Modeling Atmospheric Circulations with Soundproof Equations

Piotr K Smolarkiewicz,[*]
[*]National Center for Atmospheric Research, Boulder, Colorado, U.S.A.

- A lesson learned from "Predicting weather, climate and extreme events" (JCP, 2008, vol. 227): there is no set of equations uniformly adopted throughout the NWP community, and all operational models differ in some respect already at the theoretical level.

- In spite of the ongoing debate (on the preferred theoretical formulation), there seems to be a belief that anelastic (viz. soundproof) equations are inappropriate for predicting weather and climate.

- However, anelastic models evolve, progress, expand their predictive skill and range of expertise, and keep producing sound results.

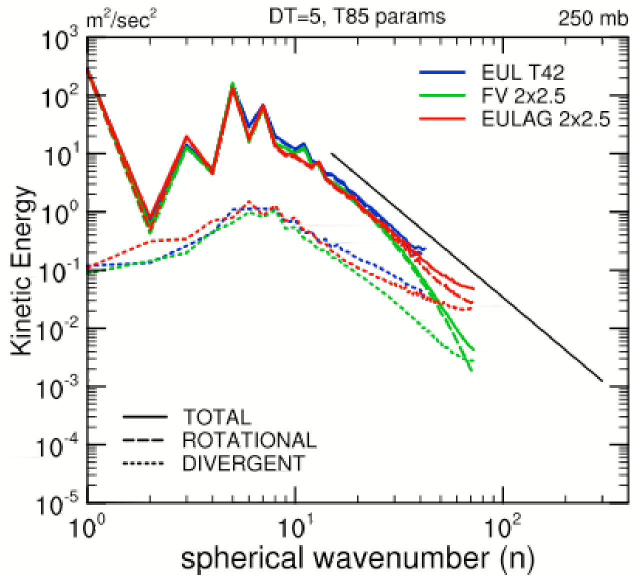


Figure 1: Aqua planet simulation with three different models, courtesy of Dave Williamson; see also Abiodun, Prusa & Gutowski, *Climate Dynamics* 2008

• The aim of this presentation is to draw attention to some recent developments and to some not-necessarily obvious aspects of soundproof models.

Conservative integrals of adiabatic Durran's equations†

• Soundproof equations — including classical incompressible Boussinesq model (Spiegel & Veronis, *Astrophys. J.*, 1960) — underlie the majority of research in low-Mach-number flows under gravity, such as atmospheres (planetary and stellar) and oceans.

• Apart from filtering out sound waves, anelastic models truncate baroclinic production of vorticity, in essence, to horizontal gradients of buoyancy; thus admitting thermally driven circulations only in vertical planes (viz., $\parallel$ g).

• The Durran (*J. Atmos. Sciences* 1989, *J. Fluid Mech.* 2008) sound-proof system is different, as it retains the unabbreviated (viz. non-Boussinesq) form of the momentum equation; thus separating baroclinicity from compressibility per se.

†After Smolarkiewicz & Dörnbrack, 2008 *Int. J. Numer. Meth. Fluids*, **56**, 1513-1519 $\Rightarrow$ *IJNMF08*.

• For decades, anelastic models were successful in advancing the understanding of geophysical and stellar flows; however, their range of validity is not yet fully understood — the underlying scale analysis is not a generally discriminating tool.

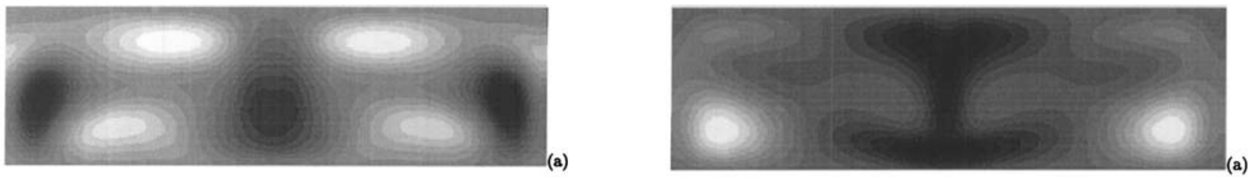


Figure 2: DNS of Rayleigh-Bénard convection in cryogenic gas; $Ra \sim 2 \cdot 10^4$.

• Arguable examples range from bifurcating Rayleigh-Bénard convection in cryogenic gas (Robinson & Chan, *Phys. Fluids* 2004) to "climate" of the solar convection zone (Robinson & Chan, *Month. Notes Roy. Astron. Soc.* 2001)

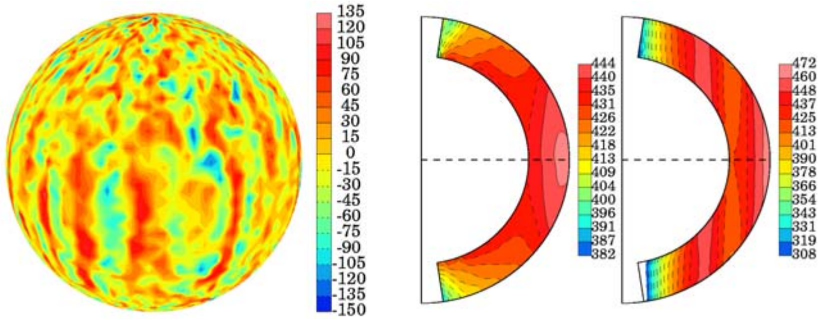


Figure 3: Solar convection. Radial velocity and two alternate solutions for angular velocity (DNS at $Re \approx 80$ versus LES; Elliot & Smolarkiewicz, *IJNMF* 2002)).

• In the literature, the disparities between fully compressible and anelastic simulations are often attributed to compressibility per se (viz. $\partial\rho'/\partial t$), thus repudiating anelastic models.

• The distinct vorticity dynamics of the Durran *pseudo-incompressible* system makes it a unique theoretical tool that complements both the standard anelastic and fully compressible models.

Durran vs. Lipps-Hemler (*JAS* 1982) sound-proof equations

$$\nabla \bullet (\rho^* \mathbf{v}) = 0 \;, \tag{1}$$
$$\frac{D\mathbf{v}}{Dt} = -\theta \nabla \pi' - \mathbf{g}\frac{\theta'}{\theta_e} - \mathbf{f} \times \mathbf{v}' \equiv -\nabla \phi' - \mathbf{g}\frac{\theta'}{\theta_e} + \phi' \nabla (\ln \theta) - \mathbf{f} \times \mathbf{v}'$$
$$\frac{D\theta'}{Dt} = -\mathbf{v} \bullet \nabla \theta_e$$

- The two (adiabatic) systems differ in a few items:
  i) $\rho^* = \rho_b \theta_b$ in (1), but $\rho^* = \rho_b$ in the anelastic system;
  ii) the term $\propto \nabla(\ln \theta)$ on the rhs of the momentum equation;
  iii) ambient $\theta_e(\mathbf{x})$ in the buoyancy denominator, in lieu of the base state $\theta_b(z)$;
  iv) $\mathbf{v}' \equiv \mathbf{v} - (\theta/\theta_e)\mathbf{v}_e$ in (1), but $\mathbf{v}' \equiv \mathbf{v} - \mathbf{v}_e$ in the anelastic system.

- The deeper the studied Earth's atmosphere or larger the stratification, the greater will be solution departures from the familiar behaviors of anelastic codes.

Conservative Integrals (http://www.mmm.ucar.edu/eulag)

- For each dependent variable ψ, a template algorithm

$$\psi_{\mathbf{i}}^{n+1} = LE_{\mathbf{i}}(\psi^{n} + 0.5\delta t R^{n}) + 0.5\delta t R_{\mathbf{i}}^{n+1} \;\equiv\; \widehat{\psi}_{\mathbf{i}} + 0.5\delta t R_{\mathbf{i}}^{n+1}; \tag{2}$$

gives $\mathcal{O}(\delta t^3)$ solution $\psi_{\mathbf{i}}^{n+1}$ in $(t^{n+1}, \mathbf{x}_{\mathbf{i}})$ grid points, using two-time-level flux-form non-oscillatory advection scheme MPDATA for LE transport operator.

- (2) is implicit for all dependent variables in (1). To retain this proven structure for the Durran system, (2) is executed as

$$\begin{aligned}
\theta'|_{\mathbf{i}}^{n+1,\nu} &= \widehat{\theta'}_{\mathbf{i}} - 0.5\delta t \left(\mathbf{v}^{n+1,\nu} \bullet \nabla\theta_e\right)_{\mathbf{i}} \\
\mathbf{v}_{\mathbf{i}}^{n+1,\nu} &= \hat{\mathbf{v}}_{\mathbf{i}} - 0.5\delta t \left[\theta^{n+1,\nu-1}\nabla\pi'|^{n+1,\nu} + \mathbf{g}\frac{\theta'|^{n+1,\nu}}{\theta_e} + \mathbf{f}\times\left(\mathbf{v}^{n+1,\nu} - \frac{\theta^{n+1,\nu-1}}{\theta_e}\mathbf{v}_e\right)\right]_{\mathbf{i}}
\end{aligned} \tag{3}$$

where $\nu = 1,..,m$ numbers the outer iterations, and at each iteration the implied linear elliptic problem is solved with preconditioned GCR algorithm. For $\theta^{n+1,0}$ we use the homogeneous solution (2) for $\psi \equiv \theta = \theta' + \theta_e$ and $R \equiv 0$.

Results

• For shallow ($\approx$ 10 km deep) mesoscale motions the differences between the compressible, anelastic and Durran's systems were shown insignificant compared to truncation errors of discrete integrals (Nance & Durran, *JAS* 1994).

• In *IJNMF08* we reported the differences immaterial even for deep (60 km) mesoscale atmospheres and shallow (10 km) planetary orographic flows.

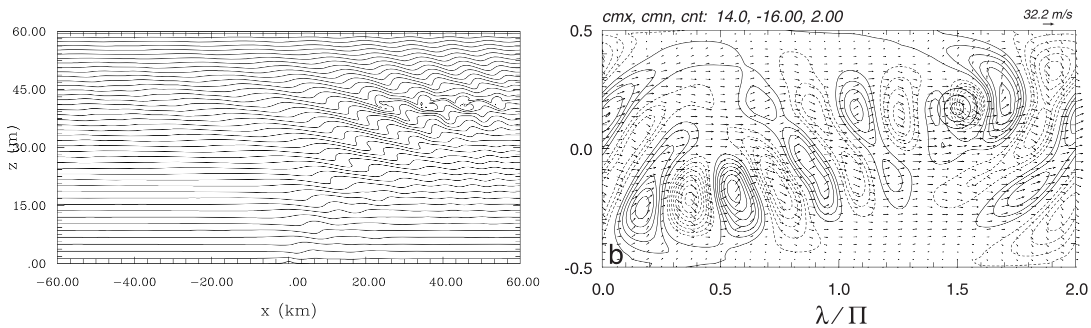
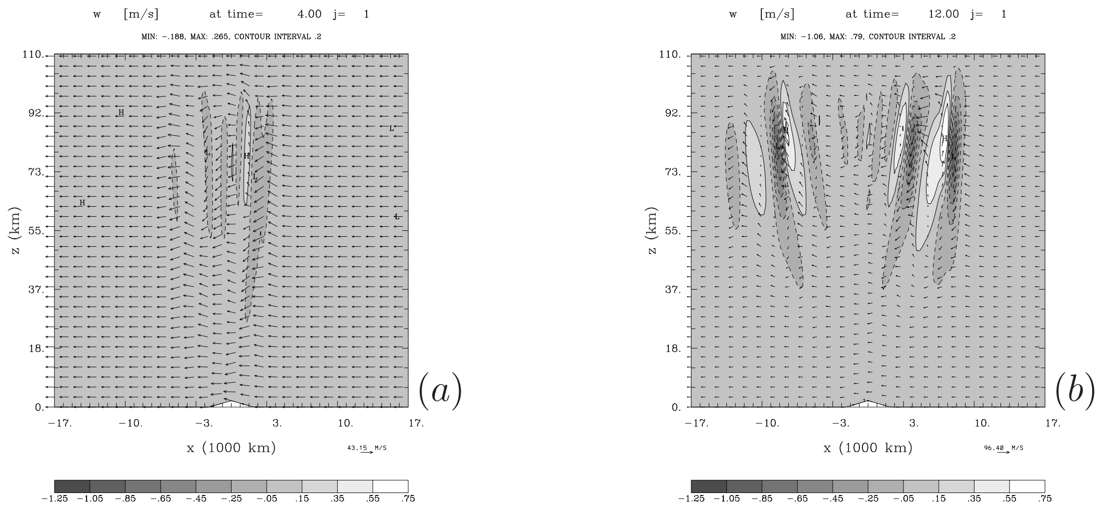
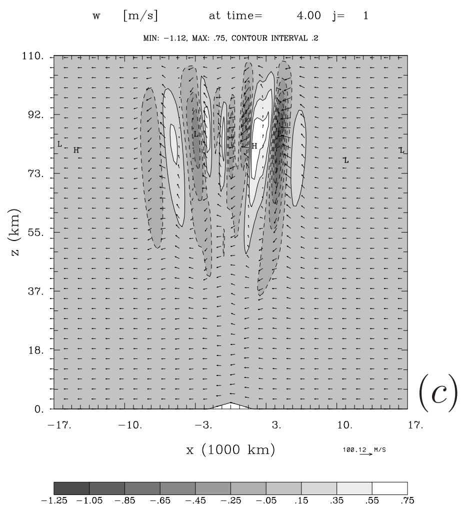


Figure 4: Meso- and planetary-scale orographic flows.

• This is consistent with the normal mode analysis of Davies et al. (*QJR* 2003), predicting substantial differences between the two soundproof systems (and fully compressible equations) only for deep planetary atmospheres $\Longrightarrow$





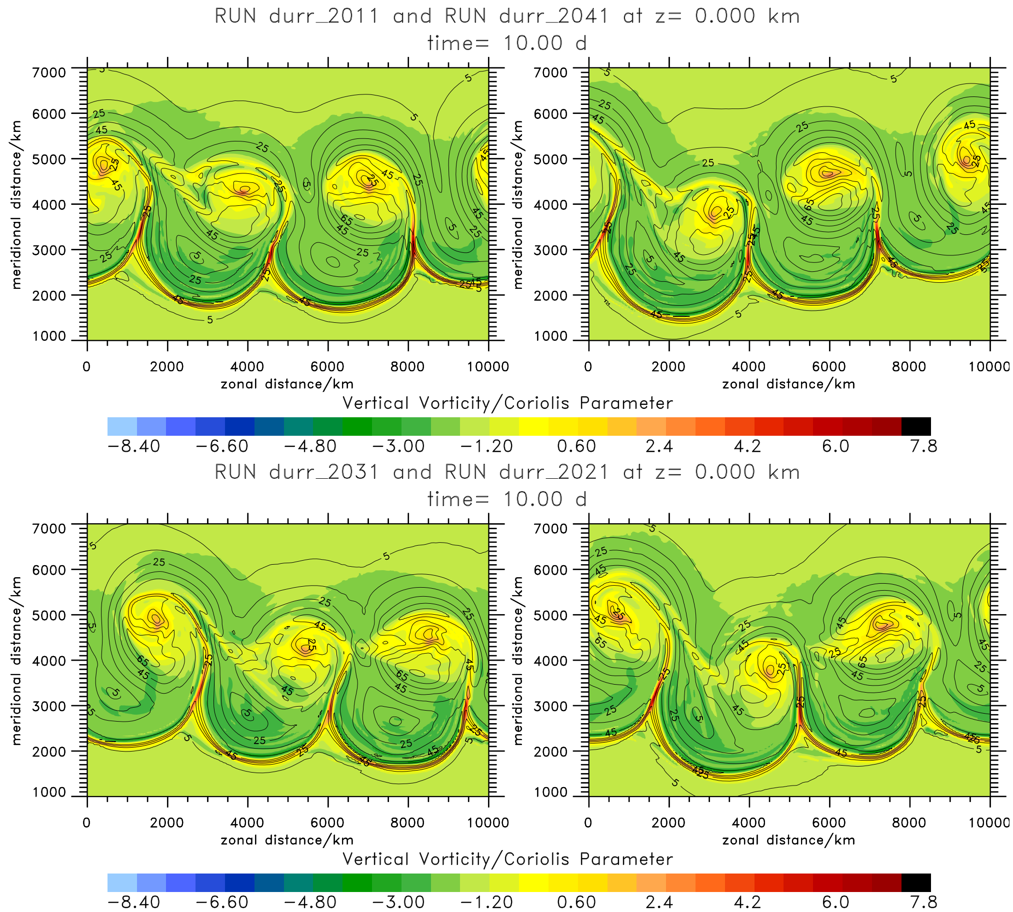
Vertical velocity for anelastic (a), (b) and Durran (c) solutions — idealized 2D deep planetary inertia-gravity wave at a mid latitude; (a) and (c) are after 4h, and (b) is after 12h starting from a potential orographic flow. Grey scale and contour intervals are the same in all panels, but wind vectors scale with maximal flow magnitude. Mountain height and width allude to the continent of North America.

Baroclinic lifcycle experiments

- Each system has slightly different thermal wind balance $\theta_e(y,z) \Rightarrow u_e(y,z)$

$$\begin{aligned} \mathrm{Durran} \quad \Rightarrow \quad 0 = & \quad -\theta_e \nabla(\pi_e - \pi_b) - \mathbf{g}(\theta_e - \theta_b)/\theta_b - \mathbf{f} \times \mathbf{v}_e \; , \\ \mathrm{Lipps - Hemler} \quad \Rightarrow \quad 0 = & \quad -\nabla(\phi_e - \phi_b) - \mathbf{g}(\theta_e - \theta_b)/\theta_b - \mathbf{f} \times \mathbf{v}_e \; , \end{aligned}$$

so it is difficult to assess aspects of equations rather than of initial conditions



Normalized surface vertical vorticity after 10 days; Lipps-Hemler anelastic solution (left) and Durran solution (right); Lipps-Hemler thermal-wind initialization (top) and Durran thermal-wind initialization (bottom).

Remarks

• For adiabatic dynamics the differences between consistent numerical solutions of the Durran and Lipps-Hemler anelastic equations appear immaterial for a broad range of problems, but become substantial for very deep planetary modes.[‡]

• Whether this conclusion holds for diabatic dynamics remains largely unknown — likely, because it adds substantial complexity to the resulting boundary value problem for pressure, due to diabatic source appearing in the mass-continuity equation:

$$\nabla \bullet (\rho^* \mathbf{v}) = \mathcal{H} \;\Rightarrow \tag{4}$$

$$\nabla \bullet (\rho^* \mathbf{v}^s) = 0, \quad \mathbf{v}^s \equiv \mathbf{v} - \frac{\nabla z}{\rho^*} \int_0^z \mathcal{H}\, d\xi\ .$$

Thus (4) defines a vertically adapting coordinate $d\,\overline{z} = \dfrac{dt}{\rho^*} \displaystyle\int_0^z \mathcal{H}\, d\xi$.

[‡]Noteworthy, very deep planetary inertia-gravity waves are misrepresented much more in the Durran system than in the Lipps-Hemler anelastic model (Davies et al 2003).

Comments on soundproof models

• It is important to distinguish between *anelastic system* of equations and *anelastic model* (code). An anelastic model can include a variety of optional soundproof systems, such as Ogura-Phillips, Lipps-Hemler, Bacmeister-Schoeberl, Durran as well as incompressible/compressible Boussinesq, incompressible Euler, Voigt (viscoelastic) and even fully compressible Euler equations.

• Soundproof models do not have to depend on horizontally homogeneous reference profiles $\Psi_b = \Psi_b(z)$. New formulation of the pseudo-incompressible model (Durran *J. Fluid Mech.* 2008) allows for $\Psi_b = \Psi_b(\mathbf{x}, t)$.

• The efficacy of soundproof models can be greatly enhanced by admitting an ambient state — a particular solution to a subset $\mathcal{F}_e(\Psi_e, \Psi_b) = 0$ of a governing set $\mathcal{F}(\Psi, \Psi_b) = 0$ — to formulate the perturbational form $\mathcal{F}'(\Psi, \Psi_b, \Psi_e) = 0$

• The geometry, but not the anelasticity per se, inhibits some physics. In a time dependent curvilinear framework, $\nabla \cdot \rho_b(z)\mathbf{v} = 0$ becomes $\partial_t \rho^* + \nabla \cdot \rho^* \mathbf{v}^* = 0$, with $\rho^* := G(\mathbf{x}, t)\rho_b$, thus facilitating a variety of boundary-condition models.

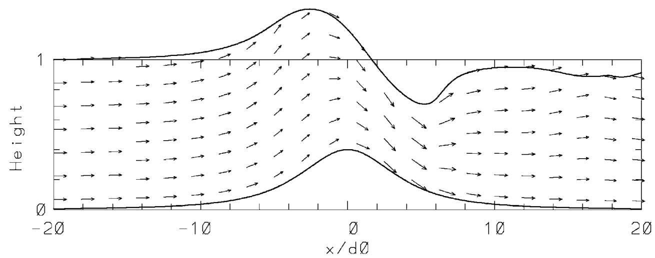


Figure 5: Flow past a sea mount (incompressible Euler eqs.); Wedi & Smolarkiewicz 2003 (JCP).

Conclusions

⋆ Davies et al (QJR, 2003) performed a thorough normal-mode analysis of various equation systems, and quantified their departures from fully compressible equations. Regardless of the conclusions derived, their results extend (upscale) the validity of anelastic equations for atmospheric circulations.

⋆ It can be difficult to design conclusive comparisons of compressible and soundproof equations. It is easy to fall in comparing numerics and/or model setups.

⋆ There is no hope for solving practical fluid equations analytically. Thus, the utility of a theoretical model depends on the quality of numerics. Incidentally, soundproof equations admit more flexibility (than fully compressible equations) in designing accurate large-time-step integration methods.

⋆ The research record of a single anelastic model (EULAG; Prusa et al. 2008 *Comuput. Fluids*) for a range of scales from planetary to microphysical, and for a

Figure 6: Geophysical turbulence; scales of motion $\mathcal{O}(10^7)$, $\mathcal{O}(10^4)$, and $\mathcal{O}(10^{-2})$ m.

range of applications from solar convection, via oceanic and atmospheric circulations, to laboratory and wind tunnel studies documents the versatility of sound-

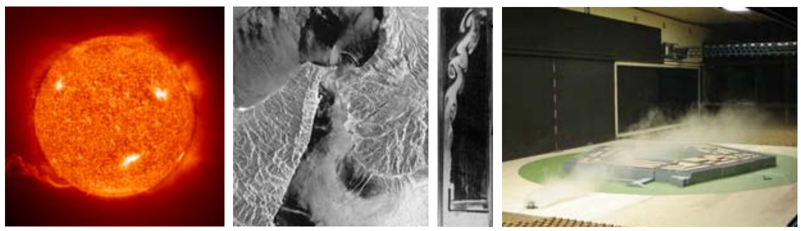


Figure 7: A range of applications

proof equations, and exhibits their potential for unified numerically-consistent Earth-system models.