Vortical-Wave Mode Interactions in Atmosphere-Ocean Flows

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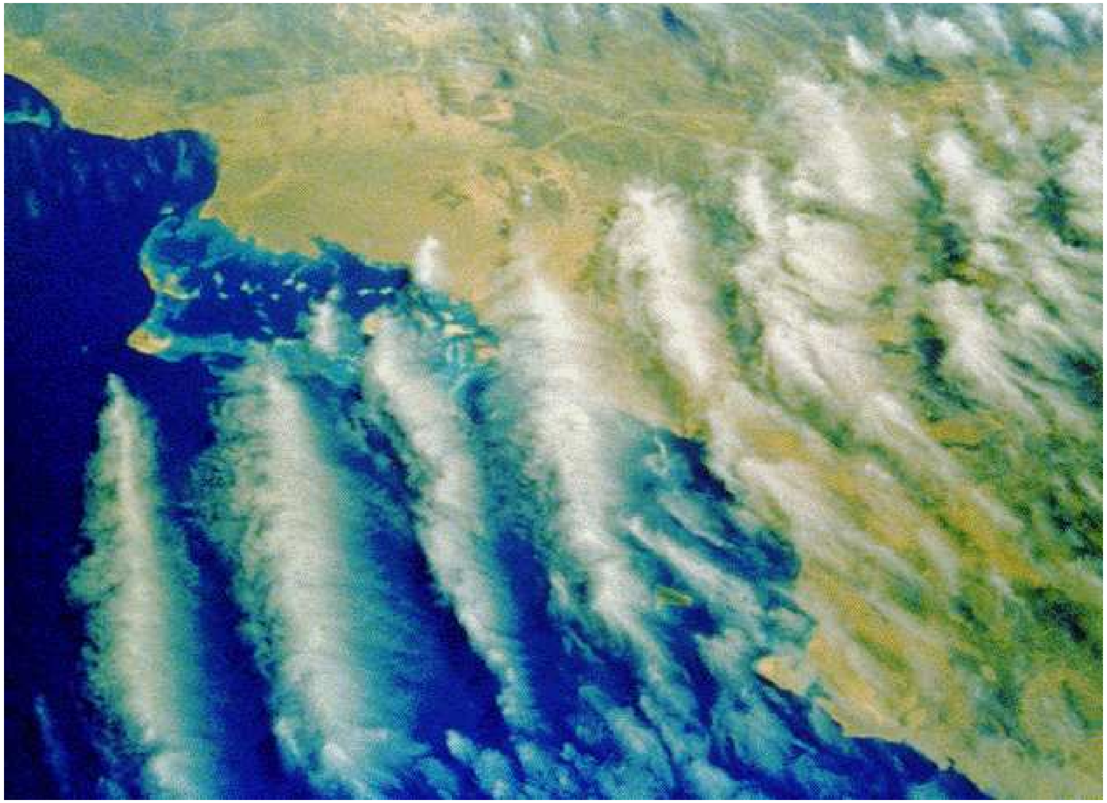

Roll clouds in the jet stream over Saudi Arabia/Red Sea

Supported by NSF and DOE

Vortical-Wave Mode Interactions in Atmosphere-Ocean Flows

- What is a vortical mode, what is a wave mode, what are vortical-wave mode interactions?

- When are vortical-wave mode interactions important?

- Can we use this framework to understand balanced and unbalanced components of geophysical flows?

Overview:

Rotating stratified fluid flow is a starting point to understand atmosphere-ocean phenomena.

In the linear limit, the governing equations possess:

- a so-called ‘vortical’ linear eigenmode
- wave eigenmodes

The simplest ‘reduced models’ (e.g. QG models)

- keep nonlinear interactions between vortical modes
- neglect wave mode interactions.

This strategy works to describe large-scale flows on short time scales, e.g. short-term weather prediction.

Overview continued:

However, wave mode interactions contribute in many physical situations, e.g. flow over topography,

and influence large-scale coherent flows on long times

e.g. they can *generate* jets, vortices and layers.

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We introduce a framework to construct an understanding of all wave-vortical mode interactions.

New, non-perturbative models are derived ’intermediate’ between QG and the full governing equations.

Outline

I. Analytical Properties of the Governing Equations

- Solution as a superposition of linear eigenmodes
- Reduced models

II. Example Numerical Results Depending Crucially on Wave Mode Interactions

- large-scale forcing, small-scale forcing, decay

III. Derivation of new PDE reduced models to understand wave-vortical mode interactions

IV. Numerical results for new PDE models

Conservation laws for vertically stratified flow rotating about the vertical $\hat{\mathbf{z}}$-axis:

$$momentum: \quad \frac{D\mathbf{u}}{Dt} + f\hat{\mathbf{z}} \times \mathbf{u} = -\nabla\phi - N\theta\hat{\mathbf{z}} + \nu\nabla^2\mathbf{u}$$

$$mass: \quad \nabla \cdot \mathbf{u} = 0$$

$$energy: \quad \frac{D\theta}{Dt} - Nw = \kappa\nabla^2\theta, \quad \theta = \frac{g}{N\rho_o}\rho'$$

$$f = 2\Omega, \qquad Ro = \frac{U}{fL}$$

$$\rho = \rho_o - bz + \rho', \quad \rho' \ll \rho_o, |bz|, \quad N^2 = \frac{gb}{\rho_o}, \quad Fr = \frac{U}{NH}$$

Rossby and Froude numbers in geophysical flows

Pedlosky (1986) estimates:

- $Ro \approx 0.14$ for typical synoptic-scale winds at mid-latitudes
  $U \approx 10 \ \text{m} \ \text{s}^{-1}$, $L \approx 1000 \ \text{km}$
- $Ro \approx 0.07$ in the western Atlantic
  $U \approx 5 \ \text{cm} \ \text{s}^{-1}$, $L \approx 100 \ \text{km}$

Typical values are $N/f \approx 100$ in the stratosphere and $N/f \approx 10$ in the oceans.

Flows with $N/f \approx L/H ==> Fr \approx 0.1$ (Burger number unity).

Solutions in the unforced, linear, inviscid limit

$$[\mathbf{u}, \theta]^T(\mathbf{x}, t; \mathbf{k}) = \phi(\mathbf{k}) \exp\left[i\left(\mathbf{k}\cdot\mathbf{x} - \sigma(\mathbf{k})t\right)\right] + \text{ c.c.}$$

with eigenmodes $\phi(\mathbf{k})$ and eigenvalues $\sigma(\mathbf{k})$.

- Wave modes $\phi_+(\mathbf{k})$ and $\phi_-(\mathbf{k})$ with

$$\sigma_\pm(\mathbf{k}) = \pm\frac{(N^2k_h^2 + f^2k_z^2)^{1/2}}{k}$$

- A non-wave (vortical or geostophic) mode $\phi_0(\mathbf{k})$ with

$$\sigma_0(\mathbf{k}) = 0$$

.

Slow wave modes (as important as slow vortical modes!)

- Rotation-dominated flows

$$\sigma_{\pm}(\mathbf{k}) \approx \pm \frac{f k_z}{k}$$

slow when $k_z = 0$, e.g. vortical columns.

- Stratification-dominated flows

$$\sigma_{\pm}(\mathbf{k}) \approx \pm \frac{N k_h}{k}$$

slow when $k_h = 0$, e.g. horizontal shear layers (VSHF)

Eigenmode representation for nonlinear flows

Since $\boldsymbol{\phi}_s(\mathbf{k})$, $s = \pm, 0$ form an orthogonal basis

$$[\mathbf{u}, \theta]^T(\mathbf{x}, t) = \sum_{\mathbf{k}} \sum_{s} b_s(t; \mathbf{k}) \boldsymbol{\phi}_s(\mathbf{k}) \exp\left[i \left(\mathbf{k} \cdot \mathbf{x} - \sigma_s(\mathbf{k}) t \right) \right]$$

and the equations become

$$\frac{\partial}{\partial t} b_{s_\mathbf{k}} = \sum_{\triangle} \sum_{s_\mathbf{p}, s_\mathbf{q}} C^{s_\mathbf{k} s_\mathbf{p} s_\mathbf{q}}_{\mathbf{kpq}} \; b^*_{s_\mathbf{p}} \, b^*_{s_\mathbf{q}} \exp\left[i \left(\sigma_{s_\mathbf{k}} + \sigma_{s_\mathbf{p}} + \sigma_{s_\mathbf{q}} \right) t \right]$$

27 interaction types, including 3-wave interactions

Exact and near resonances dominate: $|\sigma_{s_\mathbf{k}} + \sigma_{s_\mathbf{p}} + \sigma_{s_\mathbf{q}}| \ll 1.$

Reduced Models

Reduced models resulting from restriction of the sum

$$\frac{\partial}{\partial t} b_{s_{\mathbf{k}}} = \sum_{\triangle} \sum_{s_{\mathbf{p}}, s_{\mathbf{q}}} C^{s_{\mathbf{k}} s_{\mathbf{p}} s_{\mathbf{q}}}_{\mathbf{kpq}} \; b^*_{s_{\mathbf{p}}} \; b^*_{s_{\mathbf{q}}} \exp\left[i \left(\sigma_{s_{\mathbf{k}}} + \sigma_{s_{\mathbf{p}}} + \sigma_{s_{\mathbf{q}}} \right) t \right]$$

automatically conserve energy because each triad $(\mathbf{k}, \mathbf{p}, \mathbf{q})$ satisfies the detailed balance:

$$C^{s_{\mathbf{k}} s_{\mathbf{p}} s_{\mathbf{q}}}_{\mathbf{kpq}} \; + \; C^{s_{\mathbf{p}} s_{\mathbf{q}} s_{\mathbf{k}}}_{\mathbf{pqk}} \; + \; C^{s_{\mathbf{q}} s_{\mathbf{k}} s_{\mathbf{p}}}_{\mathbf{qkp}} = 0$$

3D Pure Rotation

keeping only slow wave modes with $k_z = 0 ==>$ symmetric 2D flow for (u,v); w is a passive scalar.

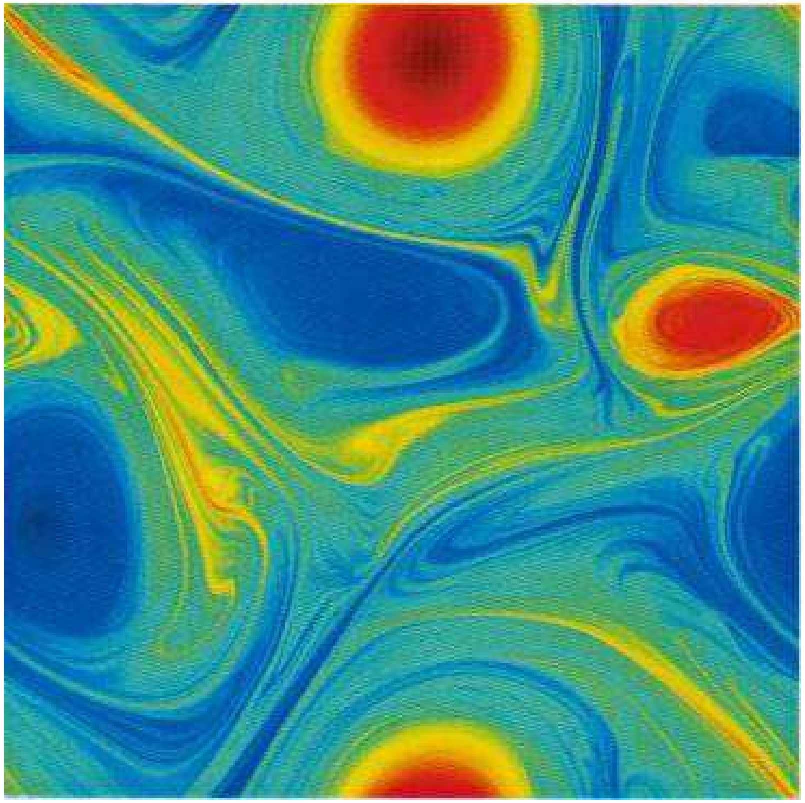
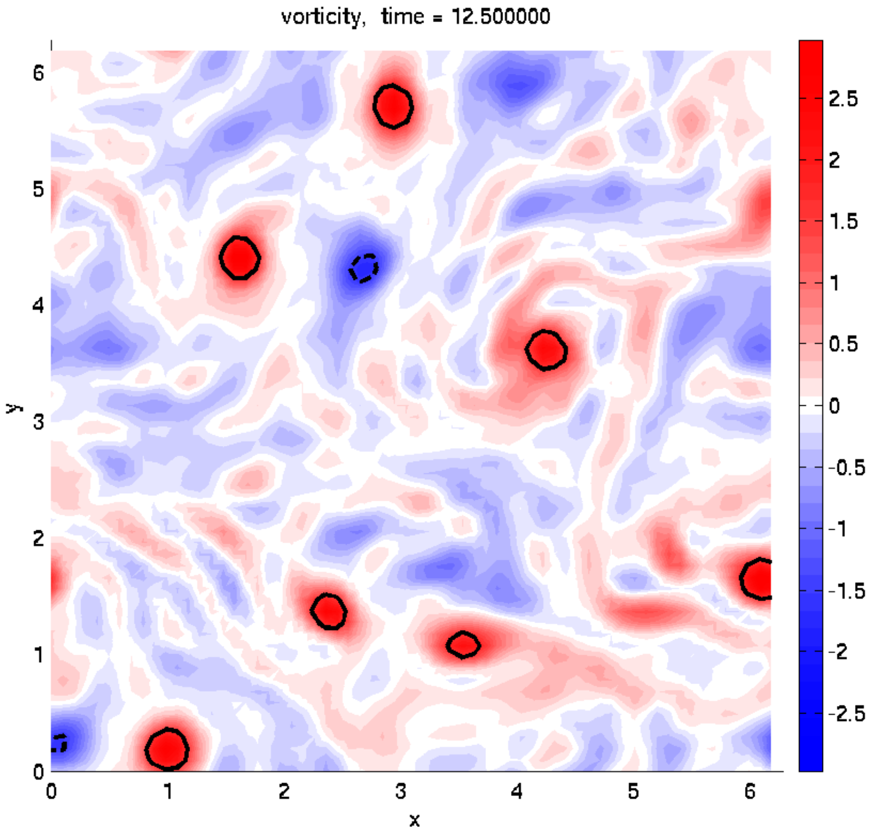
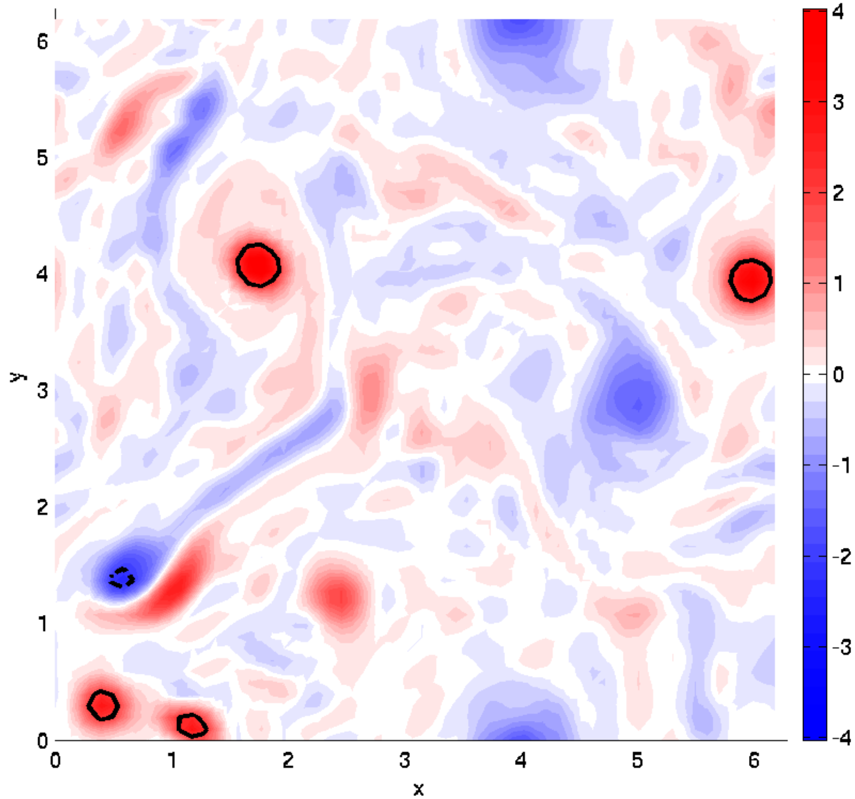


Fig. shows (symmetric) 2D decay with large-scale drag.

Embid & Majda 1996, 1998, Babin et al. 2002 all interactions with $|\sigma_{s_\mathbf{k}} + \sigma_{s_\mathbf{p}} + \sigma_{s_\mathbf{q}}| < Ro ==>$ cyclone dominance (but is not a PDE !).



Full



Near resonances (12%)

Smith & Lee 2005

Reduced Models for 3D Boussinesq

Keeping only slow vortical mode interactions $==>$ the symmetric 3D quasi-geostrophic equation.

In Fourier space

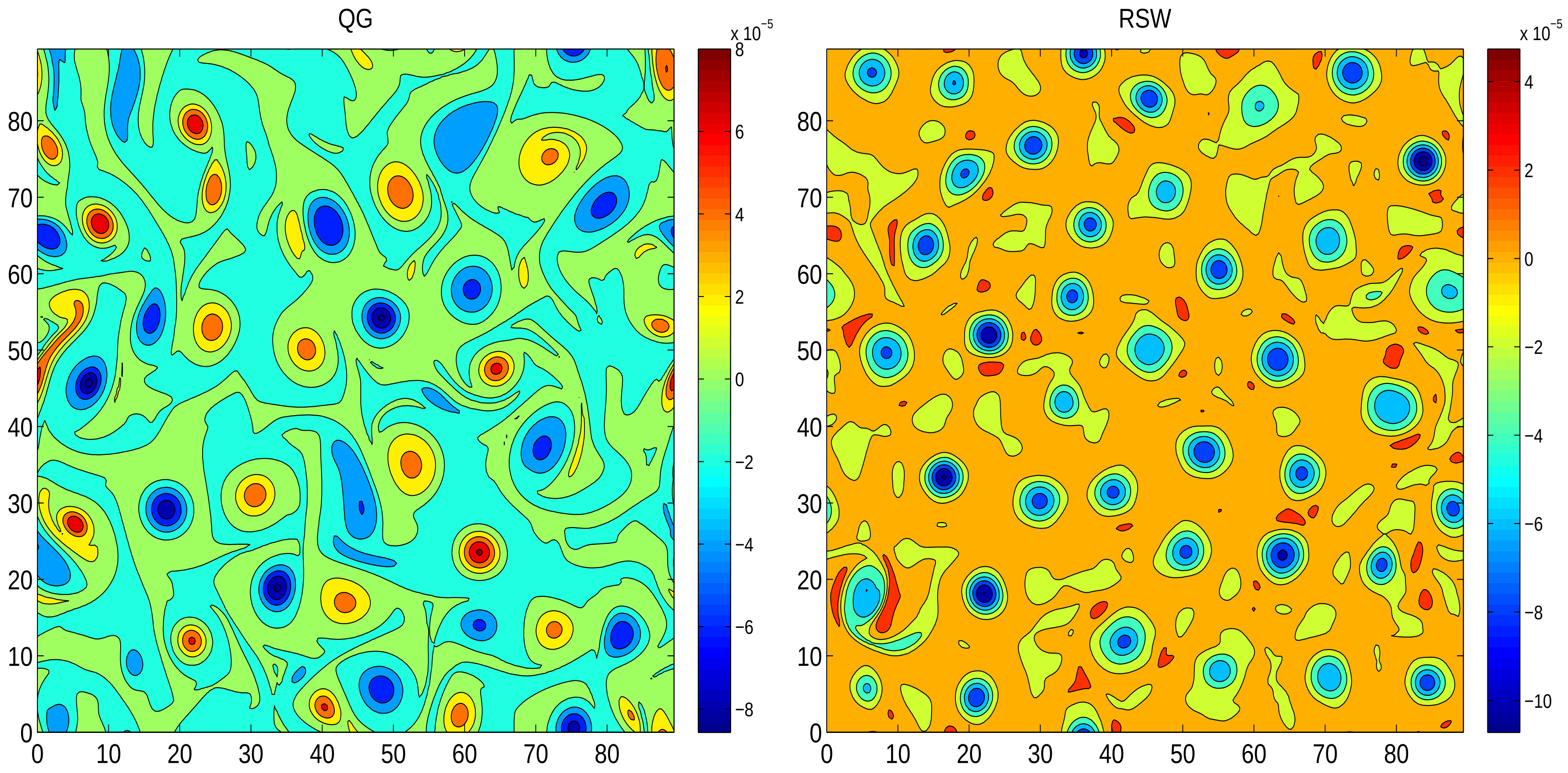
$$\frac{d}{dt} b_0(t; \mathbf{k}) = \sum_{\triangle} C^{000}_{\mathbf{kpq}} \; b^*_0(\mathbf{p}) \; b^*_0(\mathbf{q})$$

An inverse transform gives 3DQG

$$\left(\frac{\partial}{\partial t} + \mathbf{u}_H \cdot \boldsymbol{\nabla}\right) q = 0, \qquad q = \left(\nabla^2_H + \frac{f^2}{N^2}\frac{\partial^2}{\partial z^2}\right)\psi(\mathbf{x}, t)$$

$$\nabla^2_H = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad \mathbf{u}_H = \hat{\mathbf{z}} \times \boldsymbol{\nabla}\psi, \quad \theta = -\frac{f}{N}\frac{\partial \psi}{\partial z}$$

Example 1: Anticyclone dominance in RSW decay



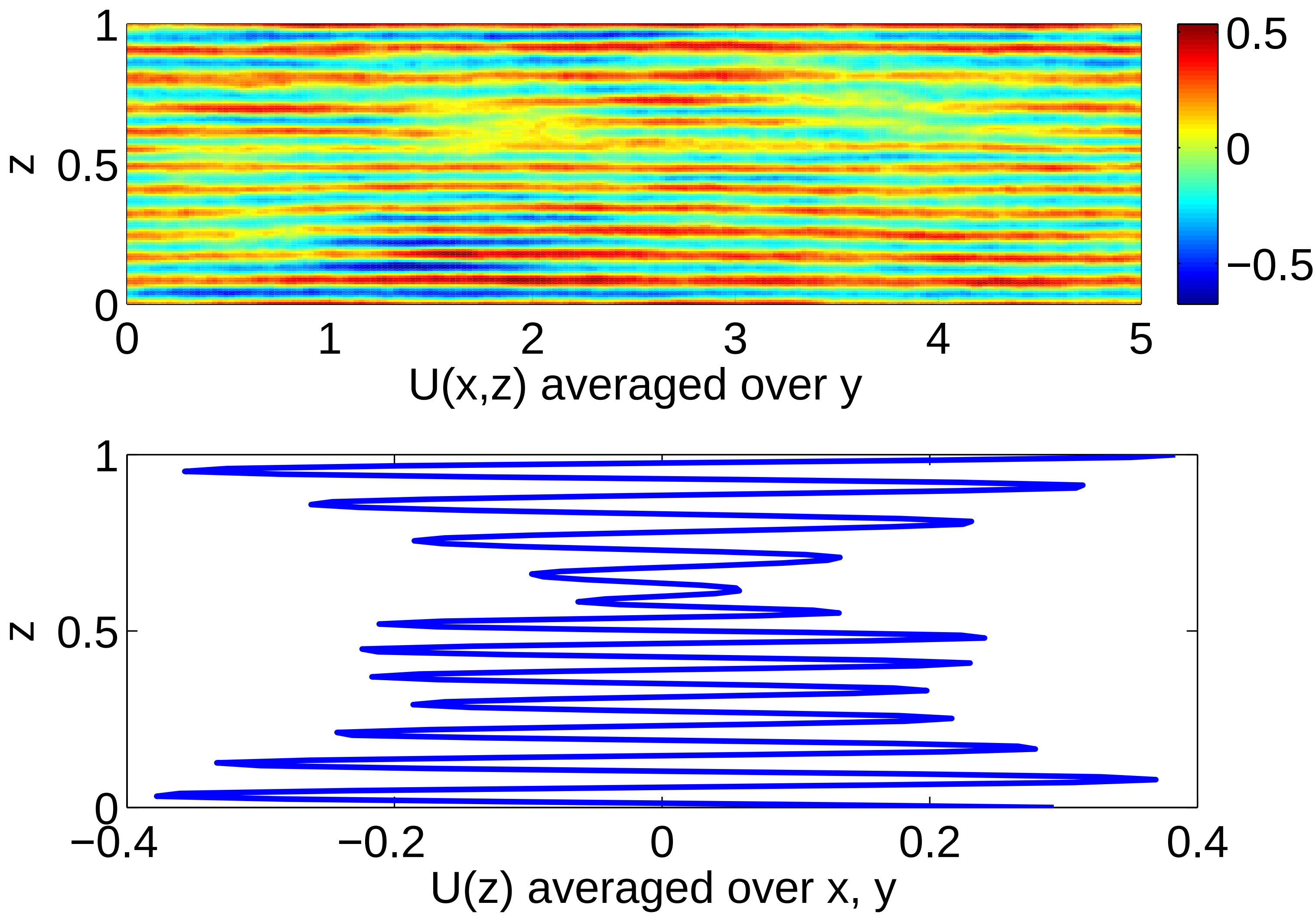
2DQG model

Full RSW

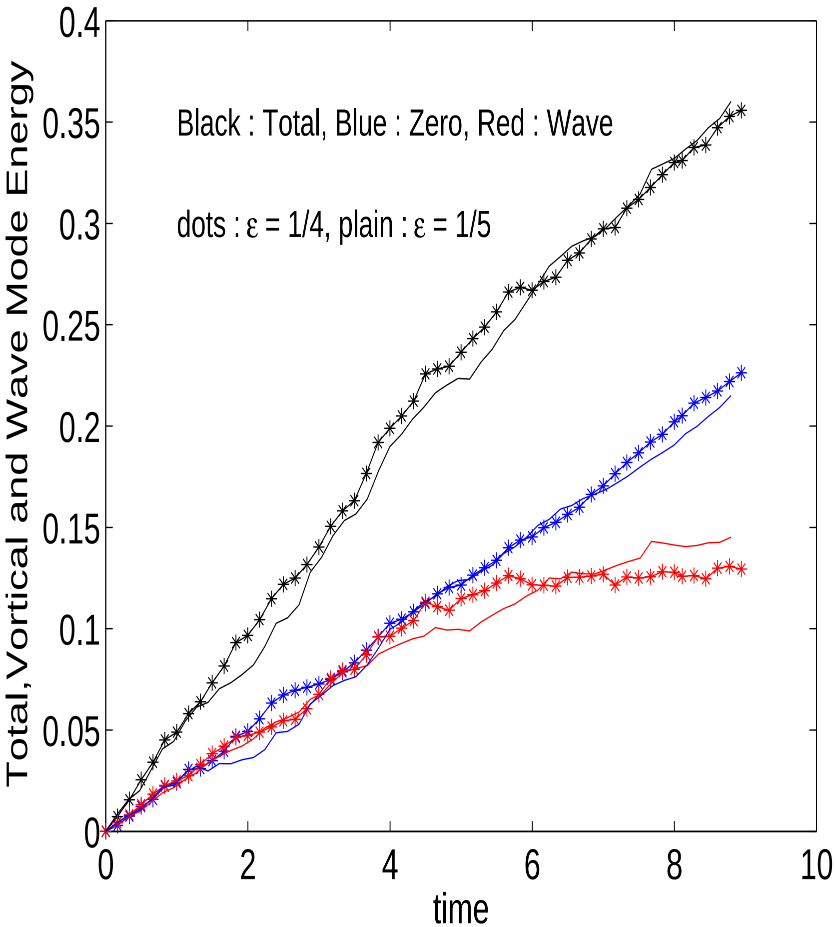
$$Ro = U(fL)^{-1} = 0.4,\ Fr = U(gH)^{-1/2} = 0.25$$

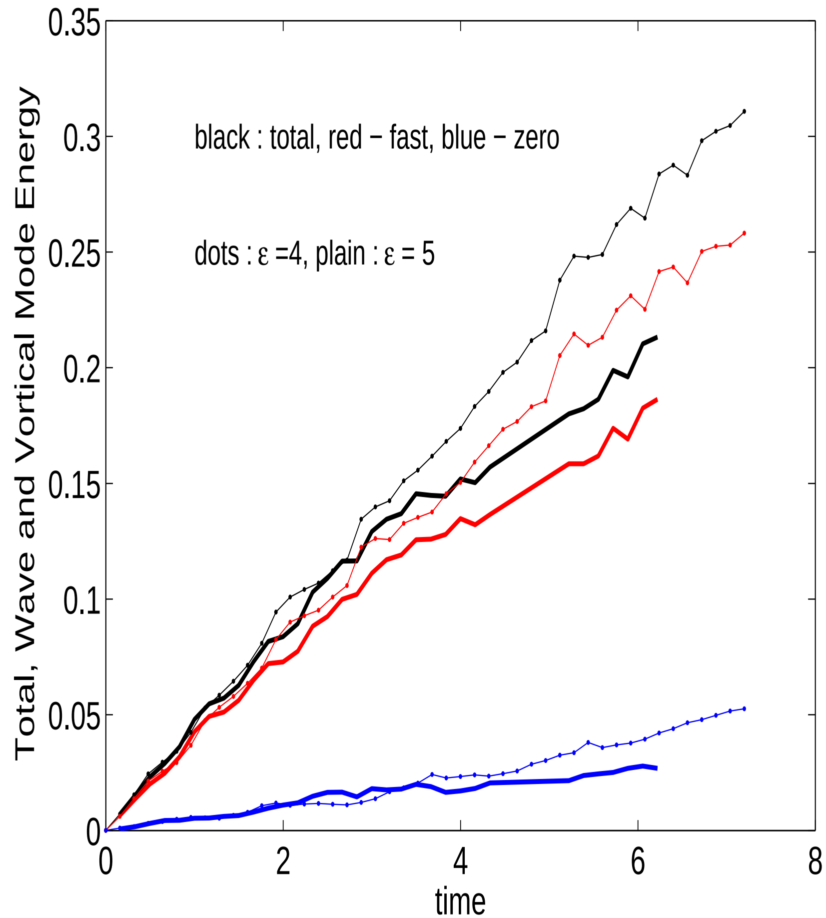
Example 2: Generation of VSHF; Bu = fL/(NH)=1; H/L=1/5

Ro = U/(fL) = 0.1; Fr = U/(NH) = 0.1



Example 3: Asymmetry with respect to $f/N = 1$ (QG-like)





f/N = 1/4, 1/5

f/N = 4, 5

Bartello 1995, Sukhatme & Smith 2008

Part III. PDE Intermediate Models

Intermediate models add more physics to QG

Improving upon 2DQG: Allen, Barth & Newberger 1990; Spall & McWilliams 1992; Yavneh & McWilliams 1994; Warn, Bokhove, Shepherd & Vallis 1995; Vallis 1996

Improving upon 3DQG: Allen 1991, 1993; Muraki, Snyder & Rotunno 1999; Muraki & Hakim 2001

Previous intermediate models are perturbative in nature

A hierarchy of NEW Intermediate Models

- Derived by adding subsets of wave-vortical mode interactions to QG
- Non-perturbative
- Include near-resonant triads
- Provides a framework for understanding the coupling between balanced and unbalanced flow components

  e.g. Kuo, Allen, Polvani 99; Ford, McIntyre, Norton 00; Majda 03

Symbolically

The Full Equations:

$$0 \quad | \; 00 \;\oplus\; 0+ \;\oplus\; 0- \;\oplus\; ++ \;\oplus\; +- \;\oplus\; -- \tag{1}$$

$$+ \quad | \; 00 \;\oplus\; 0+ \;\oplus\; 0- \;\oplus\; ++ \;\oplus\; +- \oplus\; -- \tag{2}$$

$$- \quad | \; 00 \;\oplus\; 0+ \;\oplus\; 0- \;\oplus\; ++ \;\oplus\; +- \oplus\; -- \tag{3}$$

QG (vortical mode interactions only):

$$0 \quad |00$$

Two NEW Models

PPG (add to QG interactions involving exactly 1 wave):

$$0 \quad | \; 00 \;\oplus\; 0+ \;\oplus\; 0- \qquad (ppg1)$$

$$+ \quad | \; 00 \qquad \text{and} \qquad - \quad | \; 00 \qquad (ppg2, 3)$$

P2G (add to PPG interactions involving exactly 2 waves):

$$0 \quad | \; 00 \;\oplus\; 0+ \;\oplus\; 0- \;\oplus\; ++ \;\oplus\; +- \;\oplus\; -- \qquad (p2g1)$$

$$+ \quad | \; 00 \;\oplus\; 0+ \;\oplus\; 0- \qquad (p2g2)$$

$$- \quad | \; 00 \;\oplus\; 0+ \;\oplus\; 0- \qquad (p2g3)$$

QG and PPG Rotating Shallow Water (RSW) Equations

QG: $\partial Q/\partial t + J(\Psi, Q) = 0$

$$\text{PPG}: \qquad \frac{\partial \nabla^2 \chi}{\partial t} - \nabla^2 V = 2J\left(\frac{\partial A}{\partial x}, \frac{\partial A}{\partial y}\right), \quad (1)$$

$$\frac{\partial Q}{\partial t} + J(\Psi, Q) + \nabla\chi \cdot \nabla Q + \langle u \rangle \frac{\partial Q}{\partial x} + \langle v \rangle \frac{\partial Q}{\partial y} + Q\nabla^2\chi = 0, \quad (2)$$

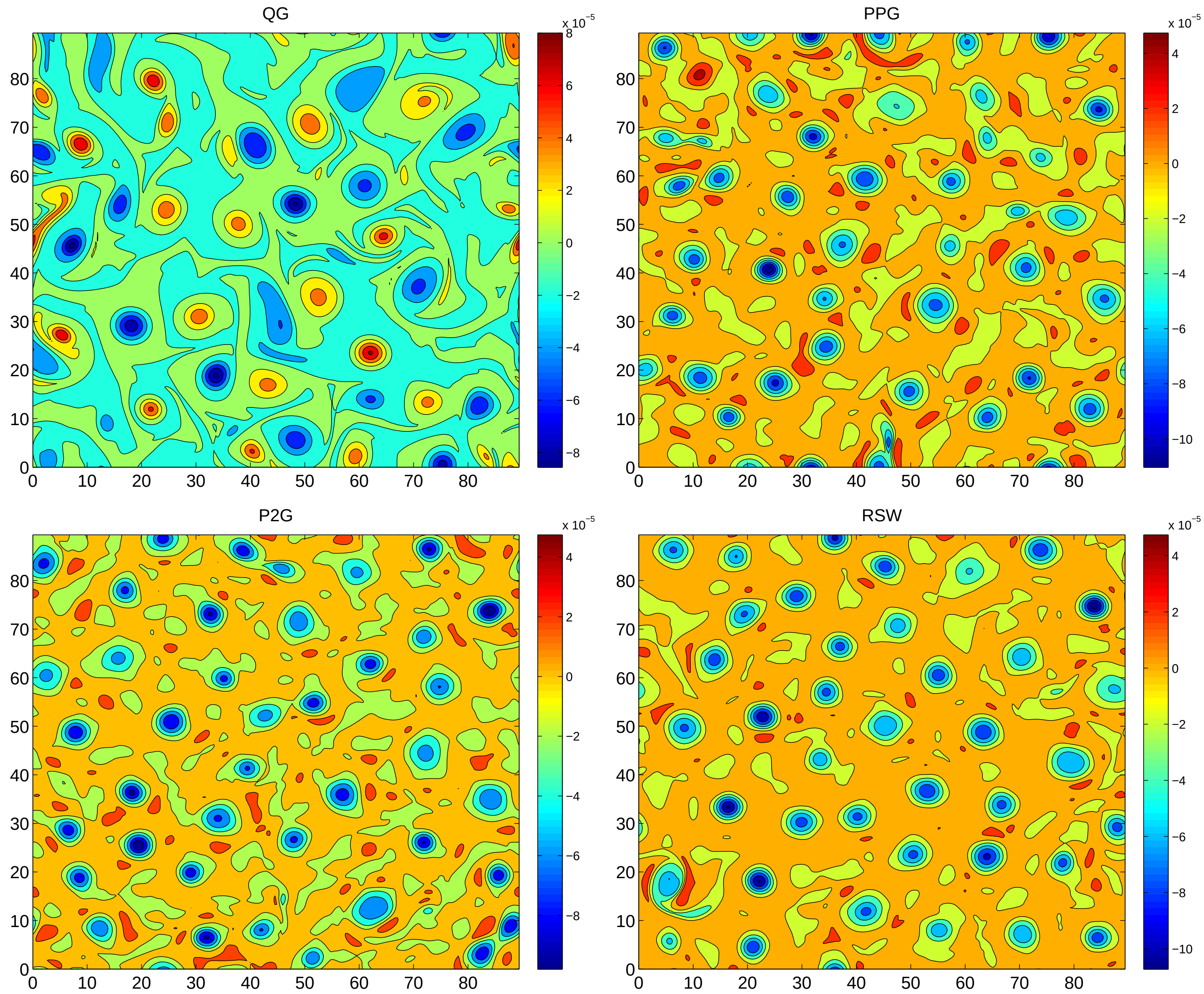
$$\frac{\partial \nabla^2 V}{\partial t} - c^2\nabla^4\chi + f^2\nabla^2\chi = fJ(A, Q) \quad (3)$$

$Q = (\nabla^2 - \frac{f^2}{gH})\Psi, \quad u = \chi_x - \Psi_y, \quad v = \chi_y + \Psi_y, \quad \nabla^2\chi = u_x + v_y$

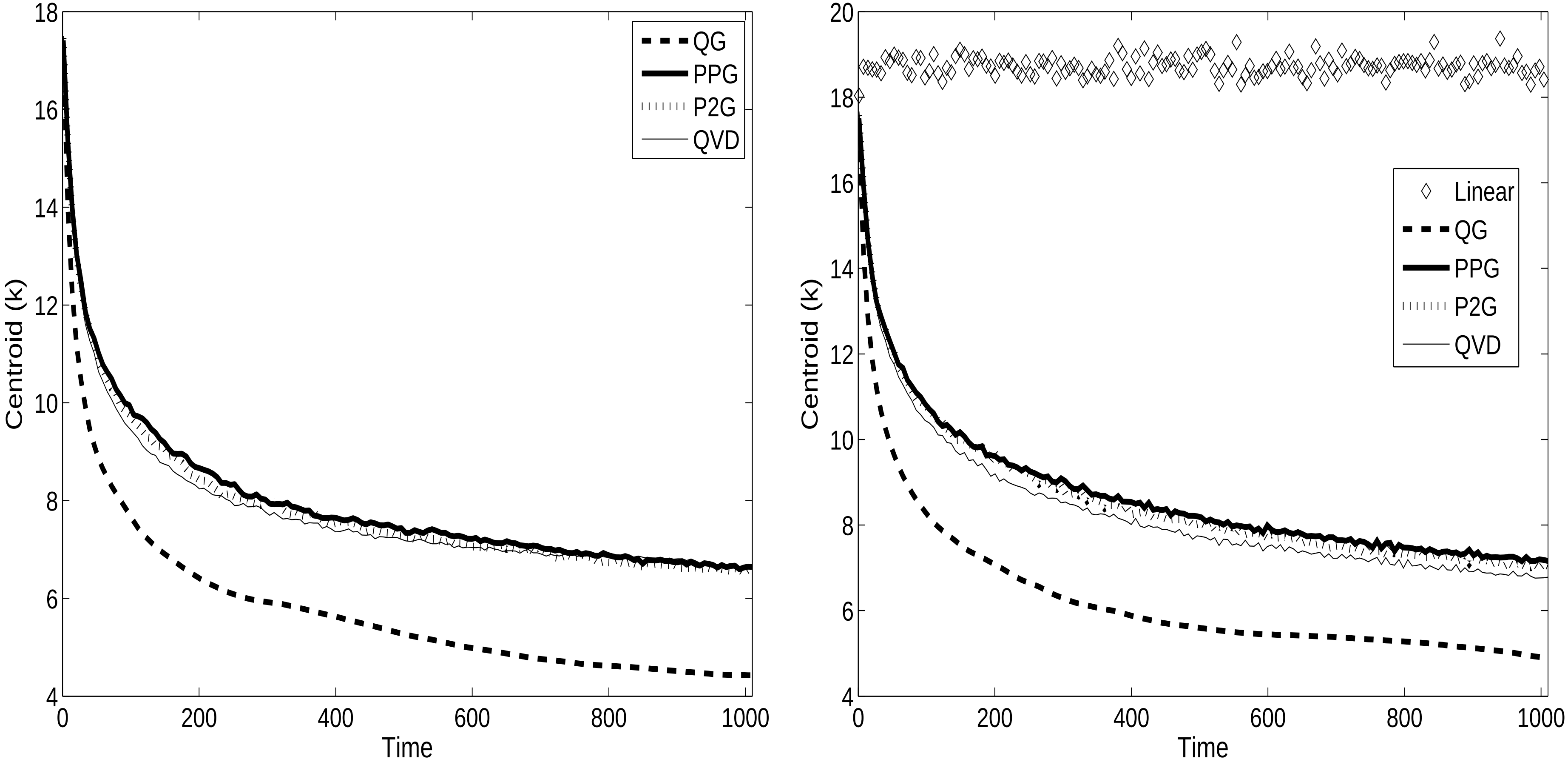
$\nabla^2 V = \nabla^2(f\Psi - gh)$ is a measure of geostrophic imbalance (Vallis 96); also called geostrophic departure (Warn 95); ageostrophic vorticity (Mohebalhojeh & Dritschel 01)

$A \equiv (f^2 - c^2\nabla^2)^{-1}c^2 Q$

RSW decay, Ro=0.4, Fr = 0.25, divergence-free unbalanced i.c.



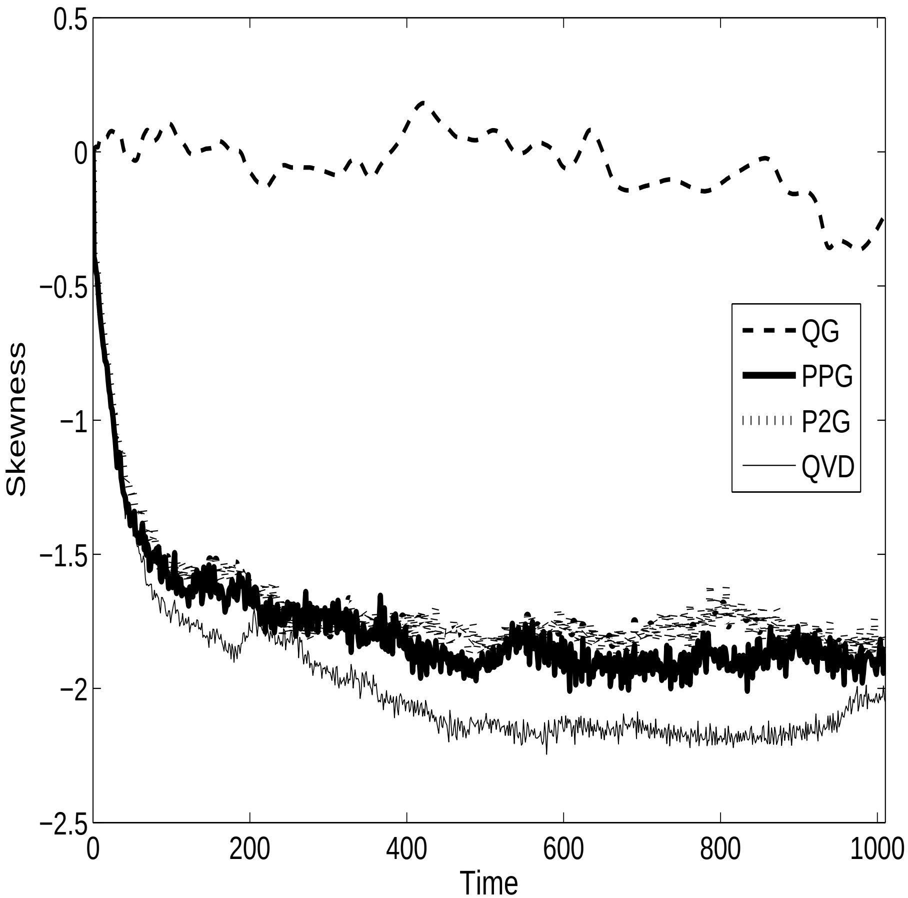
Centroid in RSW decay; divergence-free unbalanced i.c.

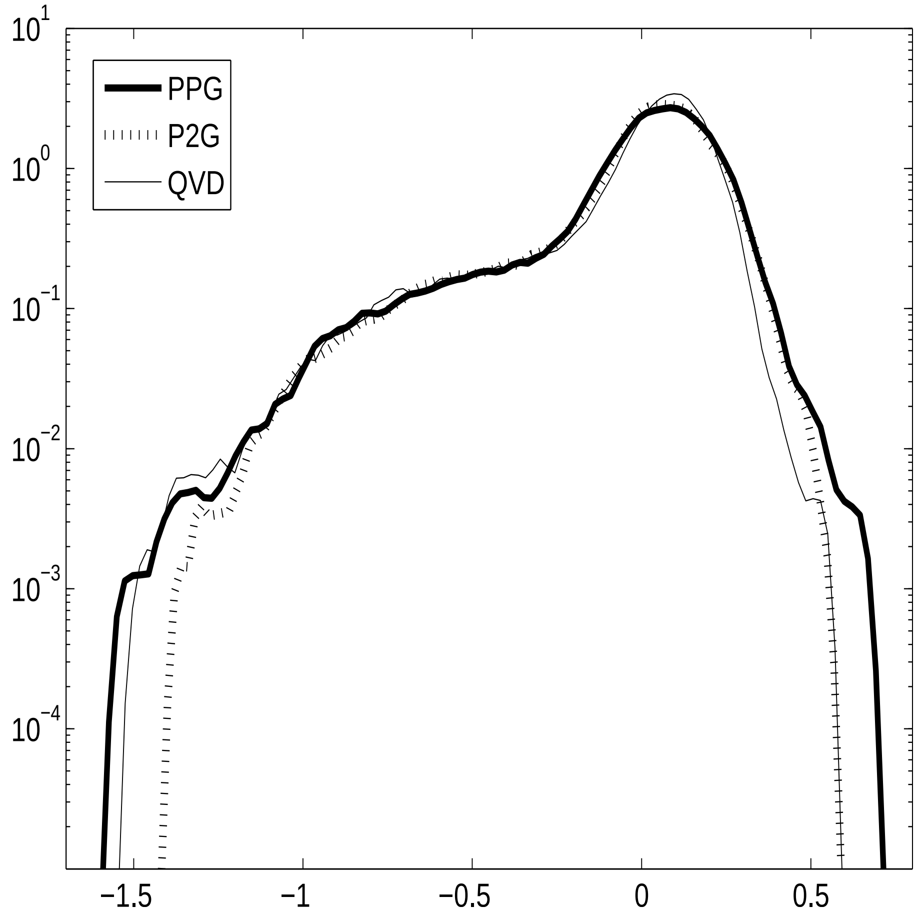


Ro = 0.4, Fr = 0.25 Ro = 0.25, Fr = 0.2

$$\text{Cent}(k) = (\sum_{\mathbf{k}} k(|u_{\mathbf{k}}|^2 + |v_{\mathbf{k}}|^2) / \sum_{\mathbf{k}}(|u_{\mathbf{k}}|^2 + |v_{\mathbf{k}}|^2)$$

Vorticity skewness in RSW decay; divergence-free unbalanced i.c.

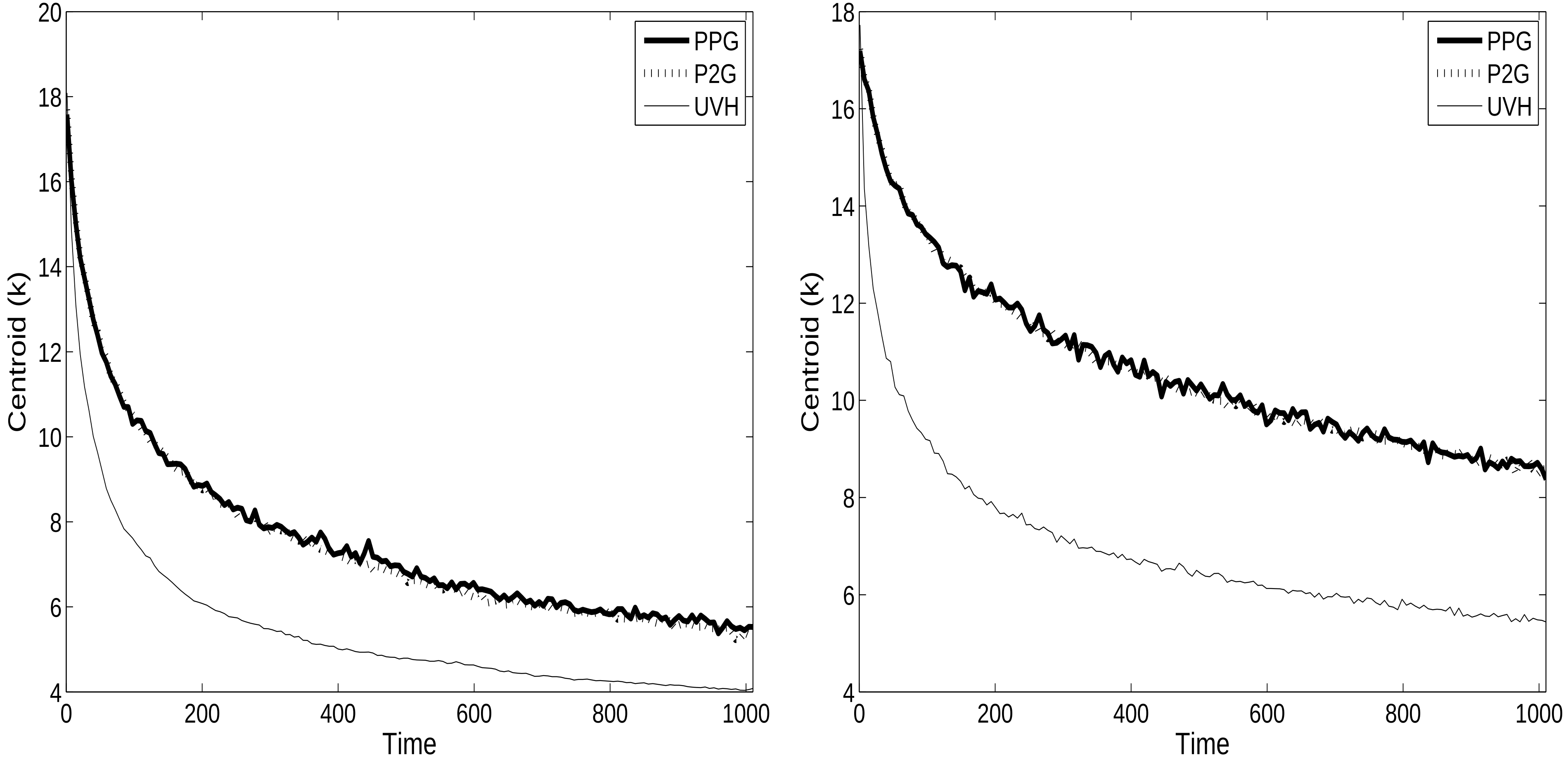




Ro = 0.4, Fr = 0.25

Centroid in RSW decay with divergent initial conditions

Ro = 1, Fr = 0.3

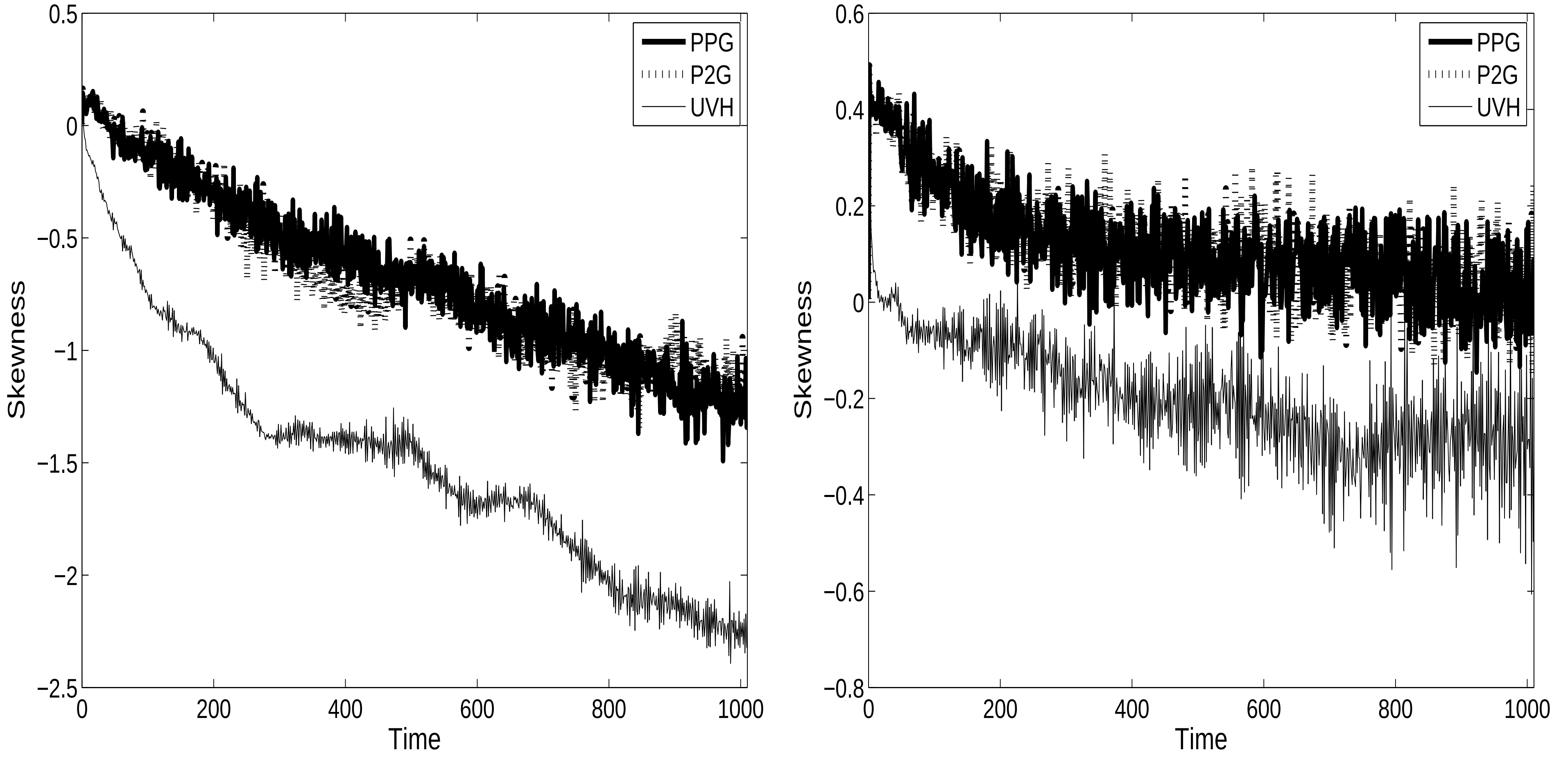


25% divergence-free

5% divergence-free

Skewness in RSW decay with divergent initial conditions

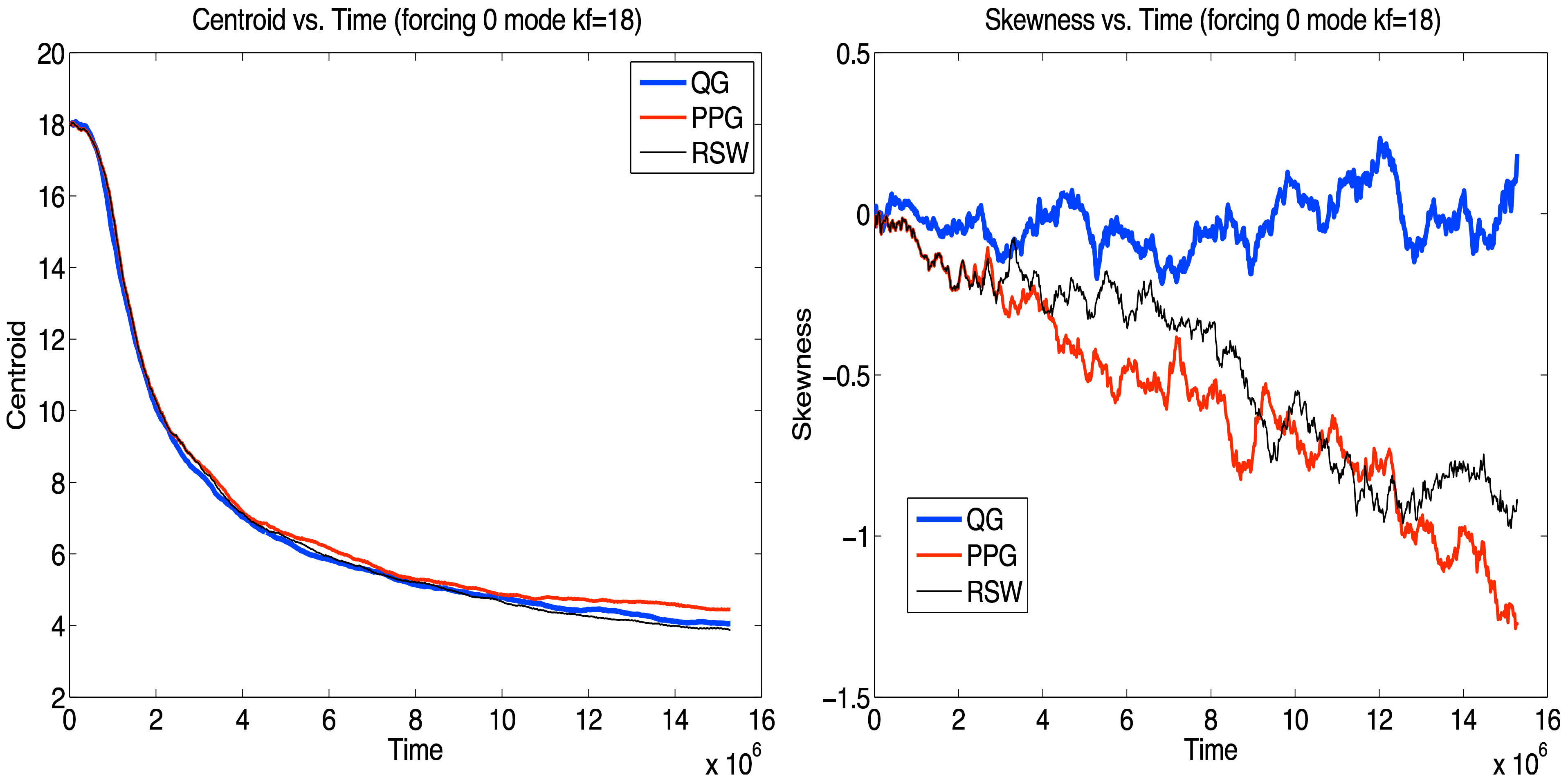
Ro = 1, Fr =0.3



25% divergence-free   5% divergence-free

Anticyclones in forced RSW (white-noise forcing of PV modes)

$$Ro = \varepsilon^{1/3}(k_f/\pi)^{2/3} f^{-1} = 0.05,\ Fr = (\varepsilon\pi)^{1/3} k_f^{-1/3} (gH)^{-1/2} = 0.03$$



Differences between RSW and RB

- Unlike for RB, truncations of RSW are not guaranteed to maintain energy as an integral invariant (Warn 86).
- RSW intermediate models with both energy conservation and Lagrangian invariance of PV do not perform as well as models conserving only PV (Allen 93).
- Model performance is not necessarily linked to conservation properties (Allen 93).

Differences between RSW and RB

- In RSW, two wave modes cannot transfer energy into or out of a vortical mode, i.e.

$$C^{0+-}_{\mathbf{kpq}} = C^{0++}_{\mathbf{kpq}} = C^{0--}_{\mathbf{kpq}} = 0$$

  (true only for exact resonances in RB)

- In RSW the catalytic resonances with $(s_{\mathbf{k}}, s_{\mathbf{p}}, s_{\mathbf{q}}) = (\pm, 0, \pm)$ only exchange energy in the same wavenumber shell

- In RSW, there are no exact 3-wave resonances (Majda 96)

Rotating Boussinesq Equations

- In RB, two wave modes can transfer energy into or out of a vortical mode, except for exact resonances.
- In RB the catalytic resonances with $(s_{\mathbf{k}}, s_{\mathbf{p}}, s_{\mathbf{q}}) = (\pm, 0, \pm)$ exchange energy between scales (on a cone in wavevector space)
- In RB, there are exact 3-wave resonances

Rotating Boussinesq Equations

- In RB, two wave modes can transfer energy into or out of a vortical mode, except for exact resonances.
- In RB the catalytic resonances with $(s_{\mathbf{k}}, s_{\mathbf{p}}, s_{\mathbf{q}}) = (\pm, 0, \pm)$ exchange energy between scales (on a cone in wavevector space)
- In RB, there are exact 3-wave resonances

Richer behavior in RB from interactions involving one vortical mode and two wave modes (including catalytic exact/near resonances), and from 3-wave exact/near resonances.

Testable prediction for RB

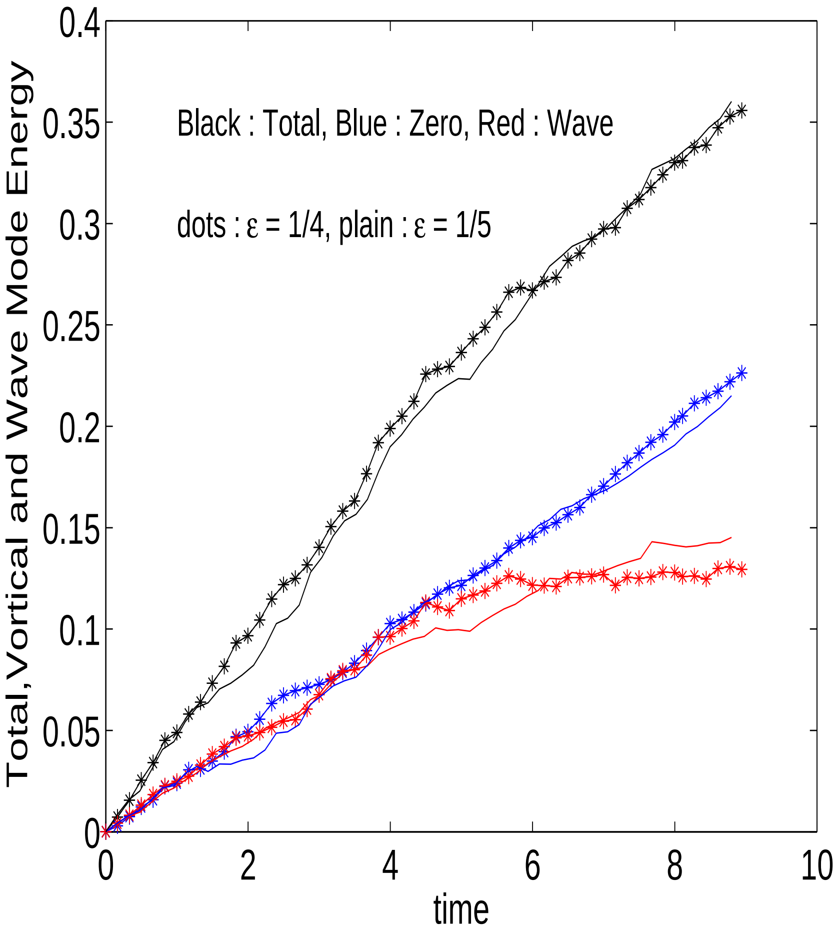
We predict the P2SG model can reproduce asymmetry with respect to f/N = 1 in the forward transfer range:

$$0 \quad | \; 00 \;\oplus\; ++ \;\oplus\; +- \;\oplus\; -- \tag{1}$$

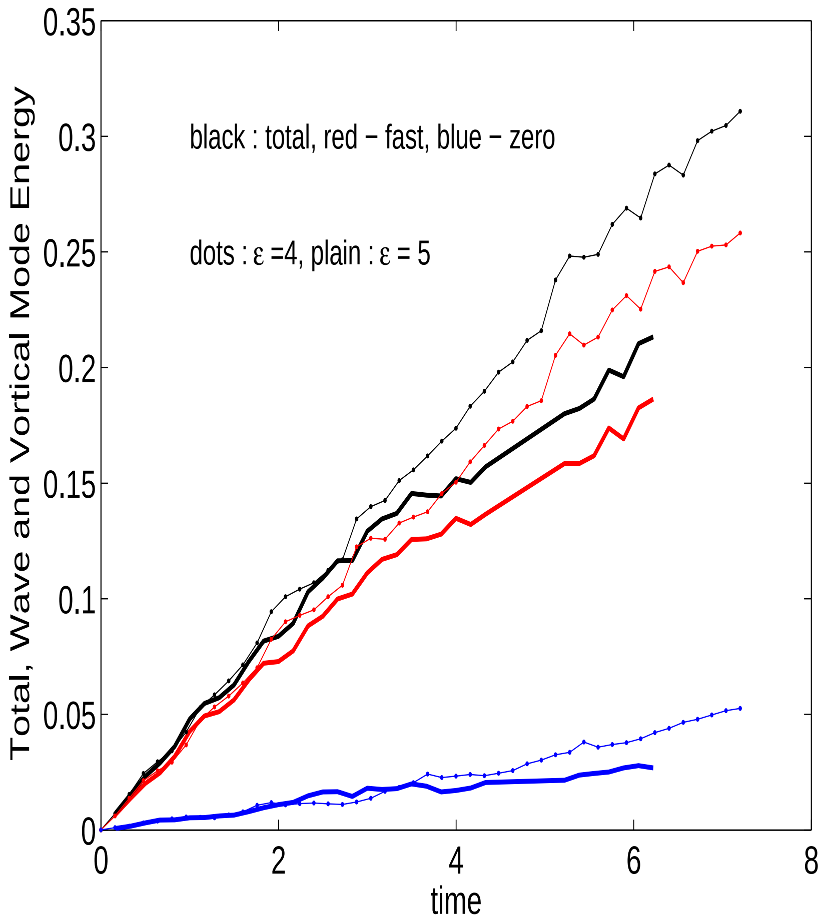
$$+ \quad |0+ \;\oplus\; 0- \tag{2}$$

$$- \quad |0+ \;\oplus\; 0- \tag{3}$$

Example 3: Asymmetry with respect to $f/N = 1$ (QG-like)



f/N = 1/4, 1/5



f/N = 4, 5

Conclusions/Discussion

- A nonperturbative method to derive a hierarchy of models including wave-vortical interactions
- Provides a framework for complete understanding of balanced and imbalanced flow components
- Interactions involving exactly one gravity wave responsible for anticyclone dominance in RSW
- Stay tuned for results on RB