Rapidly Rotating Rayleigh-Bénard Convection

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Outline

- Rotationally constrained flow
  - Rayleigh-Bénard convection (Bénard 1900; Rayleigh 1916)
    - Challenges for experiments and direct numerical simulation (DNS) of full Navier-Stokes equations in rapidly rotating limit
  - Geophysical fluid dynamics

- Derivation of a reduced model for convection in rapidly rotating limit

- DNS of reduced system: method & results
  - Julien et al. JFM vol 555 (2006); Sprague et al. JFM vol 551 (2006)

- Future work
Rotationally Constrained Convection

Ra–Ta Parameter Space

Convection

$Ra \propto \Delta T$ (thermal forcing)

$Ra_{\text{crit}} \approx 8.7T a^{2/3}$ (linear stability)

(Chandrasekhar, 1961)

Conduction (No Fluid Motion)

$Ta \propto \Omega^2$ (rotation rate)
Rotationally Constrained Convection

$Ra - Ta$ Parameter Space

Convection

$Ra \propto \Delta T$ (thermal forcing)

Conduction (No Fluid Motion)

$Ra_{\text{crit}} \approx 8.7 Ta^{2/3}$ (linear stability)

$Az \approx 0.77 Ta^{1/6}$

$Ta \propto \Omega^2$ (rotation rate)
Rotationally Constrained Convection

$Ra - Ta$ Parameter Space

$Ra \propto \Delta T$ (thermal forcing)

$Ro_{conv} = \sqrt{Ra/(PrTa)}$

$= t_{rot}/t_{free-fall}$

$Ra \propto \Delta T$ (thermal forcing)

$Ta \propto \Omega^2$ (rotation rate)
Rotationally Constrained Convection

*Ra–Ta Parameter Space*

\[ \text{Ro}_{\text{conv}} = \sqrt{\text{Ra} / (PrTa)} \]

\[ = t_{\text{rot}} / t_{\text{free-fall}} \]

Parameter Space for Rotationally Constrained Convection
Rotationally Constrained Convection

\( Ra - Ta \) Parameter Space: Experiments

Ocean & Astrophysical Systems: \( Ra, Ta \Rightarrow 10^{17} - 10^{20} \)

\( Ro_{\text{conv}} = 0.01 - 1 \)

Vorobieff & Ecke (2002)

Sakai (1997)
Rotationally Constrained Convection ($R_{\text{conv}} \ll 1$)

Experiments by Sakai (1997); Vorobieff & Ecke (2002) show features of rotationally constrained convection:

- intense vortical structures spanning layer of fluid
- cyclonic and anticyclonic vortical structures
- vortex-vortex interaction

Experimental Challenge:
Visualization/measurement of 3-D data

Temperature (Sakai, 1997)
($Ra \approx 10^7$, $R_{\text{conv}} \approx 0.1$, $Pr \approx 7$)
**Rapidly Rotating Convection: Vortex Structure**

- Hot vortices have cyclonic vorticity below mid-plane; anti-cyclonic above mid-plane (opposite for cold vortices)
- Sakai alludes to geostrophic balance in interior: pressure forces balance Coriolis forces (our results support this!)

(Sakai, 1997)
Rapidly Rotating Convection: Challenges for Direct Numerical Simulation (DNS)

- Ekman boundary layers become increasingly thin as the rotation rate is increased ($\delta_E \sim E^{1/2} \ll 1$): must resolve in DNS

- Fast inertial waves exist ($\omega \sim E^{-1}$), which hinder explicit time integration
Rapidly Rotating Convection: Challenges for Direct Numerical Simulation (DNS)

- Ekman boundary layers become increasingly thin as the rotation rate is increased ($\delta_E \sim E^{1/2} \ll 1$): must resolve in DNS
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**DNS Simulation (Julien et al. 1996)**

- $Ro_{\text{conv}} = 0.75$, $Ra \approx 10^7$, $Pr \approx 1$
- Temperature (red/blue) and vertical *cyclonic* vorticity (yellow)
- In physical experiments, *anti-cyclonic* vortices emerge as $Ro_{\text{conv}} \lesssim 0.2$ (Vorobieff & Ecke 2002; Sakai 1997)

So, how do we numerically investigate convection in the regime $Ro_{\text{conv}} \ll 1$?
Governing Equations

- Scales used for nondimensionalization: $L, U, \tilde{T}, P$

- **Boussinesq approximation** in a rotating coordinate frame $\hat{z}$:

\[
D_t \mathbf{u} + Ro^{-1} \hat{z} \times \mathbf{u} = -\overline{P} \nabla p + \Gamma \hat{z} + Re^{-1} \nabla^2 \mathbf{u} \\
D_t \left( \theta - \frac{1}{Fr^2} \overline{\rho}(z) \right) = Pe^{-1} \nabla^2 \theta \\
\n\n\nwhere $\mathbf{u} = (u, v, w)$ is the velocity, $D_t = \partial_t + \mathbf{u} \cdot \nabla$, $p$ is pressure, and $\theta$ is the buoyancy anomaly (temperature)

- **Important Nondimensional Parameters**:

  - $Ro = U/2\Omega L$ \hspace{1cm} Rossby Number \hspace{1cm} $Fr = \frac{U}{N_0 L}$ \hspace{1cm} Froude Number
  - $Re = \frac{UL}{\nu}$ \hspace{1cm} Reynolds Number \hspace{1cm} $Pe = \frac{UL}{\kappa}$ \hspace{1cm} Péclet Number
  - $\Gamma = \frac{BL}{U^2}$ \hspace{1cm} Buoyancy Number \hspace{1cm} $\overline{P} = \frac{P}{\rho_0 U^2}$ \hspace{1cm} Euler Number
Asymptotic Theory: NH-QGE

Multiple scales expansion in the vertical direction and in time:

\[ \partial_z \to \frac{1}{A_Z} \partial_Z, \quad \partial_t \to \partial_t + \frac{1}{A_\tau} \partial_\tau \]

Large Scale: \( Z = A_Z^{-1} z \)

Slow Time: \( \tau = A_\tau^{-1} t \)

Field variables are separated into \textit{average} (over fast/short scales) and \textit{fluctuating} components:

\[ \mathbf{v}(x, Z, t, \tau) = (u, p, \theta)^T = \mathbf{\nabla}(Z, \tau) + \mathbf{v}'(x, Z, t, \tau), \]

where

\[ \mathbf{\nabla} := \lim_{\tilde{t}, V \to \infty} \frac{1}{\tau V} \int_{\tilde{t}, V} \mathbf{v} \, dx \, dt, \quad \mathbf{\nabla}' = 0. \]
Asymptotic Theory: NH-QGE

- Relate aspect ratio to $Ro \equiv \epsilon$: $AZ = \epsilon^{-1}$

- Find:
  
  $$A_\tau = \epsilon^{-2}, \quad \overline{P} = O(\epsilon^{-2}), \quad \Gamma = O(\epsilon^{-1})$$

- Scaling chosen
  
  - For isotropic velocity field: $u_0 \sim v_0 \sim w_0$
  
  - For fluid motions to feed back and adjust mean stratification:
    
    $$Fr = \epsilon^{\frac{1}{2}}$$

- Remark I: If $AZ < O(\epsilon^{-1})$ vertical motions are weak.
  Hydrostatic-QGE recovered for columnar regime.

- Remark II: If $\overline{P} \sim \epsilon^{-1}, \Gamma = 1$ no feedback occurs. Dynamics consists of nonlinear propagating inertial-gravity waves (Smith & Waleffe JFM 2002).

- Expand all fields in powers of $\epsilon$:

  $$\mathbf{v} = \mathbf{v}_0 + \epsilon \mathbf{v}_1 + \epsilon^2 \mathbf{v}_2 + \ldots$$
Asymptotic Theory: NH-QGE

Leading-Order Results:

- Hydrostatic balance: \( \frac{\partial}{\partial Z} p_0 = \tilde{\Gamma} \theta_0 \), \( \mathbf{u}_0 = 0 \)
- Temp. & press. fluctuations occur at first order \((\theta'_0 = 0, p'_0 = 0)\)
- Momentum (geostrophic balance)

\[ \hat{z} \times u'_0 = -\nabla p'_1 \quad \Rightarrow \quad \begin{cases} (\hat{z} \cdot \nabla) p'_1 = 0 \\ (\hat{z} \cdot \nabla) u'_0 = 0 \\ \nabla_\perp \cdot u'_{0\perp} = 0 \end{cases} \]

All dependent variables are governed by Taylor-Proudman constraint on small scales (invariance along axis of rotation):

Solution: \( u'_0 = \hat{z} \times \nabla \psi(x, y, Z, t) + W(x, y, Z, t)\hat{z}, \quad p'_1 = \psi(x, y, Z, t) \)
Asymptotic Theory: NH-QGE

Geostrophy:
\[ \hat{z} \times u'_0 + \nabla p'_1 = 0, \]

Nonhydrostatic Quasigeostrophic equations obtained from solvability conditions applied to:
\[ \hat{z} \times u'_1 + \nabla p'_2 = F(u'_0, \theta'_1), \quad \nabla \cdot u'_1 + \partial_Z w'_0 = 0. \]

Vertical Velocity \((W)\) & Vertical Vorticity \((\omega = \nabla^2_{\perp} \psi)\):
\[ \partial_t W + J(\psi, W) + \partial_Z \psi = \tilde{\Gamma}_1' + \Re^{-1} \nabla^2_{\perp} W \]
\[ \partial_t \omega + J(\psi, \omega) - \partial_Z W = \Re^{-1} \nabla^2_{\perp} \omega \]
Stream-Function Formulation: Closed System

**Vertical Velocity** \((W)\) & **Vertical Vorticity** \((\omega = \nabla^2_\perp \psi)\):

\[
\begin{align*}
\partial_t W + J (\psi, W) + \partial_Z \psi &= \tilde{\Gamma} \theta_1' + Re^{-1} \nabla^2_\perp W \\
\partial_t \omega + J (\psi, \omega) - \partial_Z W &= Re^{-1} \nabla^2_\perp \omega
\end{align*}
\]

**Fluctuating and mean temperature equations:**

\[
\begin{align*}
\partial_t \theta_1' + J (\psi, \theta_1') + W \partial_Z \left( \theta_0 - \tilde{\Gamma}^{-1} \rho \right) &= Pe^{-1} \nabla^2_\perp \theta_1' \\
\partial_x \theta_0 + \partial_Z \left( \theta'_1 W \right) &= Pe^{-1} \partial_{ZZ} \theta_0
\end{align*}
\]

where \(J (\psi, f) \equiv \partial_x \psi \partial_y f - \partial_x f \partial_y \psi = u_{0\perp} \cdot \nabla_\perp f\)
Stream-Function Formulation: Closed System

Vertical Velocity \((W = \nabla^2 \perp \phi)\) & Vertical Vorticity \((\omega = \nabla^2 \perp \psi)\):

\[
\begin{align*}
\partial_t W + J(\psi, W) + \partial_Z \psi &= \tilde{\Gamma} \theta_1' + Re^{-1} \nabla^2 \perp W \\
\partial_t \omega + J(\psi, \omega) - \partial_Z W &= Re^{-1} \nabla^2 \perp \omega
\end{align*}
\]

Fluctuating and mean temperature equations:

\[
\begin{align*}
\partial_t \theta_1' + J(\psi, \theta_1') + W \partial_Z \left(\bar{\theta}_0 - \tilde{\Gamma}^{-1} \bar{\rho}\right) &= Pe^{-1} \nabla^2 \perp \theta_1' \\
\partial_t \bar{\theta}_0 + \partial_Z \left(\bar{\theta}_1' W\right) &= Pe^{-1} \partial_Z \bar{\theta}_0
\end{align*}
\]

Conserves energy: 

\[
E = \frac{1}{2} \int_D |\nabla \perp \psi|^2 + \tilde{\Gamma} \frac{\theta^2_1}{\partial_Z \left(\bar{\theta}_0 - \tilde{\Gamma}^{-1} \bar{\rho}\right)} dxdydz
\]

Conserves PV:

\[
\Pi \equiv \nabla^2 \perp \psi + J \left(\phi, \frac{\theta_1'}{\partial_Z \left(\bar{\theta}_0 - \tilde{\Gamma}^{-1} \bar{\rho}\right)}\right) + \partial_Z \left(\frac{\theta_1'}{\partial_Z \left(\bar{\theta}_0 - \tilde{\Gamma}^{-1} \bar{\rho}\right)}\right)
\]
**Application to Rotating RBC**

- **Vertical Velocity** \((W)\) & **Vertical Vorticity** \((\omega = \nabla^2_\perp \psi)\):

\[
\begin{align*}
\partial_t W + J (\psi, W) + \partial_Z \psi &= \frac{\tilde{R}a}{Pr} \theta_1' + \nabla^2_\perp W \\
\partial_t \omega + J (\psi, \omega) - \partial_Z W &= \nabla^2_\perp \omega
\end{align*}
\]

- **Fluctuating- and mean-Temperature equations:**

\[
\begin{align*}
\partial_t \theta_1' + J (\psi, \theta_1') + W \partial_Z \bar{\theta}_0 &= Pr^{-1} \nabla^2_\perp \theta_1' \\
\partial_{\tau} \bar{\theta}_0 + \partial_Z \left( \theta_1' W \right) &= Pr^{-1} \partial_{ZZ} \bar{\theta}_0
\end{align*}
\]

- **RBC nondimensionalization**

- \(Ta = 4\Omega^2 H^4/\nu^2, \quad Ra = g\alpha \Delta TH^3/\nu \kappa, \quad Pr = \nu/\kappa\)
- \(Ro = Ta^{-1/6}, \quad Re = 1, \quad Pe = Pr, \quad \tilde{\Gamma} = \epsilon^4 \tilde{R}a/Pr \equiv \tilde{R}a/Pr\)
Exact Single-Mode Solutions

- Bassom & Zhang GAFD ’94; Julien & Knobloch PoF ’96, ’99; JFM ’98
- **Separable Solutions:** \( W = A(Z) h(x, y) \), with \( \nabla^2_{\perp} h + k^2_{\perp} h = 0 \)

\[
\partial_{ZZ} A + \left( \frac{k^2_{\perp} \overline{RaNu}}{1 + \frac{Pr^2}{k^2_{\perp}} A^2} - k^6_{\perp} \right) A = 0, \quad A(0) = A(1) = 0
\]

![Graphs showing separable solutions](image)
Numerical Method for DNS of Reduced Model

- Spectral spatial discretization: periodic Fourier modes in the horizontal; Chebyshev-Tau in the vertical
  - Nonuniform grid-point distribution in vertical is well suited to resolving the thin thermal boundary layers
- Impenetrable, stress-free boundary conditions
- Mixed implicit/explicit third-order Runge-Kutta time integration (Spalart et al., JCP, 1991)
- Employs CRAY SHMEM libraries for parallelization; solved on CRAY and/or SGI supercomputers
- Typical models have $64^3$ to $512^2 \times 256$ grid points
- Solutions evolve on fast ($t$) and slow ($\tau$) time scales; we neglect variation on slow time scale $\Rightarrow$ numerical solutions only valid in statistically steady-state regime
Results: Topological Change of Flow: Columnar Regime

\[ \tilde{Ra} = Ta^{-2/3} Ra = 20,\ Pr = 7 \text{ (water)}, \tilde{Ra}_{\text{crit}} \approx 8.7 \]

Above shows Temperature Anomaly \( \theta'_1 \)

Columnar structure is clear

Hot vortices have cyclonic vorticity below mid-plane; anti-cyclonic above mid-plane (opposite for cold vortices)

Cyclonic and anti-cyclonic vorticity balanced due to symmetry in governing equations; not present in full Boussinesq equations
Results: Topological Change of Flow: Shielded-Vortex Regime

\[ \tilde{Ra} = Ta^{-2/3}Ra = 40, \; Pr = 7 \; (Sakai \; '97, \; \tilde{Ra} \approx 35) \]

- Columnar structure is clear; vortices are shielded by opposite-signed ‘sleeves’ extending across layer
- Vortices are in constant, but slow horizontal motion
- Columns highly efficient at heat transport; columns responsible for transporting 60% of heat flux
Results: Topological Change of Flow: Geostrophic Turbulence Regime

\[ \tilde{Ra} = Ta^{-2/3} \]

\[ Ra = 80, \ Pr = 7 \]

As thermal forcing is increased, lateral mixing plays significant role

Columnar structure and shielding destroyed

Geostrophic turbulence regime characterized by hot (cold) plumes emanating from the lower (upper) thermal boundary layers
Results: $\tilde{Ra} = Ro^4 Ra = 40, Pr = 7$ (water)

- Vortical columns weakly interacting
- Zero vortical circulation $\int_0^{R^*} \omega r dr d\theta \approx 0$
- Particle model being pursued
**Results: Mean Temperature** ($\tilde{Ra} = 160$)

![Graph showing mean temperature vs. vertical position with different Pr values: $Pr = 1$, $Pr = 7$, $Pr \to \infty$, and Single-Mode. The graph indicates the transition from convection to pure conduction as $Pr$ increases.](image-url)
**Results: Mean Temperature Gradient at Midplane**

Mean-temperature profile saturates at a non-isothermal interior for all finite $Pr$ values studied.

Knobloch, 5 December 2005 – p.22/25
**Results: Heat Transport**

- **Nusselt Number** ($Nu = \partial_Z \bar{\theta}_0 |_{Z=1}$): Measure of convective heat transport ($Nu = 1 \implies$ conductive heat transport)

![Graph showing Nu as a function of time for different Pr values.](image)

- Results move quickly to statistically steady state
- $t_{rot} = 4\pi Pr^{-1} Ta^{-1/6}$; results shown for many rotation times
Summary

Numerical simulation of reduced PDEs allows exploration of parameter range currently inaccessible with DNS of rotating Boussinesq equations.

Reduced PDEs capture dynamics seen in experiments:
- Coherent structures spanning the layer of fluid
- Structures composed of cyclonic & anticyclonic vorticity

Important findings:
- Mean-temperature gradient saturates at nonzero value for all Prandtl numbers investigated due to increased lateral mixing.
- Found transition (with increasing $\tilde{Ra}$) through three regimes: (i) columnar, (ii) shielded-vortex, and (iii) geostrophic turbulence.
- Illustrated importance of lateral mixing; should be incorporated in convection parametrizations for ocean circulation models.
Future Work with Reduced PDEs

- Simulation of rapidly rotating convection on the tilted $f$-plane; more geophysically relevant
- Application of reduced PDEs to ocean deep convection (Legg, McWilliams & Gao, JPO 1998)
- Can we develop a particle model for the shielded-vortex regime of convection? (Legg & Marshall, JMR 1998),
- Simulation of recently developed equations for rapidly rotating convection in a cylinder (Sprague, et al., TSFP4 2005)
- Application of Large-Eddy Simulation (LES) to reduced equations (Barbosa & Métails, JOT 2000)