \frac{\partial w}{\partial z} \right) &= -\frac{\partial p}{\partial z} + b, \\
\boldsymbol{\nabla} \cdot \boldsymbol{u} + Fr_z^2 \frac{\partial w}{\partial z} &= 0, \\
\frac{\partial b}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} b + Fr_z^2\, w \frac{\partial b}{\partial z} &= -w.
\end{aligned}
$$

- Decoupled layers when $Fr_h \to 0,\ Fr_z \to 0$.
- 3D hydrostatic turbulence if $H \sim U/N$

Wave/vortical decomposition without rotation

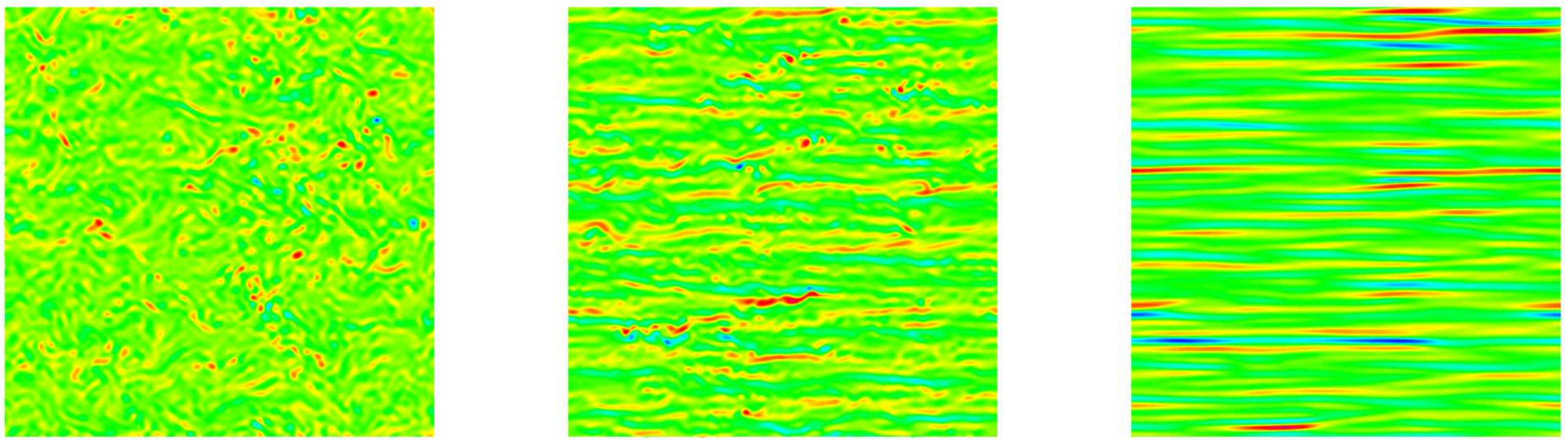
• Need to be careful: potential enstrophy only approximately quadratic when $Fr_z \ll 1$, but this may not be relevent.
• We need to restrict application to scales much larger than H_i or to particular problems, e.g. early decay of isotropic i.c.'s.
• If $Fr_z \ll 1$ we can examine inviscid truncated statistical equilibrium

$$\langle |G_{\mathbf{k}}|^2 \rangle = \frac{1}{\lambda_1 + \lambda_2 k_h^2} \qquad \langle |A_{\mathbf{k}}|^2 \rangle = \frac{2}{\lambda_1}$$

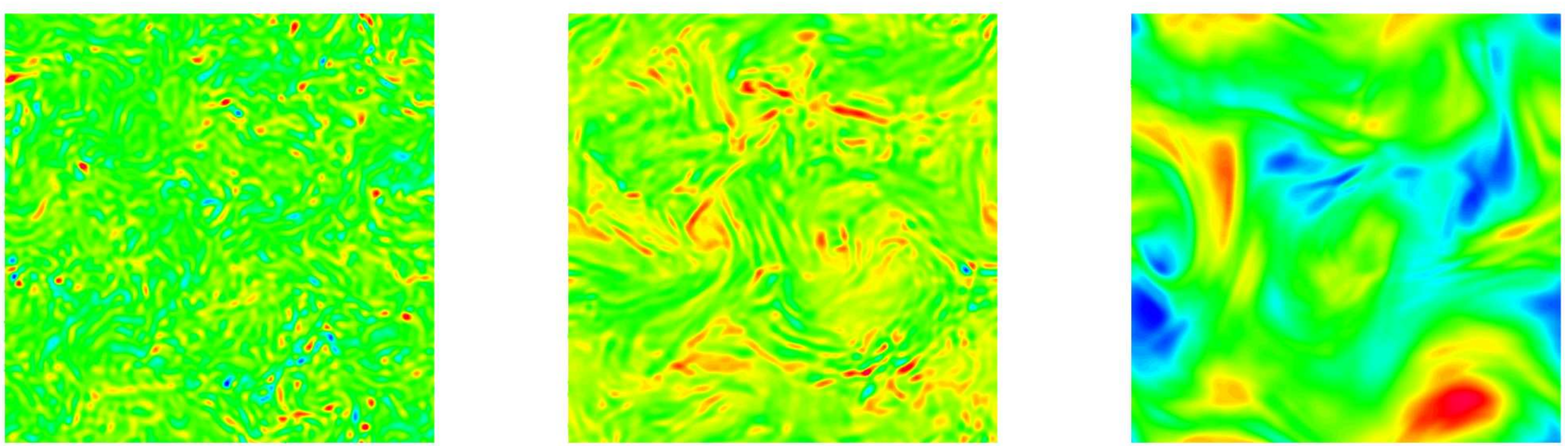
- Independent of k_z
- Can show that if $k_i \ll k_{max}$, where k_i defined by the ratio of potential enstrophy to energy, then as $k_{max} \to \infty$, the spectrum is peaked at a wavenumber $\to \sqrt{2} k_i$.
- In 2D turbulence, this wavenumber goes to zero. This argument suggest that there is no inverse cascade.

Vortical forcing ($\Omega = 0$) : perpendicular vorticity

$x - z$ plane

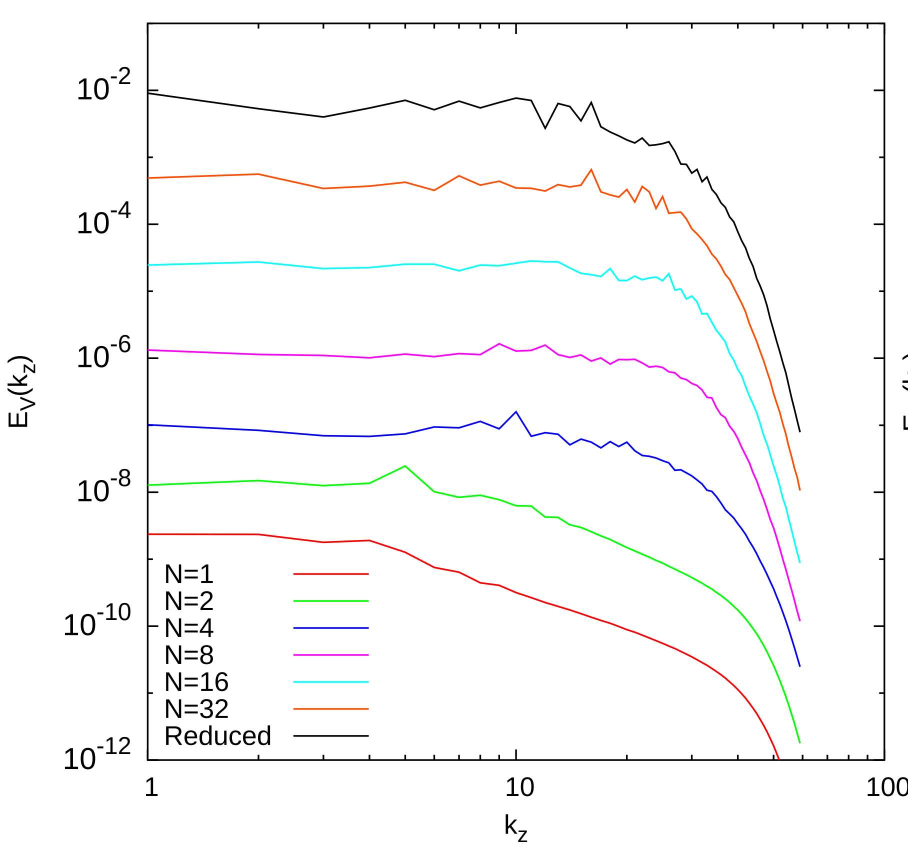


$x - y$ plane

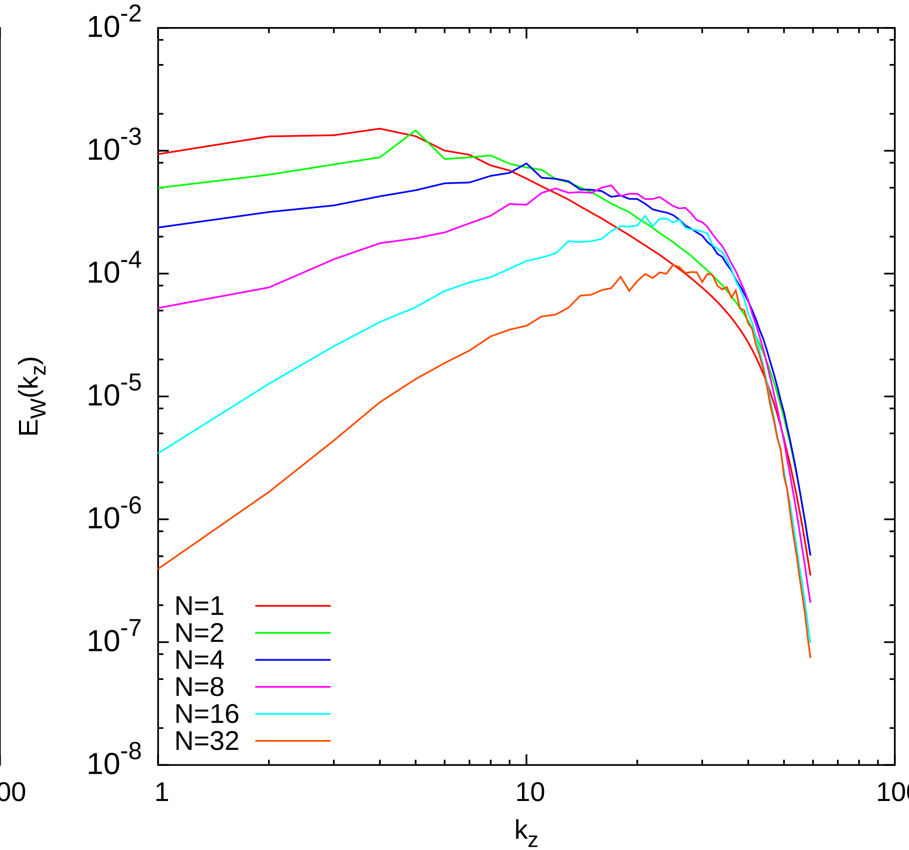


Vortical forcing ($\Omega = 0$): k_z spectra

Vortical energy



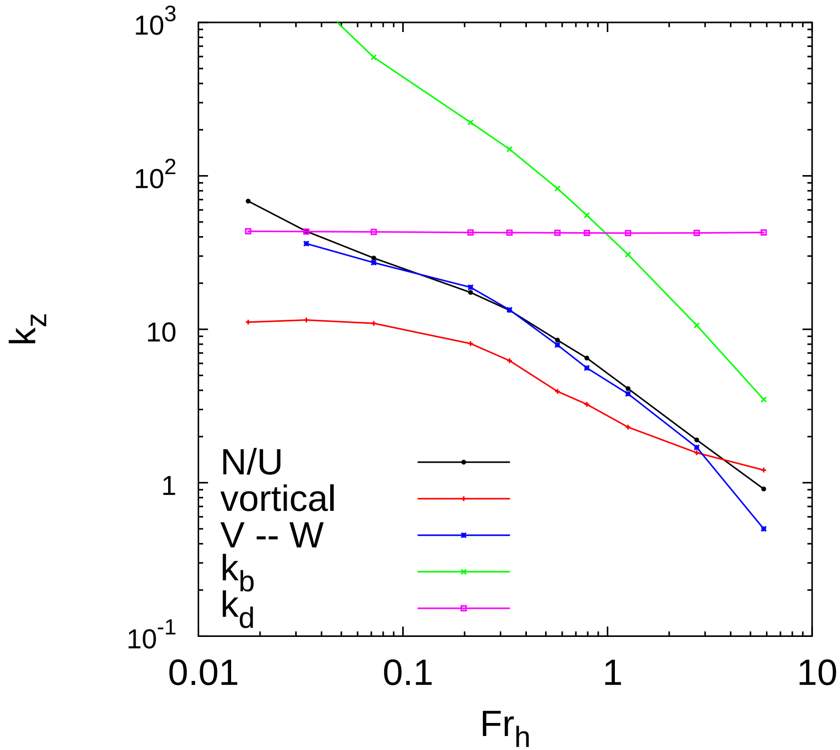
Wave energy



offset by factors of 10

Vortical Forcing ($\Omega = 0$): Length scales

Vortical forcing:



- $k_b \sim \sqrt{N^3/\epsilon}$ Ozmidov
- k_d is dissipation wavenumber

Wave forcing ($\Omega = 0$): $N^2\ k_z^{-3}$ spectra?

Dashed line: $0.1 N^2\ k_z^{-3}$ (Bouruet-Aubertot, Sommeria & Staquet 1996)

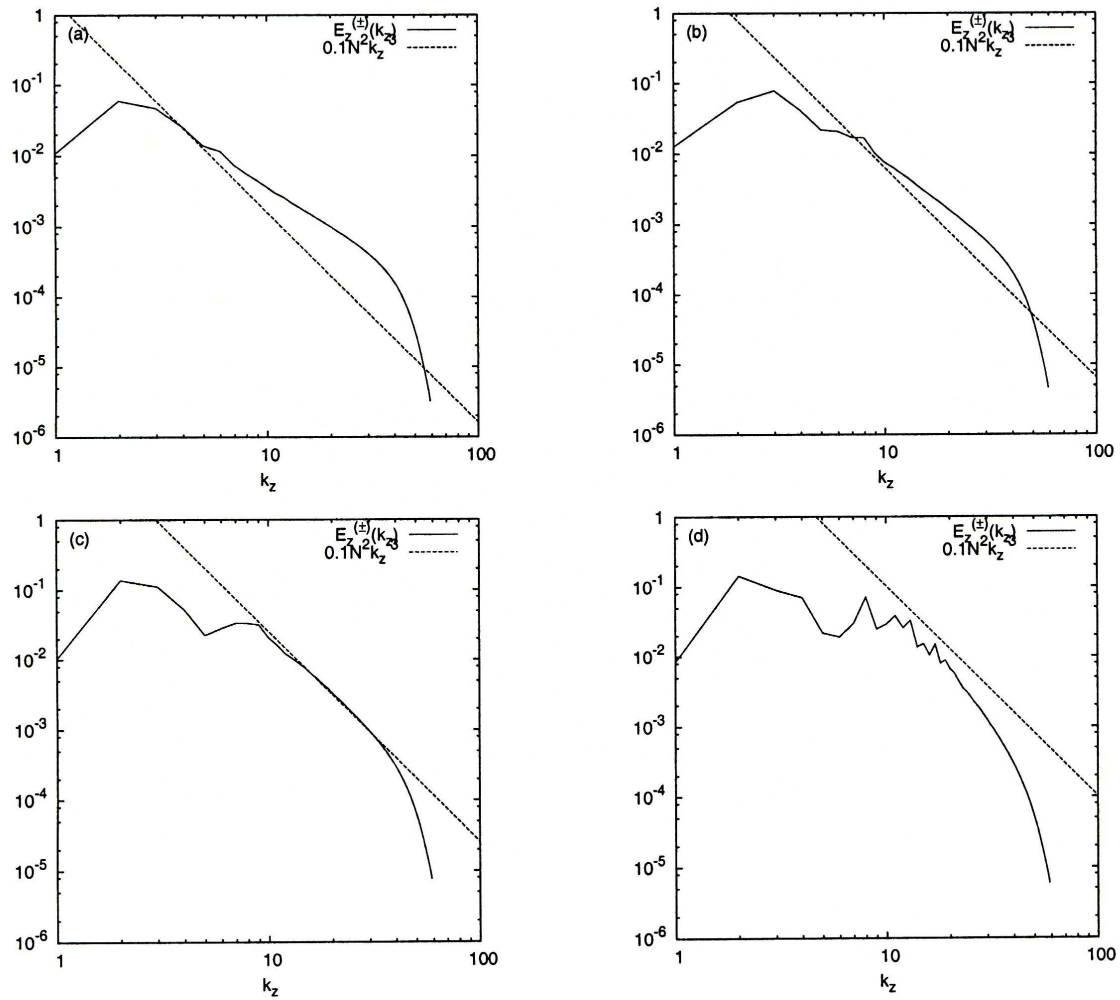
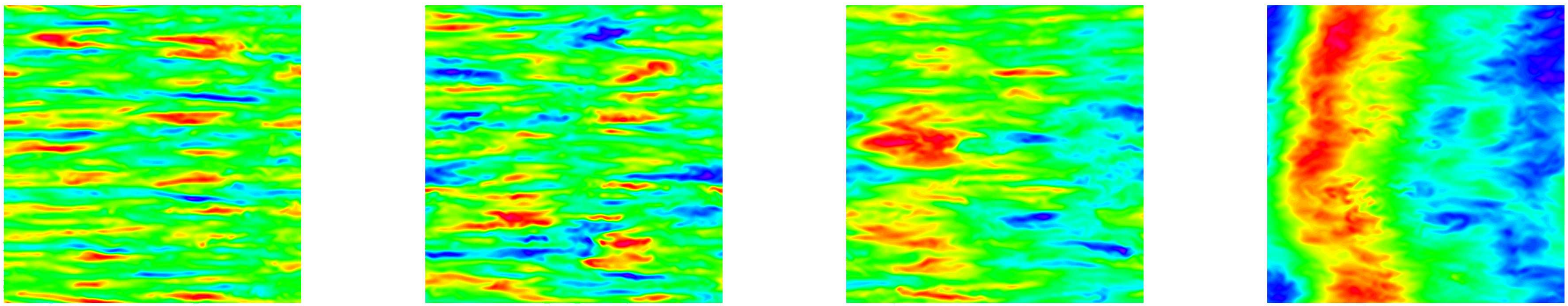


FIGURE 9. Vertical wavenumber spectra of wave energy, along with the hypothetical saturation spectrum $0.1\,N^2\,k_z^{-3}$, for (a) $N = 4$, (b) $N = 8$, (c) $N = 16$ and (d) $N = 32$ when $M = 180$ and $R = 0$.

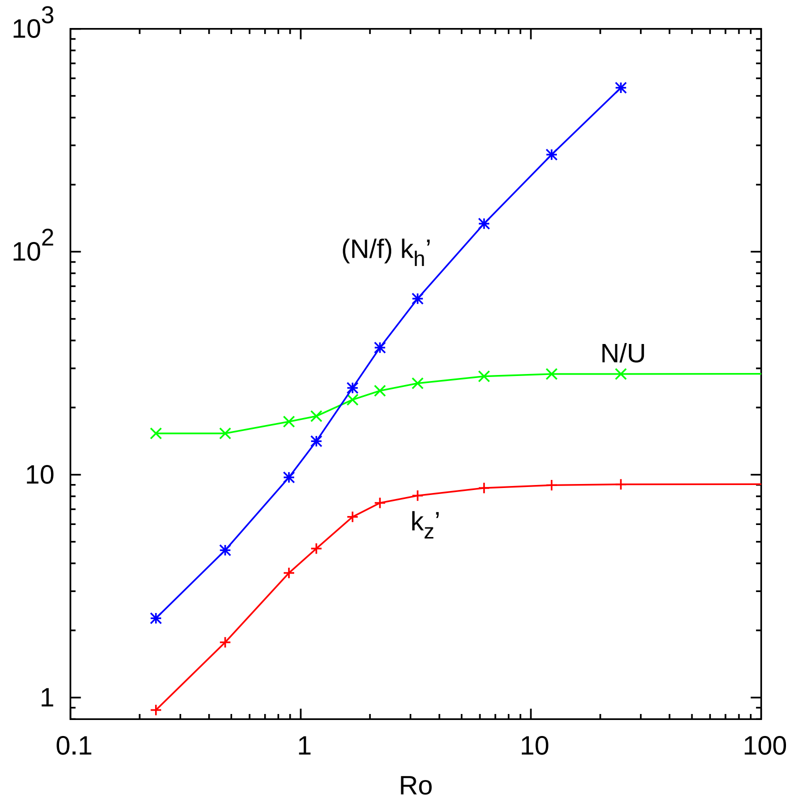
Strong stratification at various rotations



No rotation $\longrightarrow$ lots of rotation

This is the horizontal velocity component perpendicular to the screen.

Strong stratification at various rotations

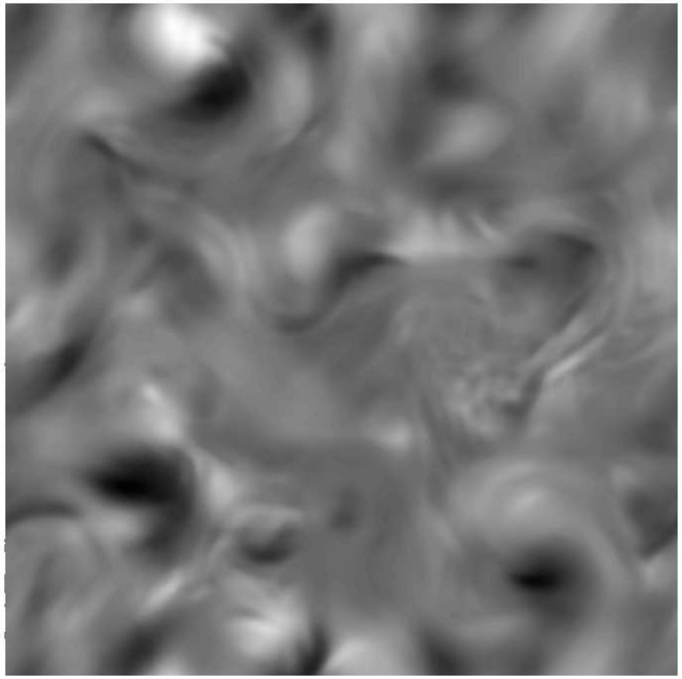


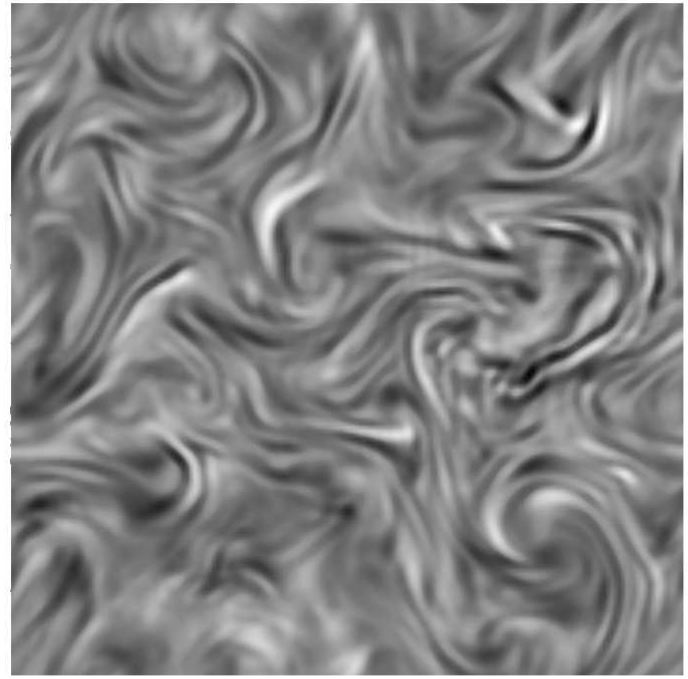
Stratified turbulence. Vertical lengthscale $= U/N$

Quasigeostrophic turbulence. Vertical lengthscale $= fL/N$ (Charney 1971)

This is how atmosphere/ocean modellers will need to set the ratio $\Delta z/\Delta x$ as resolutions increase.

Back to strong stratification and rotation





Balance at Ro=0.09 ?

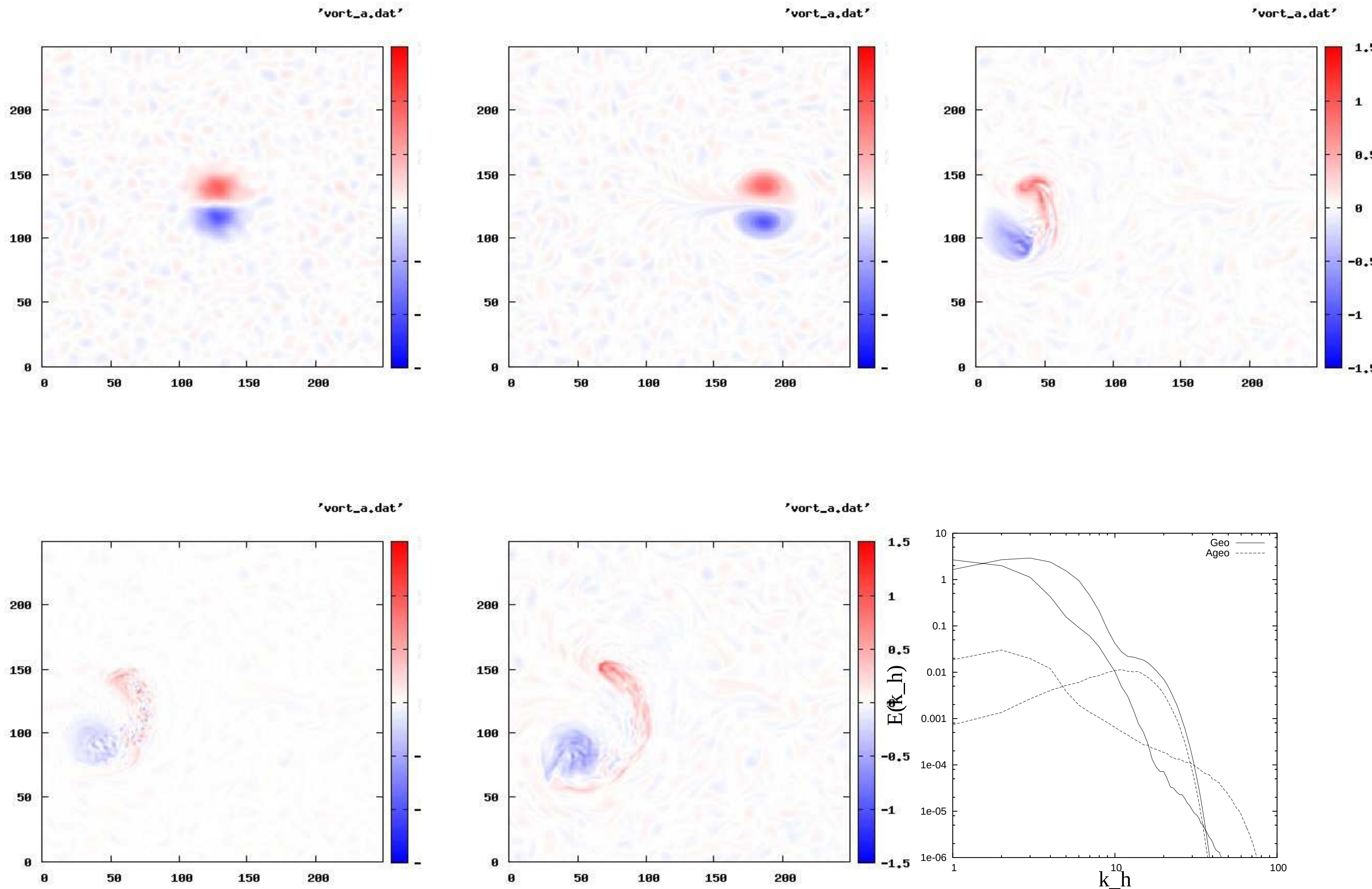
We have used the QG modes and the ω equation to diagnose vertical velocity (left).

The real vertical velocity is on the right.

If balance exists, it isn’t this simple.

Regarding recent dipoles... **(see Snyder *et al.*)**

- Boussinesq with N/f = 10, Ro=0.7. Grid: $256 \times 256 \times 16$ with $\Delta x = \Delta y = \Delta z$
- Initial flow is a 2D dipole $+$ a little 3D noise.



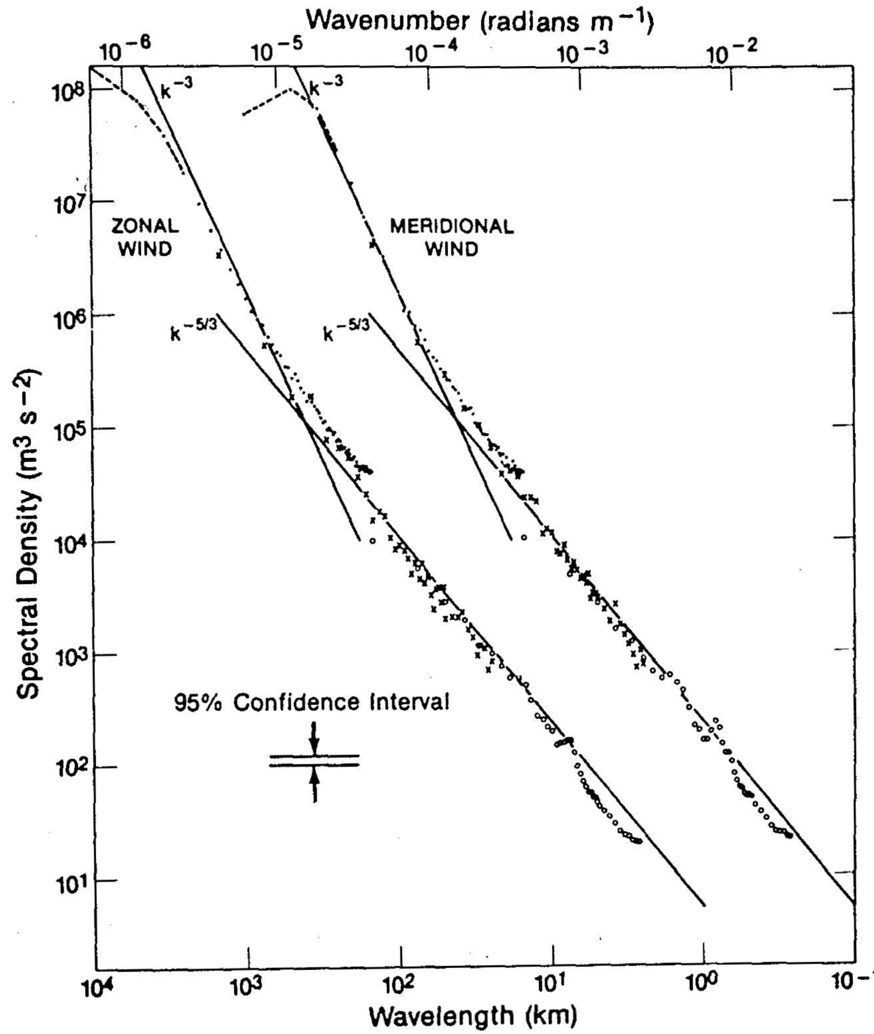
Summary of the New Stuff

- Stratified Turbulence without Rotation
  - Vortical forcing:
    - Direct cascade.
    - Simulations confirm $H \sim U/N$.
  - Wave forcing:
    - $N^2 k_z^{-3}$ not observed.
  - Both: When H not resolved by Δz weird things happen.
- Stratified Turbulence with rotation
  - Growth in the vertical integral scale as Ω increases
  - Bands of ageostrophic energy remain localised near vortices.
  - Steep vortical spectra, shallow ageo spectra *very* robust

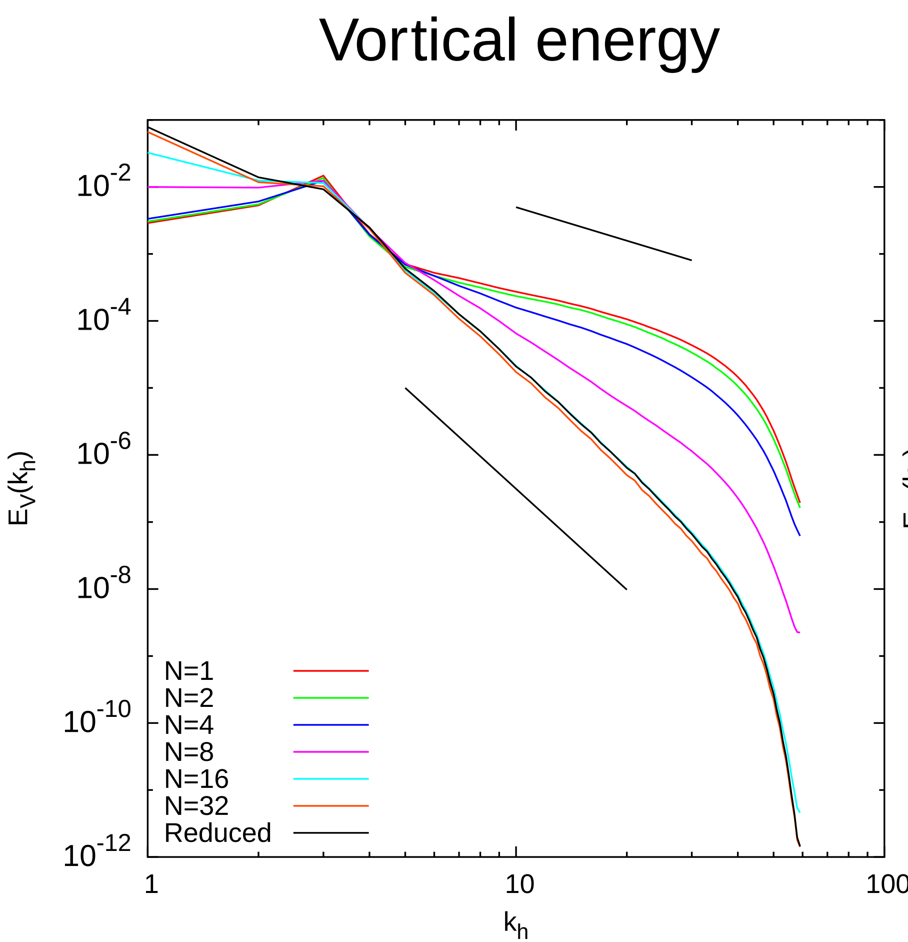
Extra stuff

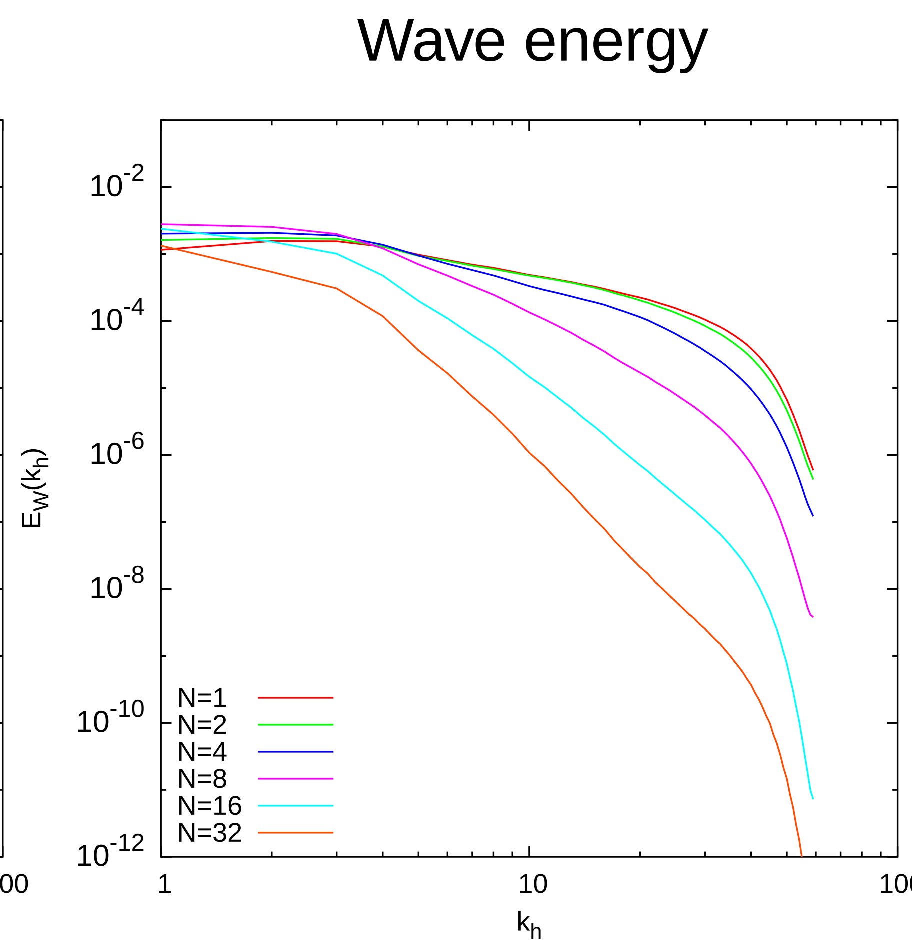
Observations: The Atmosphere

- GARP 1979
- Instruments on 7900 commercial flights
- Nastrom & Gage (1986)



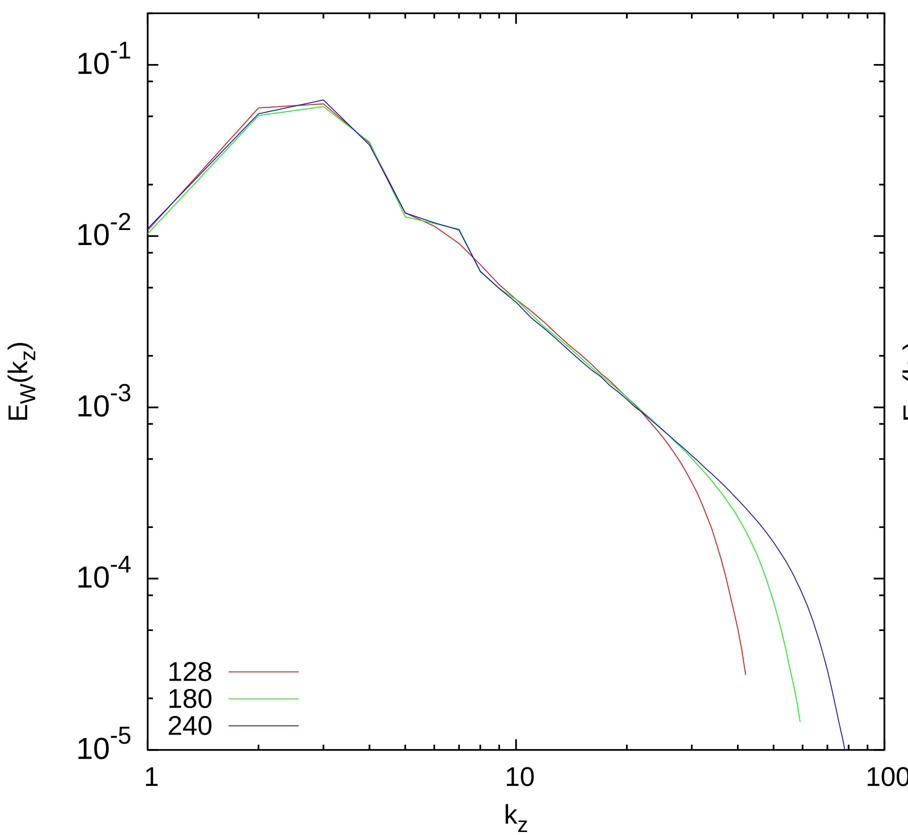
Vortical forcing ($\Omega = 0$): k_h spectra



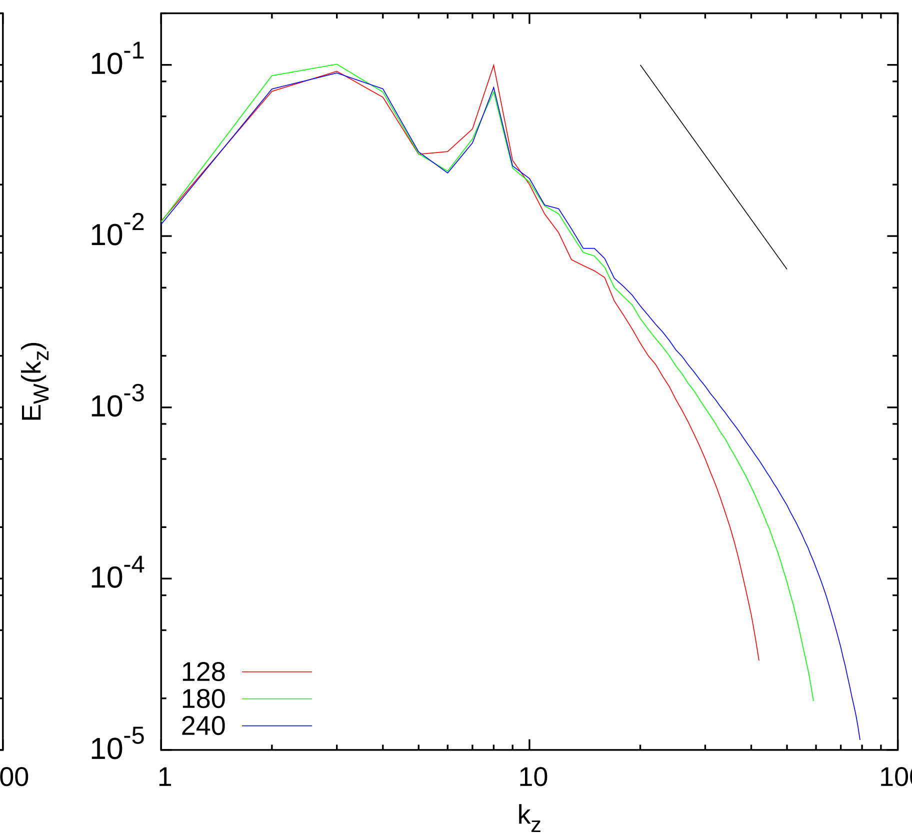


Wave forcing ($\Omega = 0$): dependence on Re

$Fr_h \sim 1$

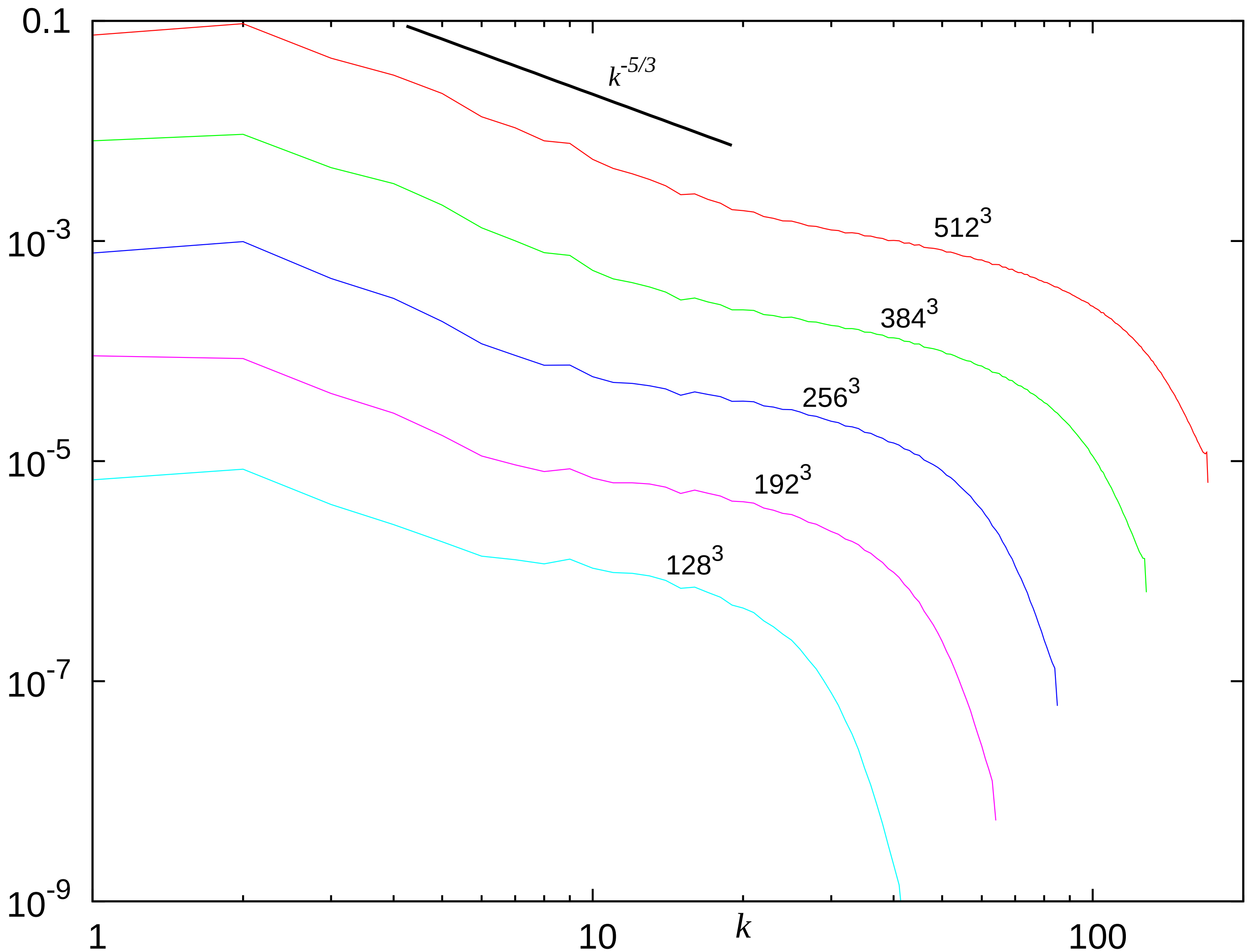


$Fr_h \sim 0.1$



- Decreasing $Fr_h \rightarrow$ steeper spectra $\rightarrow$ nonlocal interactions.

Bottleneck at Δ^4



Spyksma (2007)