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Majda - Expanded Class Notes
Fall Courant 2007

13 Appendix II. Non-dimensionalization of the Equations

Here, we briefly present the derivation of the non-dimensionalized equations. We start from the Boussinesq equations

$$\frac{D\mathbf{u}}{Dt} + f\mathbf{u}^{\perp} = -\nabla_h p + S_{\mathbf{u}}, \tag{98}$$

$$\frac{Dw}{Dt} = -\frac{\partial p}{\partial z} - g\frac{\rho'}{\rho_0}, \tag{99}$$

$$\frac{D\theta_t}{Dt} = S_\theta, \tag{100}$$

$$\nabla_h \cdot \mathbf{u} + w_z = 0, \tag{101}$$

with

$$\theta_t = \Theta_0 + \theta_{bg}(z) + \theta, \tag{102}$$

$$\Theta_0 = 300 \text{ K}, \tag{103}$$

$$\frac{d\theta_{bg}}{dz} = \frac{N^2\Theta_0}{g}, \tag{104}$$

$$\frac{\rho'}{\rho_0} = -\frac{\theta}{\Theta_0}, \tag{105}$$

and

$$N = -\frac{g}{\rho}\frac{d\rho}{dz} = 10^{-2}s^{-1}$$

is the Brunt-Väisälä buoyancy frequency. Thus the system (98)-(101) becomes

$$\frac{D\mathbf{u}}{Dt} + f\mathbf{u}^{\perp} = -\nabla_h p + S_{\mathbf{u}}, \tag{106}$$

$$\frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + g\frac{\theta}{\Theta_0}, \tag{107}$$

$$\frac{D\theta}{Dt} + w\frac{d\theta_{bg}}{dz} = S_\theta, \tag{108}$$

$$\nabla_h \cdot \mathbf{u} + w_z = 0, \tag{109}$$

Let

$$\mathbf{u} = U\widehat{\mathbf{u}} \tag{110}$$

$$w = W\widehat{w} \tag{111}$$

$$p = P\widehat{p} \tag{112}$$

$$\theta = \Theta\widehat{\theta} \tag{113}$$

$$\mathbf{x} = L\widehat{\mathbf{x}} \tag{114}$$

$$z = H\widehat{z} \tag{115}$$

$$t = T\widehat{t} \tag{116}$$

$$S_\theta = \Sigma_\theta \widehat{S}_\theta \tag{117}$$

$$S_\mathbf{u} = \Sigma_\mathbf{u} \widehat{S}_\mathbf{u} \tag{118}$$

with

$$L = H \tag{119}$$

$$U = W = \frac{L}{T} \tag{120}$$

$$\Theta = \frac{N\Theta_0 L}{gT} \tag{121}$$

$$\Sigma_\theta = \frac{\Theta}{T} \tag{122}$$

$$\Sigma_\mathbf{u} = \frac{L}{T^2} \tag{123}$$

$$P = \frac{WH}{T} = \frac{L^2}{T^2} \tag{124}$$

Note that in the above nondimensionalization, the horizontal and vertical scales and velocities are comparable and the unit of time is given by the advection time scale, $T = \frac{L}{U}$.

Define

$$\text{Fr} = \frac{U}{NL} = (NT)^{-1} \tag{125}$$

$$\text{Ro} = \frac{U}{fL} = (fT)^{-1} \tag{126}$$

Substituting (110)-(118) in (106)-(109) we obtain

$$\frac{L}{T^2}\frac{\widehat{D}\widehat{\mathbf{u}}}{D\widehat{t}} + \frac{L}{T} f\widehat{\mathbf{u}}^{\perp} = -\frac{1}{L}\widehat{\nabla}_h\left(\frac{L^2}{T^2}\widehat{p}\right) + \frac{L}{T^2}\widehat{S}_{\mathbf{u}}, \tag{127}$$

$$\frac{L}{T^2}\frac{\widehat{D}\widehat{w}}{D\widehat{t}} = \frac{1}{L}\frac{\partial}{\partial\widehat{z}}\left(\frac{L^2}{T^2}\widehat{p}\right) + g\frac{N\Theta_0 L}{gT}\frac{\widehat{\theta}}{\Theta_0}, \tag{128}$$

$$\frac{\Theta}{T}\frac{\widehat{D}\widehat{\theta}}{D\widehat{t}} + \left(\frac{L}{T}\widehat{w}\right)\frac{N^2\Theta_0}{g} = \frac{\Theta}{T}\widehat{S}_{\theta}, \tag{129}$$

$$\frac{L}{T^2}\left(\widehat{\nabla}\cdot\widehat{\mathbf{u}} + \widehat{w}_{\widehat{z}}\right) = 0, \tag{130}$$

or

$$\frac{\widehat{D}\widehat{\mathbf{u}}}{D\widehat{t}} + fT\widehat{\mathbf{u}}^{\perp} = -\widehat{\nabla}_h\widehat{p} + \widehat{S}_{\mathbf{u}}, \tag{131}$$

$$\frac{\widehat{D}\widehat{w}}{D\widehat{t}} = -\frac{\partial\widehat{p}}{\partial\widehat{z}} + NT\widehat{\theta}, \tag{132}$$

$$\frac{\widehat{D}\widehat{\theta}}{D\widehat{t}} + \frac{L}{\Theta}\frac{N^2\Theta_0}{g}\widehat{w} = \widehat{S}_{\theta}, \tag{133}$$

$$\widehat{\nabla}\cdot\widehat{\mathbf{u}} + \widehat{w}_{\widehat{z}} = 0. \tag{134}$$

Note that $NT = \mathrm{Fr}^{-1}$ in (132). Moreover, the coefficient of $\widehat{w}$ in (133) becomes

$$\frac{L}{\Theta}\frac{N_0^2\Theta}{g} = \frac{NT}{\Theta}\frac{N\Theta_0 L}{gT} = \frac{NT}{\Theta}\Theta = NT = \mathrm{Fr}^{-1}. \tag{135}$$

Dropping the "hat" in (131)-(134), one obtains the non-dimensional system

$$\frac{D\mathbf{u}}{Dt} + \mathrm{Ro}^{-1}\mathbf{u}^{\perp} = -\nabla_h p + S_{\mathbf{u}} \tag{136}$$

$$\frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \mathrm{Fr}^{-1}\theta \tag{137}$$

$$\frac{D\theta}{Dt} + \mathrm{Fr}^{-1}w = S_{\theta} \tag{138}$$

$$\nabla_h\cdot\mathbf{u} + w_z = 0 \tag{139}$$

We now fix the reference magnitudes as following

$$\begin{aligned}
L &= H = 10km = 10^4 m && (140)\\
T &= 15\,\text{min} = 900\,\text{s} && (141)\\
\Theta &= \frac{N\Theta_0 L}{gT} = \frac{0.01\times 300\times 10^4}{10\times 900} \simeq 3\ \text{K} && (142)\\
\Sigma_\theta &= \frac{\Theta}{T} = \frac{3\ \text{K}}{15\,\text{min}} && (143)\\
\Sigma_{\mathbf{u}} &= \frac{L}{T^2} && (144)\\
P &= \frac{WH}{T} = \frac{L^2}{T^2} && (145)
\end{aligned}$$

Note that the above reference magnitudes correspond to small Froude number

$$\text{Fr}=\frac{U}{NL} = (NT)^{-1} = \frac{1}{0.01\times 900} \simeq 0.1 = \epsilon, \qquad (146)$$

with $\epsilon = 0.1$. Further,

$$\text{Ro}^{-1} = fT = 2\Omega \sin\phi_0 T = 2\left(\frac{2\pi}{24\times 3600s}\right)(\sin\phi_0)(15\times 60s) \approx \epsilon \sin\phi_0, \qquad (147)$$

where $\Omega = \frac{2\pi}{24\times 3600\ s}$ is the Earth rotation frequency. The approximate magnitude of the heating rate observed in the "hot towers" is $120\frac{{}^\circ K}{\text{hour}}$ or $\frac{30^\circ K}{15\ \text{minutes}}$ which occur on scales of order 10 kilometers through moist deep convection in the hurricane embryo (Hendricks et al. 2004; Montgomery et al. 2006). Such a heating rate $\frac{30^\circ K}{15\ \text{minutes}}$ is strong compared with the reference magnitude $\Sigma_\theta = \frac{3\ \text{K}}{15\,\text{min}}$. Hence, in this paper, S_θ is given the magnitude ϵ^{-1}, i.e. $S_\theta = \epsilon^{-1} S_\theta^*$, and the star is dropped for simplicity. Thus, we obtain

$$\begin{aligned}
\frac{D\mathbf{u}}{Dt} + \epsilon \sin\phi_0 \mathbf{u}^\perp &= -\nabla_h p + S_{\mathbf{u}} && (148)\\
\frac{Dw}{Dt} &= -\frac{\partial p}{\partial z} + \epsilon^{-1}\theta && (149)\\
\frac{D\theta}{Dt} &= -\epsilon^{-1}(w + S_\theta) && (150)\\
\nabla_h \cdot \mathbf{u} + w_z &= 0 && (151)
\end{aligned}$$

According the nondimensionalization procedure and the reference magnitude employed here, the above system corresponds to the following situation:

- Weak temperature gradient (WTG); $\Theta = \overline{\Theta}(z) + \epsilon\theta(\mathbf{x}_h, z, t)$
- Small Froude number ($\text{F}_{\text{R}} = \frac{U}{NL} = \epsilon$),
- Isotropic scale $L = H$, where L and H are horizontal and vertical spatial scale, respectively,
- Comparable horizontal and vertical velocity magnitudes $U = W$,
- Strong heat sources.