Anisotropic constraints on energy in rotating and stratified flows

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Thanks to:
Leslie Smith, Jai Sukhatme, Mark Taylor and Beth Wingate.
Question:

- How does potential enstrophy constrain the energy of rotating and stratified flow in regimes other than quasi-geostrophy?
  - strong rotation, strong stratification (equal)
  - strong rotation, moderate stratification
  - moderate rotation, strong stratification
Rotating and stratified turbulence

• Boussinesq equations for rotating, stably stratified flow (periodic or infinite boundary conditions)

\[
\frac{D}{Dt} \mathbf{u} + f \mathbf{\hat{z}} \times \mathbf{u} + \nabla p + N \theta \mathbf{\hat{z}} = \nu \nabla^2 \mathbf{u} + \mathbf{F}
\]

\[
\frac{D}{Dt} \theta - N w = \kappa \nabla^2 \theta
\]

\[\nabla \cdot \mathbf{u} = 0,\]

• conserved quantities (inviscid)

total energy \( E_T = E + P \), \( \frac{D}{Dt} \int E_T \, d\mathbf{x} = 0 \), \( E = \frac{1}{2} \mathbf{u}^2 \)

potential vorticity \( q = (\boldsymbol{\omega}_a \cdot \nabla \rho_T) \), \( \frac{Dq}{Dt} = 0 \), \( P = \frac{1}{2} \theta^2 \)

potential enstrophy \( Q = \frac{1}{2} q^2 \), \( \frac{DQ}{Dt} = \frac{D}{Dt} \int Q \, d\mathbf{x} = 0 \).
Potential enstrophy in \textit{quasi-geostrophic} turbulence

- inviscid quasi-geostrophic dynamics is described by evolution of potential vorticity:

\[
\frac{\partial q_{qq}}{\partial t} + u_{0h} \cdot \nabla q_{qq} = 0,
\]

where \( q_{qq} = f \frac{\partial \theta}{\partial z} - N \omega_3 \),

\[
Q = \frac{1}{2} |q|^2
\]

- Charney 1971: Forward (downscale) transfer of total energy is suppressed in favor of \textit{(isotropic)} forward transfer of potential enstrophy, with resultant scaling of energy spectrum:

\[
E_T(k) \propto \varepsilon^{2/3} Q^{1/3} k^{-3}
\]
Potential enstrophy in Boussinesq rotating and stratified flows

- non-dimensional potential vorticity: \( Ro = U/Lf, \ Fr = U/LN \)

\[
q = \omega \cdot \nabla \theta + Ro^{-1} \frac{\partial \theta}{\partial z} - Fr^{-1} \omega_3
\]

- Linear limit of potential vorticity is approached when either rotation or stratification are strong

\[
\tilde{q}(k) \sim f k_z \tilde{\theta} + i N k_h \tilde{u}_h
\]
Limiting cases in $N$ and $f$ ($Fr$ and $Ro$)

$$\tilde{q}(k) \approx f k_z \tilde{\theta} + i N k_h \tilde{u}_h$$

- strongly rotating and strongly stratified
  $N = f$; $N$ and $f$ large ($Fr$ and $Ro$ small)
- strongly rotating and moderately stratified
  $N \gg f$; $N$ large, $f \sim O(1)$ ($Fr$ small, $Ro \sim O(1)$)
- moderately rotating, strongly stratified
  $N \ll f$; $N \sim O(1)$, $f$ large ($Fr \sim O(1)$, $Ro$ small)
Numerical simulations

- 3d Boussinesq, unit aspect ratio, periodic, stochastic forcing of momentum at \( k=3,4,5 \), variable rotation and stratification, hyperviscous dissipation (diffusion) \( (\nabla^{16}) \) of momentum (density)

<table>
<thead>
<tr>
<th># gridpoints</th>
<th>case</th>
<th>Ro</th>
<th>Fr</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 512^3 )</td>
<td>( N = f )</td>
<td>0.004</td>
<td>0.004</td>
</tr>
<tr>
<td>( 640^3 )</td>
<td>( N&lt;&lt;f )</td>
<td>0.004</td>
<td>1</td>
</tr>
<tr>
<td>( 640^3 )</td>
<td>( N&gt;&gt;f )</td>
<td>1</td>
<td>0.004</td>
</tr>
</tbody>
</table>
$N=f$, potential enstrophy suppresses potential energy in the large aspect-ratio modes

- as $k_z/k_h \gg 1$ (large aspect-ratio modes) $\tilde{q} \simeq f k_z \tilde{\theta}$

$$Q(k_h, k_z) = \frac{1}{2} |\tilde{q}|^2 = f^2 k_z^2 P(k_h, k_z) \text{ where } P(k_h, k_z) = \frac{1}{2} |\tilde{\theta}|^2$$

- potential enstrophy must dominate potential energy as $k_z \to \infty$

$$\int_{k_z}^{\infty} Q(k_h, k_z) dk_z \gg f^2 k_z^2 \int_{k_z}^{\infty} P(k_h, k_z) dk_z$$

- assuming dependence on dissipation rate of potential enstrophy and dimensional analysis (Kraichnan, Charney)

$$P(k_h, k_z) \sim \varepsilon_Q^{2/5} k_z^{-3}$$
$N=f$, potential enstrophy suppresses horizontal kinetic energy in the small aspect-ratio modes

- in the limit as $k_z/k_h \ll 1$ (small aspect-ratio modes)
  \[ \tilde{q} = iNk_h \tilde{u}_h \]
- potential enstrophy must dominate over horizontal kinetic energy
  \[ E_h(k_h, k_z) = \frac{1}{2} |\tilde{u}_h|^2 \text{ as } k_h \to \infty \]
- assuming dependence on dissipation rate of potential enstrophy and dimensional analysis:
  \[ E_h(k_h, k_z) \sim \varepsilon^{2/5} Q^{2/5} k_h^{-3} \]
$N = f$, total potential enstrophy (lines) and quadratic part (circles) as a function of time.

$512^3$

$Ro = Fr = 0.014$

$512^3$

$Ro = Fr = 0.007$
Shell-averaged kinetic and potential energy spectra

- $512^3 \ Ro = Fr = 0.007$
Scaling of potential energy and horizontal kinetic energy spectra for $N=\ell$, aspect-ratio dependence

$k_z/k_h \gg 1$

$k_h/k_z \gg 1$
$N << f$: potential enstrophy suppresses potential energy dependence on $k_h$

• in this limit, dependence of potential vorticity on $k_h$ disappears

$$\tilde{q} \sim f k_z \tilde{\theta}$$

for all $k_h$

• potential enstrophy dominates over potential energy in large $k_z$, and assuming dependence on dissipation rate of potential enstrophy and dimensional analysis:

$$P(k_z) \sim \epsilon_Q k_z^{-3}$$

for all $k_h$
\( N \ll f \): potential enstrophy suppresses potential energy dependence on \( k_h \)
\(N \gg f\), potential enstrophy suppresses horizontal kinetic energy dependence on \(k_z\)

- in this limit, the dependence on \(k_z\) disappears since
  \[
  \tilde{q} = iNk_h \tilde{u}_h \quad \text{for all } k_z
  \]
- again potential enstrophy must dominate over horizontal kinetic energy \(E_h\) for large \(k_h\)
- assuming dependence on dissipation rate of potential enstrophy and dimensional analysis:
  \[
  E_h(k_h) \sim \epsilon Q k_h^{-3} \quad \text{for all } k_z
  \]
\( N \gg f \), potential enstrophy suppresses horizontal kinetic energy dependence on \( k_z \)
Summary


• Case of unequal rotation and stratification (SK, in preparation 2008)
  • potential enstrophy conservation in the limit of strong rotation and moderate stratification: suppresses potential energy in large $k_z$, eliminates dependence on $k_h$
  • potential enstrophy conservation in the limit of moderate rotation and strong stratification: suppresses horizontal kinetic energy in large $k_h$, eliminates dependence on $k_z$