Theoretical Studies of Strongly Nonlinear Langmuir Circulation

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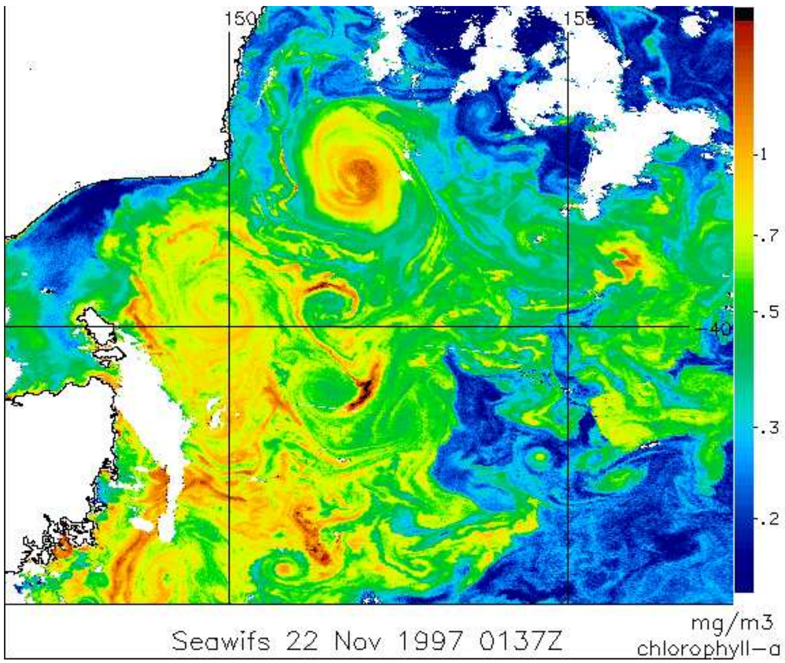
Ocean Surface Mixed Layer

- Crucial small-scale process: sub-mesoscale dynamics of ocean surface boundary (or "mixed") layer – the O(100)-m deep ocean surface layer which connects atmosphere to the deep ocean.

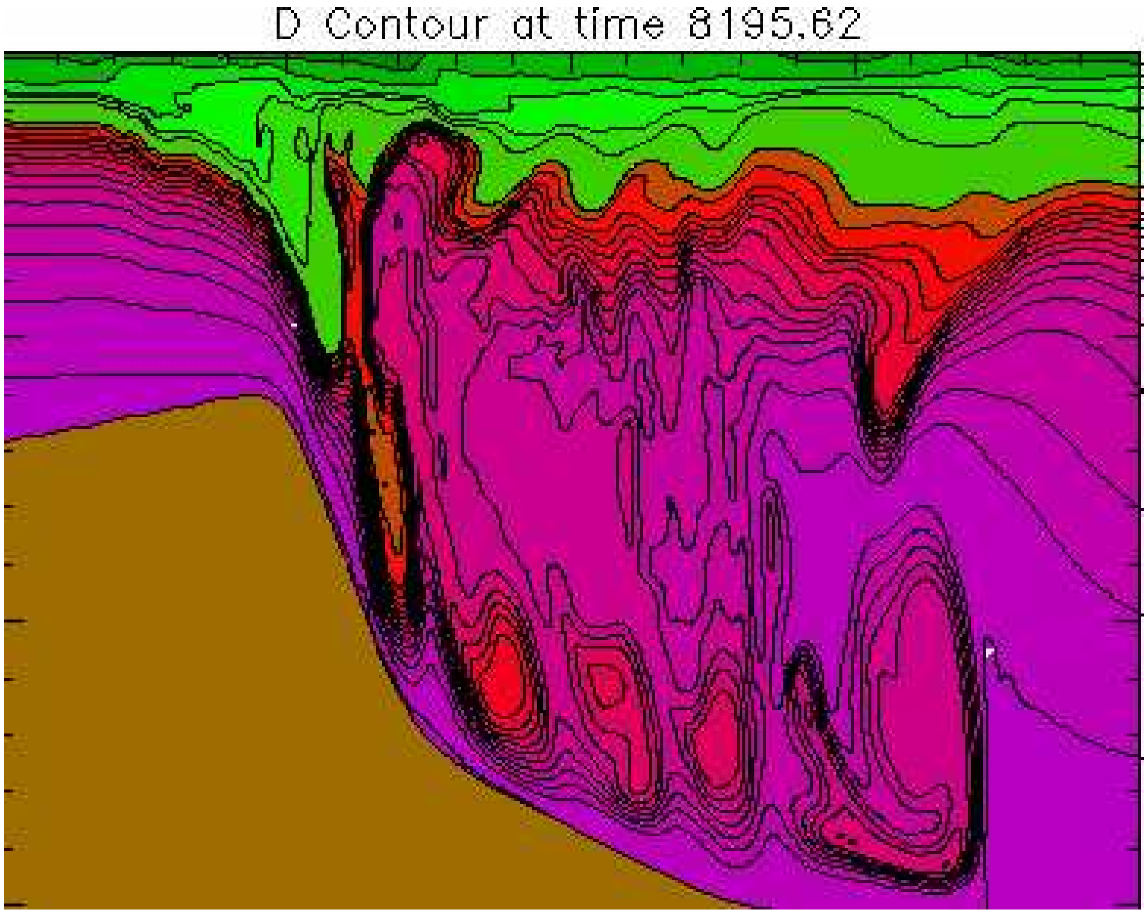
- According to the NSF report *Ocean Sciences at the New Millennium* (2001), **"Forecasting the evolution, in space and time, of the mixed layer is perhaps the biggest individual challenge in ocean prediction."**

- Key challenges include: (1) estimating mixing and transport across the mixed layer; (2) understanding interaction between the mixed layer and stratified ocean interior; (3) connecting mixed-layer/mesoscale dynamics.

Mesoscale Eddies & Spiral Vortices (NASA)

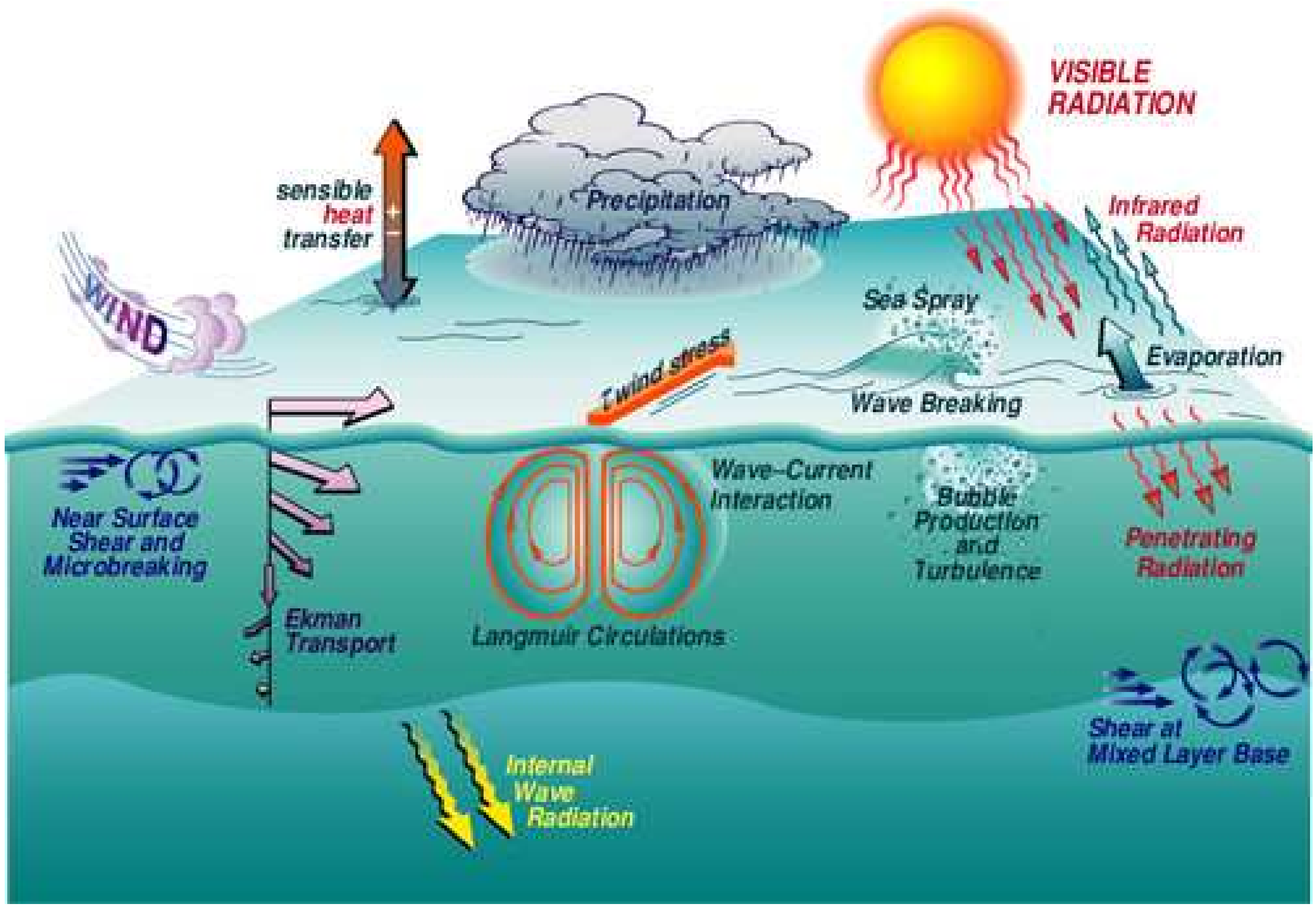


Submesoscale Phenomena – Vertical Transport/Mixing



- Image from numerical simulation by K. Lamb (U. Waterloo).

Schematic of Mixed Layer Processes (B. Weller, WHOI)



Langmuir Circulation (LC)

- Wind- and surface-wave driven convective motion (array of wind-aligned counter-rotating vortices) commonly occurring in natural water bodies.

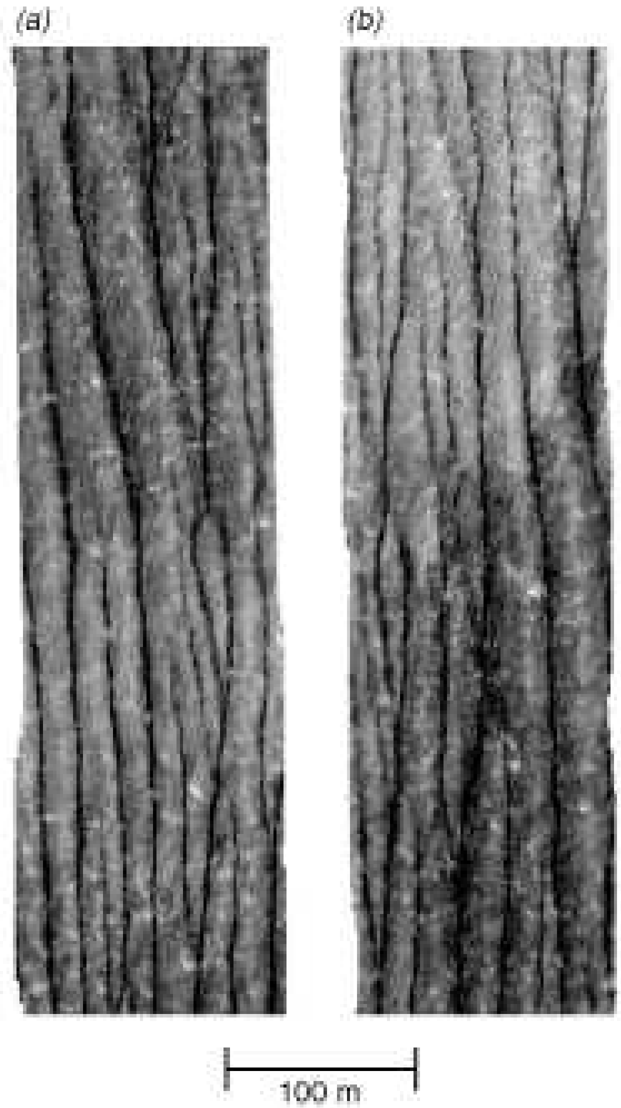
- Range from $\mathcal{O}(10\,\text{cm})$ to $\mathcal{O}(500\,\text{m})$, with $\mathcal{O}(\text{cm/s})$ velocities.

- Crucial for establishment and maintenance of oceanic mixed layer.

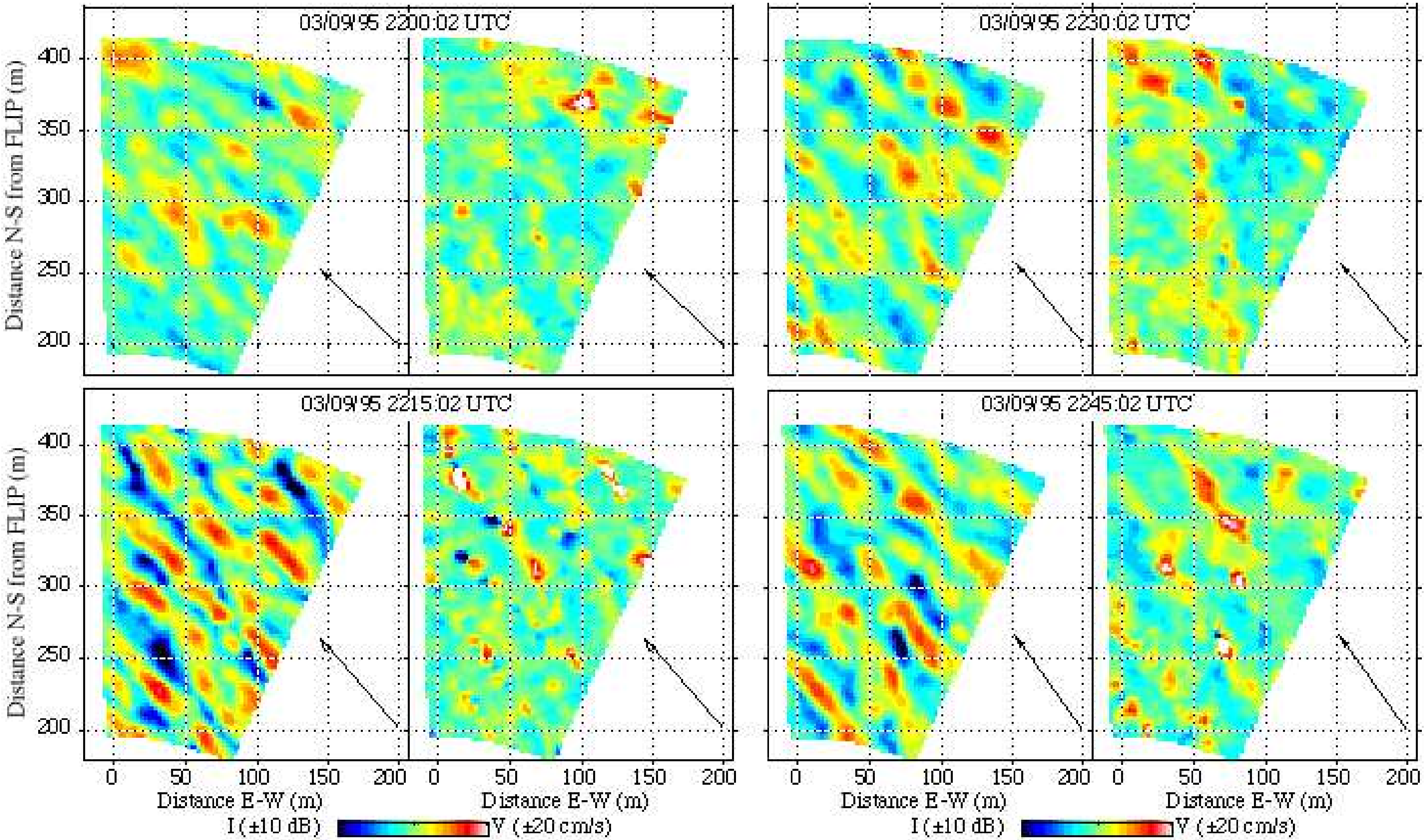
- Moderates air–sea exchanges of heat, mass and momentum, ultimately influencing weather, climate, ocean biology, pollutant (oil-spill) dispersal.

Photo of LC Windrows (A. Szeri)

IR Images of LC Windrows (G. Marmorino, NRL)



Sonar Images of LC Windrows (J. Smith, SIO)



LC Banding – Implications for Oil Spills

Craik-Leibovich (CL) Theory

Assumption

Dominant upper ocean motion due to **irrotational surface waves**: O(m/s)

Wind-driven shear flow, LC, other **rotational motions** weaker: O(cm/s)

Multiple-Time Scale Analysis $\boxed{t = \varepsilon^2 t_f}$

▷ Expansion: $\boxed{\mathbf{v}(x,y,z,t,t_f) = \varepsilon\,\mathbf{u}_w(x,y,z,t_f) + \varepsilon^2\,\mathbf{u}(x,y,z,t,t_f)}$

▷ Substitute into Navier-Stokes equations (NSE).

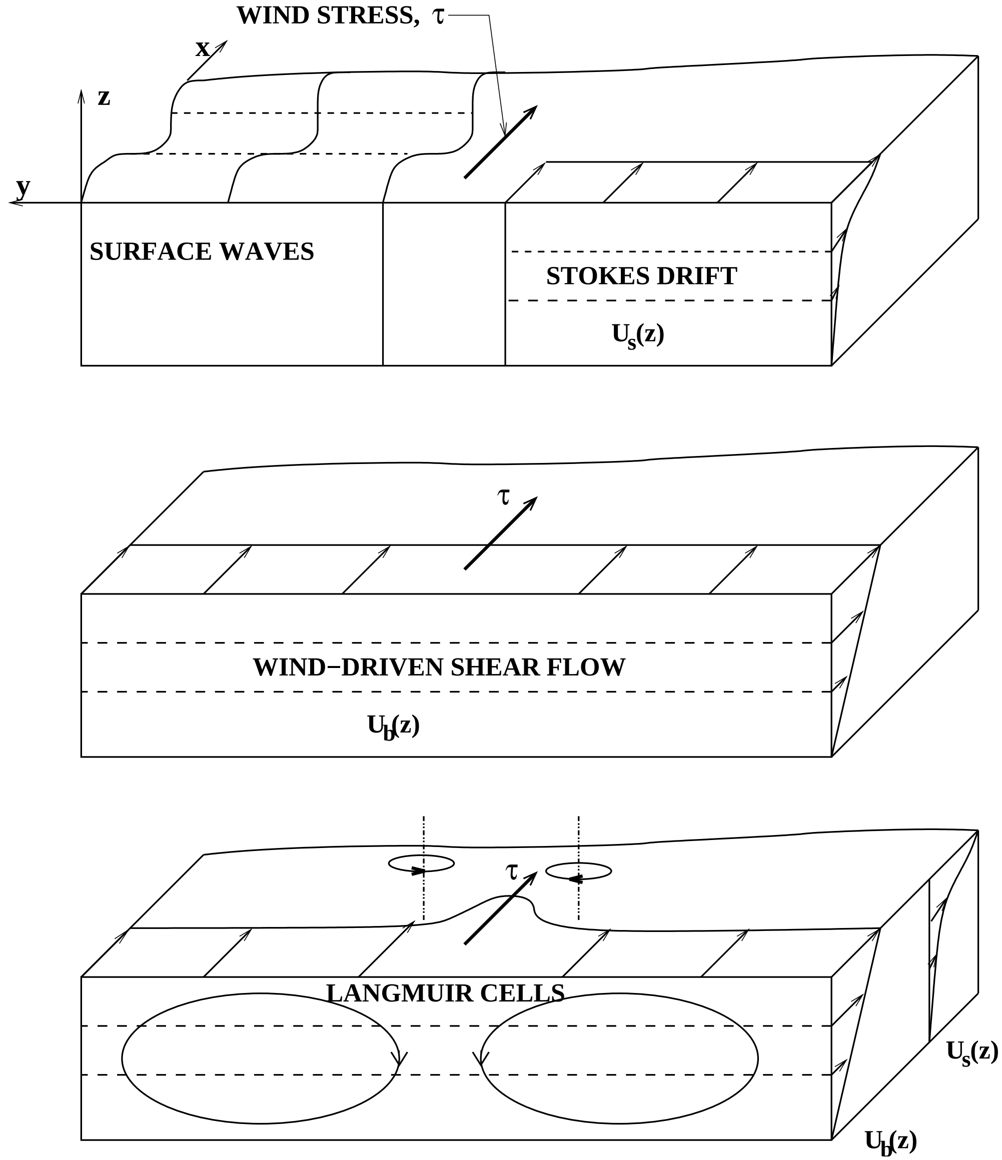
▷ Average over t_f, fast time scale associated with surface-wave period.

▷ Obtain equations formally identical to NSE... except for **vortex force**:

$$\boxed{\mathbf{u}_s \times \boldsymbol{\omega}(x,y,z,t)}$$

CL eqns show that LC arises as an **instability** of a wind-driven shear flow on which surface waves propagate.

LC Instability Mechanism $(\mathbf{u_s} \times \omega)$



Robustness of CL Theory

Following the original derivation using formal multiple scale analysis, the CL eqns have been re-derived using more sophisticated mathematical approaches:

1. Generalized Lagrangian Mean (GLM) theory – an exact theory of nonlinear wave/mean-flow interactions.

2. A time-averaged version of the Kelvin circulation theorem.

3. Lagrangian averaging of Hamilton’s principle.

Open Issues

1. $O(1)$ shear – feedback of LC on waves, rotational waves.

2. Finite-amplitude waves, wave-breaking.

3. LC wavelength selection mechanism: connection to surface waves?

4. Quantitative comparisons with field or laboratory data.

Related Work – Theory and Simulation

Quasi-Laminar Simulations of 2D Craik–Leibovich (CL) Equations

- Li & Garrett (*J. Mar. Res.* 1993, *JPO* 1995,1997)
- Gnanadesikan & Weller (*JPO* 1995)

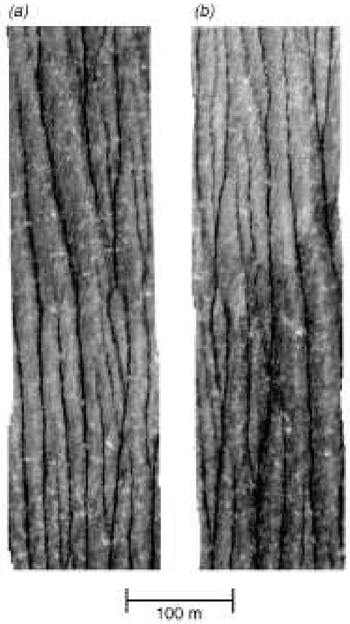
Weakly Nonlinear or Small Wavenumber 2D and 3D Investigations

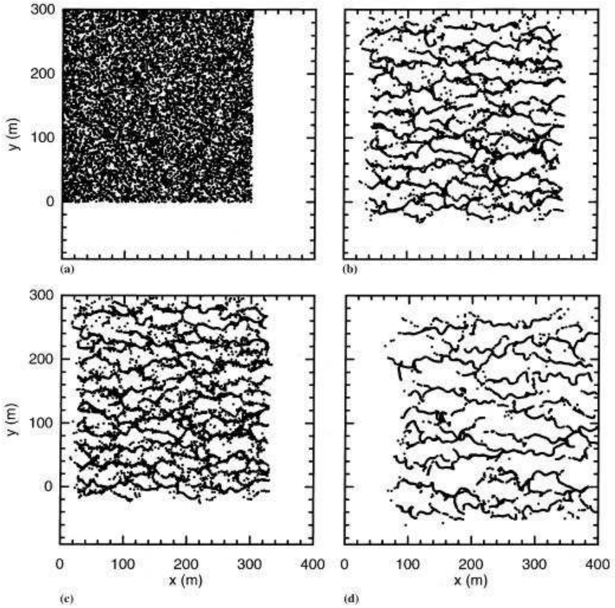
- 2D: Leibovich, Lele & Moroz (*JFM* 1989)
- 3D: Bhaskaran & Leibovich (*Phys. Fluids*, 2002)
- 3D: Cox & Leibovich (*Phys. Fluids*, 1997)

Simulations of Full 3D Craik–Leibovich (CL) Equations

- DNS: Tandon & Leibovich (*JGR*, 1995)
- LES: Skyllingstad & Denbo (*JGR*, 1995), McWilliams *et al.* (*JFM*, 1997), Tejada-Martinez & Grosch (*JFM*, 2007)

Anisotropic LC Dynamics





–A. Szeri (1996) –G. Marmorino –McWilliams *et al.* (1997)

Goals and Motivation

Objective Obtain reduced PDE model capable of describing coarse-grained, strongly anisotropic but otherwise turbulent LC dynamics.

Motivation

- Secondary stability analysis by Tandon & Leibovich (*JPO*, 1995)
- Reduced PDEs for rapidly-rotating thermal convection by Julien, Knobloch & Werne (*Theoret. Comput. Fluid Dyn.*, 1998), Sprague *et al.* (*JFM*, 2006)

Purpose

- Reveal dominant 3D physics.
- More amenable to (e.g. upper-bound) analysis.
- Less expensive numerical simulations for multi-scale process studies.
- Incorporation into formal multiscale numerical scheme.

Isotropically Scaled CL Equations

- Consider full 3D, isotropically-scaled CL equations, where two parameters $R_* \equiv u_* H/\nu_e$, $La_t = \sqrt{u_*/u_{s_0}}$ replace single parameter $La \equiv La_t R_*^{-3/2}$:

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla p + \frac{1}{La_t^2}(\mathbf{U}_s \times \boldsymbol{\omega}) + \frac{1}{R_*}\nabla^2 \mathbf{u}$$

- Two turbulence regimes:

Shear flow turbulence regime: $La_t \gg 1$ with $R_* \gg 1$.

Langmuir turbulence regime: $La_t = O(0.1)$ with $La \ll 1$.

- Motivates consideration of formal limit $La_t \to 0$ with R_* or La fixed:

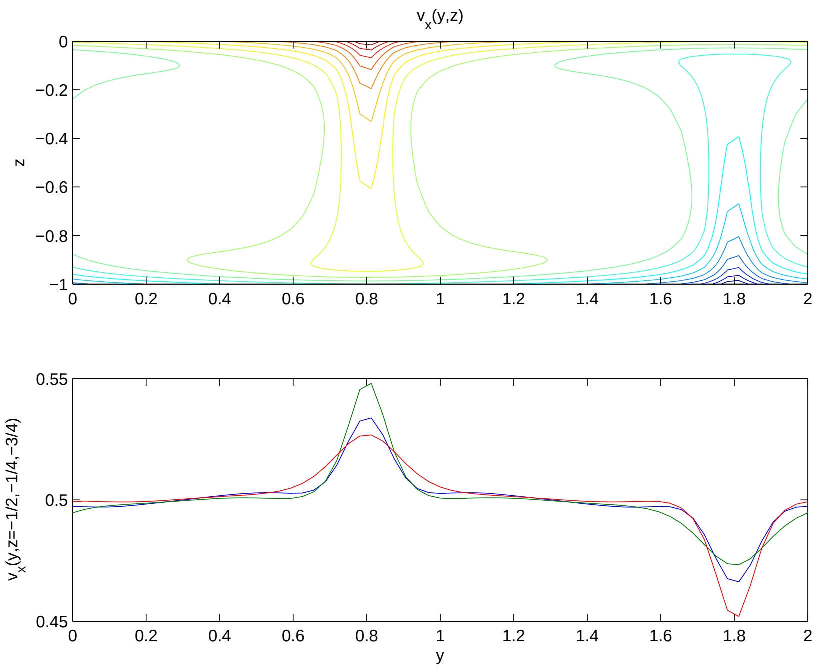
2D dynamics: $\Omega \neq 0$, $\partial_x \Omega = 0$, u–fluctuations $\ll$ (v, w)–fluctuations.

Constraint on Downwind Velocity Fluctuations

Consider 2D ($\partial/\partial x = 0$) CL downwind vorticity (Ω) equation as $La_t \to 0$...

$$\frac{\partial \Omega}{\partial t} + v\frac{\partial \Omega}{\partial y} + w\frac{\partial \Omega}{\partial z} = -\frac{1}{La_t^2}U_s'(z)\frac{\partial u}{\partial y} + \frac{1}{R_*}\nabla_\perp^2 \Omega$$

$$\Rightarrow \frac{\partial u}{\partial y} \approx 0$$



Transient Growth of Cellular Velocity Fluctuations

As $La_t \to 0$, there is a "rapid-distortion" (strong Stokes "shear") transient that drives strong cellular (v,w) velocity fluctuations.

$$
\begin{aligned}
\partial_\tau u &= 0 \quad \Rightarrow \quad u(y,z,\tau) = U_0(y,z) \\
\partial_\tau v &= -\partial_y \mathcal{P} + U_s \partial_y u \\
\partial_\tau w &= -\partial_z \mathcal{P} + U_s \partial_z u \quad \Rightarrow \quad \partial_\tau \Omega = -U_s'(z) \partial_y u
\end{aligned}
$$

$$
\boxed{\Omega(y,z,\tau) = -\left[U_s'(z)\partial_y U_0(y,z)\right]\tau + \Omega_0(y,z)}
$$

- (v,w) grow linearly with (fast) time τ, while u remains constant.

Anisotropic Velocity Scalings

- Employ anisotropic velocity scales to capture nonlinear, spatially anisotropic reduced dynamics:

$$L_x = H, \quad (L_y, L_z) = H, \quad \mathcal{T} = H/\mathcal{V}$$
$$\mathcal{U} = u_* R_*, \quad (\mathcal{V}, \mathcal{W}) = \sqrt{\mathcal{U} u_{s_0}}, \quad \mathcal{P} = \rho \mathcal{V}^2$$

- In essence, perturbing off of strictly 2D [$\partial(\cdot)/\partial x = 0$] problem.

- Identify $\boxed{\mathcal{U}/\mathcal{W} = La_t^{4/3} La^{-1/3} \equiv \epsilon La^{-1/3} \ll 1}$

(cf. Tejada-Martinez & Grosch 2007).

Rescaled CL Equations in Strong CL Vortex-Force Limit

$$\begin{aligned}
\partial_t u + \epsilon La^{-1/3}\, u\partial_x u + (\mathbf{v}_\perp \cdot \nabla_\perp)\, u &= -\epsilon^{-1} La^{1/3}\partial_x P + La\left[\partial_x^2 + \nabla_\perp^2\right] u \\
\partial_t \mathbf{v}_\perp + \epsilon La^{-1/3}\, u\partial_x \mathbf{v}_\perp + (\mathbf{v}_\perp \cdot \nabla_\perp)\, \mathbf{v}_\perp &= -\nabla_\perp P + La\left[\partial_x^2 + \nabla_\perp^2\right] \mathbf{v}_\perp \\
&+ \; U_s\left(\nabla_\perp u - \epsilon^{-1} La^{1/3}\partial_x \mathbf{v}_\perp\right) \\
\epsilon La^{-1/3}\, \partial_x u + \nabla_\perp \cdot \mathbf{v}_\perp &= 0
\end{aligned}$$

- Wind stress BC: $\partial_z u = 1$ along $z = 0, -1$.
- x-invariance at leading-order: $\partial_x P = \partial_x v = \partial_x w = 0$ and $\nabla_\perp \cdot \mathbf{v}_\perp = 0$.

Multiple Scale Expansion

1. Limit process: $\epsilon \equiv La_t^{4/3} \to 0$ with La fixed.

2. Introduce slow x scale: $X \equiv \epsilon La^{-1/3} x$ so that $\partial_x \to \partial_x + \epsilon La^{-1/3} \partial_X$.

3. Expand fields:

$$
\begin{aligned}
u(x,y,z,t) &= u_0(x,X,y,z,t) + \epsilon u_1(x,X,y,z,t) + \ldots \\
\mathbf{v}_\perp(x,y,z,t) &= \mathbf{v}_{0\perp}(X,y,z,t) + \epsilon \mathbf{v}_{1\perp}(x,X,y,z,t) + \ldots \\
P(x,y,z,t) &= P_0(X,y,z,t) + \epsilon P_1(x,X,y,z,t) + \ldots
\end{aligned}
$$

4. Substitute into PDEs, collect terms of like order and **average** over fast x.

5. Obtain closed set of equations for $\bar{u}_0 \equiv U(X,y,z,t)$, $\mathbf{v}_{0\perp} \equiv \mathbf{V}_\perp(X,y,z,t)$ and $P_0 \equiv \Pi(X,y,z,t)$.

Example: Averaging Downwind Momentum Equation

- At $O(1)$, the x momentum equation becomes:

$$\partial_T u_0 + (\mathbf{v}_{0\perp} \cdot \nabla_\perp)\, u_0 \;=\; -La^{1/3}\partial_x P_1 \,-\, \partial_X P_0 + La\left[\partial_x^2 + \nabla_\perp^2\right] u_0,$$

- Decompose all (fast) x-varying fields into fast-x average plus fluctuation, e.g.,

$$u_0(x, X, y, z, T) \;\equiv\; \bar{u}_0(X, y, z, T) + u_0'(x, X, y, z, T),$$

- Averaging in x, using the x-invariance of P_0 and $\mathbf{v}_{0\perp}$, yields an equation for $\bar{u}_0(X, y, z, T)$:

$$\partial_T \bar{u}_0 + (\mathbf{v}_{0\perp} \cdot \nabla_\perp)\, \bar{u}_0 \;=\; -\,\partial_X P_0 + La\nabla_\perp^2 \bar{u}_0.$$

Reduced PDEs

- Define:

$$D_t^{\perp}(\cdot) \;\equiv\; \partial_t(\cdot)+(\mathbf{V}_{\perp}\cdot\nabla_{\perp})(\cdot) \;\equiv\; \partial_t(\cdot)+J[(\cdot),\psi],$$

where $J[(\cdot),\psi] = \partial_z\psi\partial_y(\cdot) - \partial_y\psi\partial_z(\cdot)$.

- Reduced dynamics governed by:

$$\begin{aligned}
D_t^{\perp}U &= -\partial_X\Pi+La\nabla_{\perp}^2 U \\
D_t^{\perp}\Omega+U_s(z)\partial_X\Omega &= U_s'(z)(\partial_X V-\partial_y U)+La\nabla_{\perp}^2\Omega \\
\nabla_{\perp}^2\Pi &= 2J[\partial_y\psi,\partial_z\psi]+\nabla_{\perp}\cdot(U_s(z)\nabla_{\perp}U)+U_s'(z)\partial_X(\partial_y\psi) \\
\nabla_{\perp}^2\psi &= -\Omega, \quad \mathbf{V}_{\perp}\equiv\nabla_{\perp}\times\psi\hat{\imath}
\end{aligned}$$

- Fast x averaged BCs along $z=0,-1$: $\partial_z U = 1, \;\; \Omega = 0, \;\; \psi = 0$.
- Advection by U and stretching of Ω are subdominant processes.

Linear Stability Analysis of Reduced PDEs

- Linearize about wind-driven base state shear flow $U_B(z) = z{+}1$.
- For simplicity, set $U_s(z) = z{+}1$.
- Decompose all fields into a base state plus perturbation, e.g. $U(X,y,z,T) = U_B(z) + u(X,y,z,T)$

- Perturbations satisfy

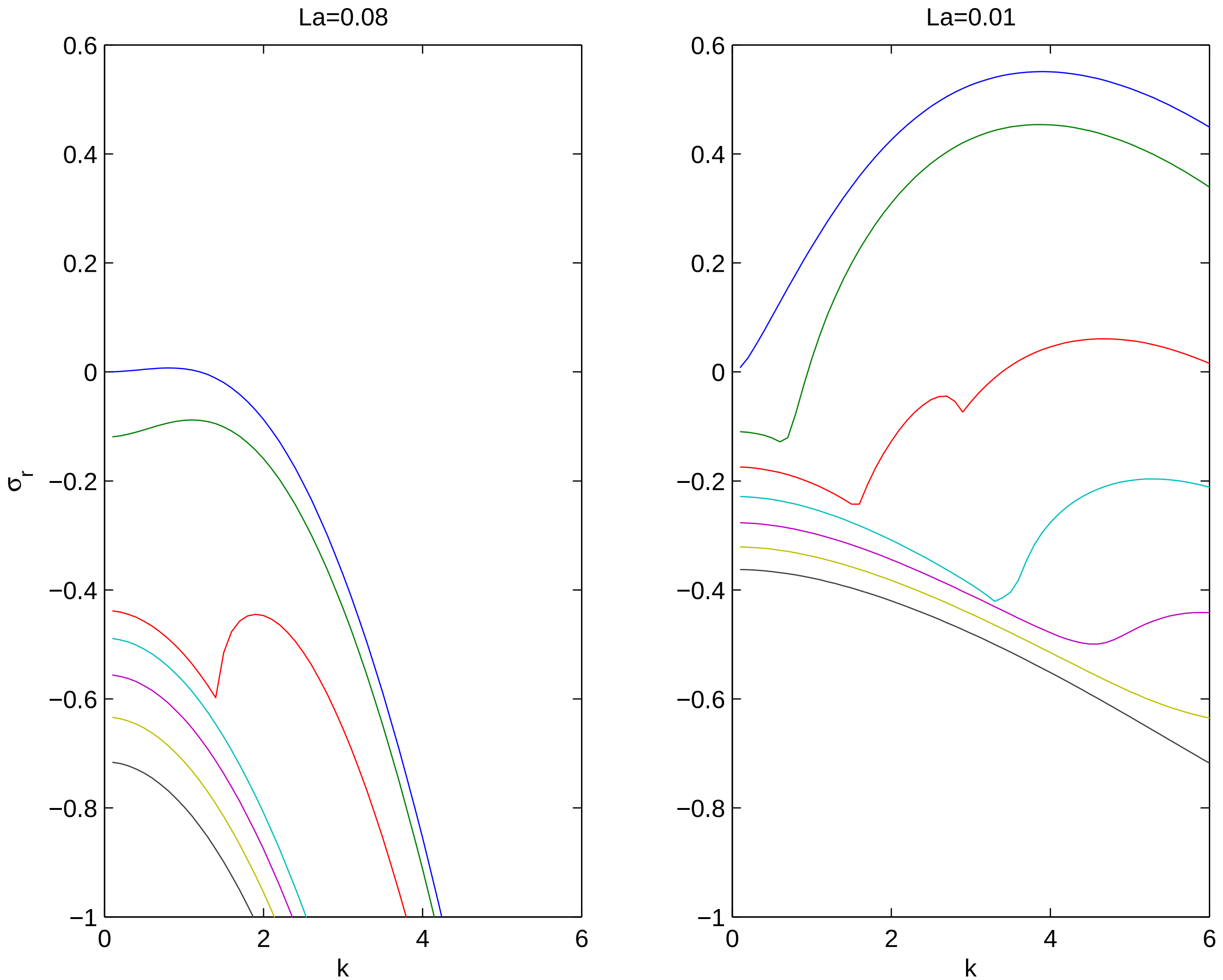
$$\begin{aligned} \partial_T u \; - \; \partial_y \phi &= -\partial_x p + La \nabla_\perp^2 u, \\ \partial_T \omega + (z{+}1)\partial_x \omega &= \partial_x(\partial_z \phi) - \partial_y u + La \nabla_\perp^2 \omega, \\ \nabla_\perp^2 \phi &= -\omega, \\ \nabla_\perp^2 p &= \nabla_\perp \cdot [(z{+}1)\nabla_\perp u] + \partial_x(\partial_y \phi), \end{aligned}$$

subject to $\partial_z u = \omega = \phi = \partial_z p = 0$ along $z = 0, -1$.

- Normal-mode *ansatz*, e.g.

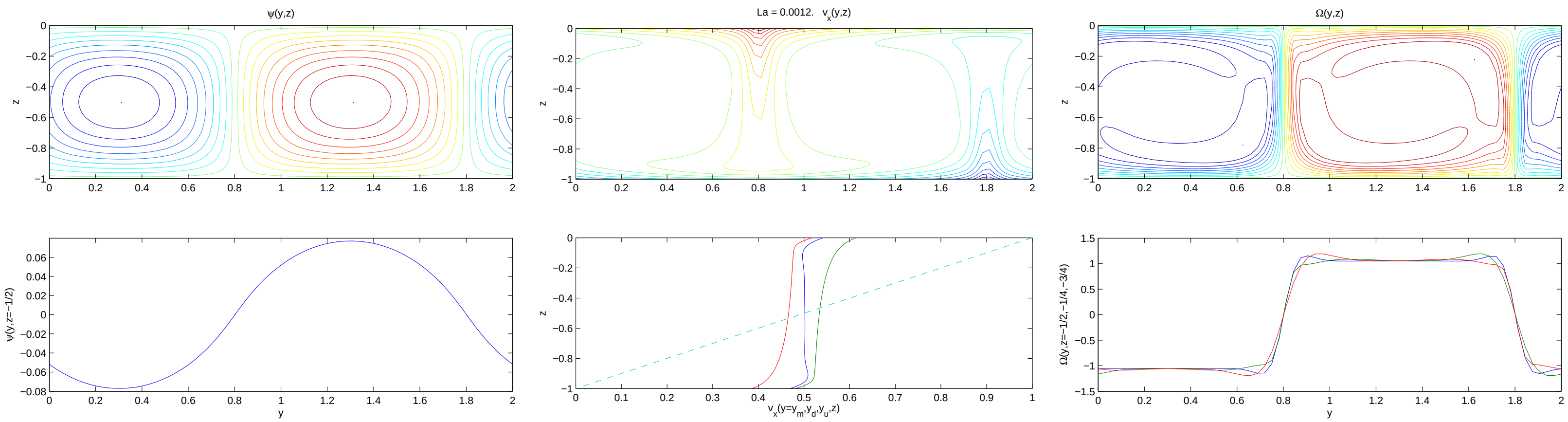
$$u(X,y,z,T) \;=\; \hat{u}(z)\mathrm{e}^{\mathrm{i}(ky+lX)}\mathrm{e}^{\sigma t} + \text{c.c.}$$

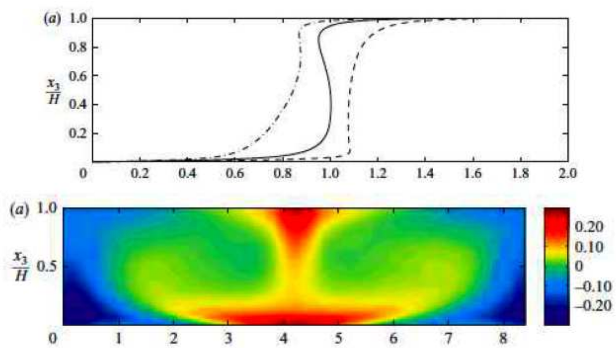
Reduced PDEs: Linear Stability Results



Strongly Nonlinear, Strictly 2D Convective States

$\psi(y,z)$ $U(y,z)$ $\Omega(y,z)$





- Steady-state $U(y,z)$ profiles show excellent qualitative agreement with $x{-}t$ averaged LES profiles of Tejada–Martinez & Grosch (2007).

Matched Asymptotic Analysis

$$\frac{\partial\psi}{\partial z}\frac{\partial U}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial U}{\partial z} = La\left(\frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}\right)$$

$$\frac{\partial\psi}{\partial z}\frac{\partial\Omega}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\Omega}{\partial z} = -U_s'(z)\frac{\partial U}{\partial y} + La\left(\frac{\partial^2 \Omega}{\partial y^2} + \frac{\partial^2 \Omega}{\partial z^2}\right)$$

$$\frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = -\Omega$$

where: $U(y,z) = U_B(z) + u(y,z) = [z+1] + u(y,z)$

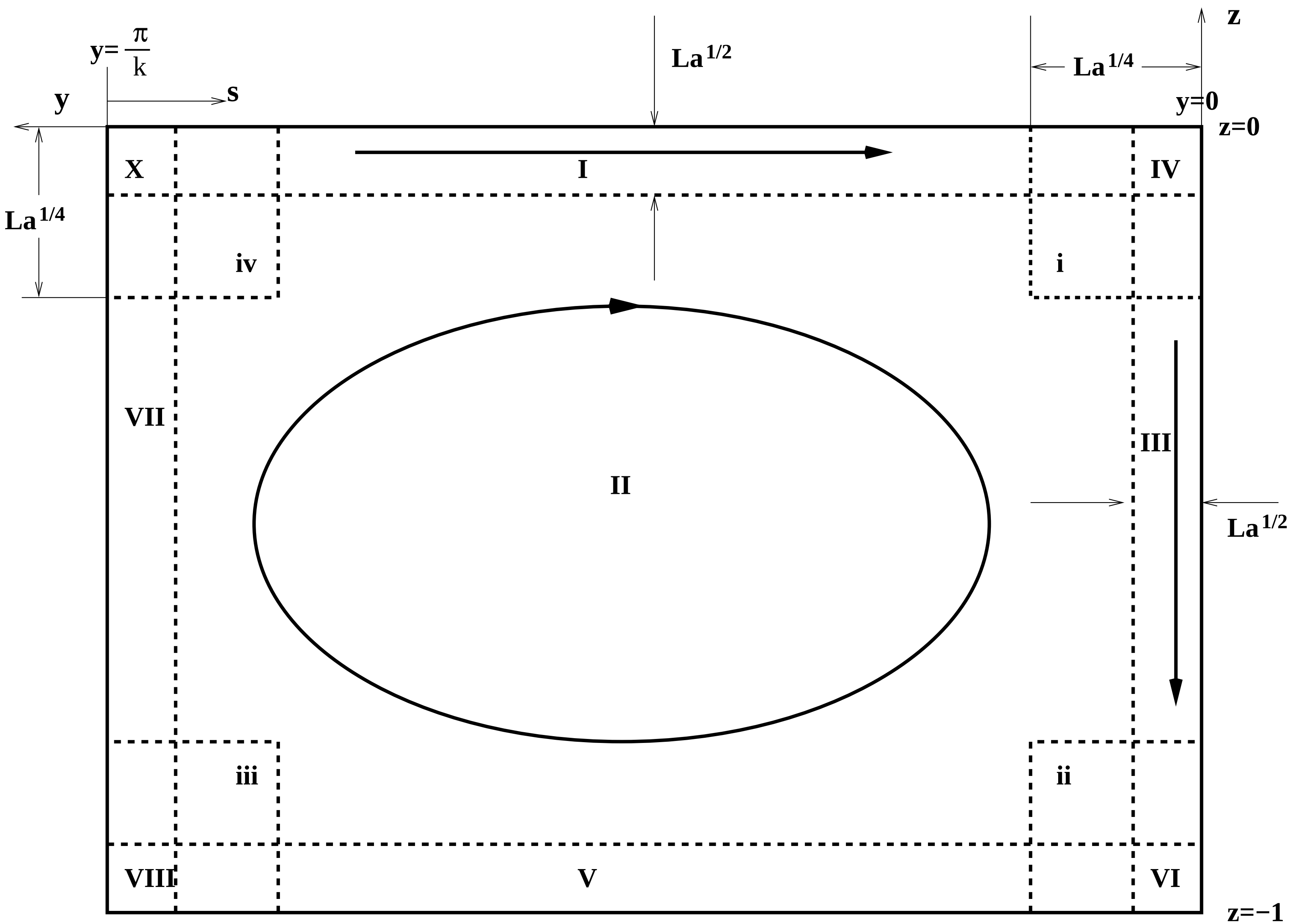
$$\underline{z = 0, -1}: \quad \frac{\partial U}{\partial z} = 1,\ \psi = 0,\ \Omega = 0$$

$$\underline{y = 0, \pi/k}: \quad \frac{\partial U}{\partial y} = 0,\ \psi = 0,\ \Omega = 0$$

Non-dimensional parameter:

$$La \equiv \frac{(\nu_e/H)^{3/2}}{u_* u_{s_0}^{1/2}} = La_t R_*^{-3/2}$$

Hypothesized Small-La Asymptotic Structure



Region II: Inviscid Vortex Core

$$U(y,z) \sim \bar{U}; \quad \Omega(y,z) \sim \bar{\Omega}; \quad \psi(y,z) \sim \psi_{II}(y,z)$$

Equations/BCs

$$\frac{\partial^2 \psi_{II}}{\partial y^2} + \frac{\partial^2 \psi_{II}}{\partial z^2} = -\bar{\Omega}$$

$$\psi_{II}(y,z) = 0 \text{ along } z = 0, -1 \text{ and } y = 0, \pi/k$$

Matching with I, III

$$\psi(y,z) \sim v_{II}(y,0)\, z \equiv V(y)\, z \quad \text{as } z \to 0^-$$

$$\psi(y,z) \sim -w_{II}(0,z)\, y \equiv -W(z)\, y \quad \text{as } y \to 0^+$$

$$V(y) = \sum_{n=1(\text{odd})}^{\infty} \left(-\frac{4\bar{\Omega}}{\pi k n^2}\right) \tanh\left(\frac{nk}{2}\right) \sin(nky)$$

$$W(z) = \sum_{n=1(\text{odd})}^{\infty} \left(-\frac{4\bar{\Omega}}{\pi k n^2}\right) \left[1 - \frac{\cosh\left(nk\left(z+\frac{1}{2}\right)\right)}{\cosh\left(\frac{nk}{2}\right)}\right]$$

Analysis in Regions I and III

Region I. Near-Surface Boundary Layer: $y = O(1)$, $z \equiv La^{1/2}Z$

$$\boxed{U(y,z) - \bar{U} \sim La^{1/2}u_I(y,Z);\ \Omega(y,z) \sim \Omega_I(y,Z);\ \psi(y,z) \sim La^{1/2}\psi_I(y,Z)}$$

- BL equations linearize since $\psi_I(y,Z)$ known.

- BL equations completely de-couple, since $\partial U/\partial y$ weak, $O(La^{1/2})$.

Region III. Downwelling Jet/Plume: $y \equiv La^{1/2}Y$, $z = O(1)$

$$\boxed{U(y,z) - \bar{U} \sim La^{1/2}u_{III}(Y,z);\ \Omega(y,z) \sim \Omega_{III}(Y,z);\ \psi(y,z) \sim La^{1/2}\psi_{III}(y,Z)}$$

- BL equations linearize since $\psi_I(y,Z)$ known.

- $\Omega(Y,z)$ coupled to $U(Y,z)$, since $\partial U/\partial y$=$O(1)$ (but converse is not true).

...Need $\bar{\Omega}$ to proceed...

Determination of Core Vorticity

1. Integrated Energy Balance.

$$La \int\int \Omega^2 \, \mathrm{d}A \;=\; -\int\int \psi \frac{\partial U}{\partial y} \, dA \qquad \text{(Exact Relation)}$$

Asymptotic Approximation:

$$\begin{aligned}
\bar{\Omega}^2 &\sim -\frac{k}{\pi La} \int\int \psi \frac{\partial U}{\partial y} dA \\
&\sim -\frac{2k}{\pi} \int_{-1}^{0} \int_{0}^{\infty} \psi_{III} \frac{\partial U_{III}}{\partial Y} \, dY \, dz \\
&= -\frac{2k}{\pi} \int_{-1}^{0} \int_{0}^{\infty} \psi_{III} \frac{\partial U_{III}}{\partial \psi_{III}} \, d\psi_{III} dz \\
&= -\frac{2k}{\pi} \int_{-1}^{0} \left[\psi_{III} U_{III} \Big|_{0}^{\infty} - \int_{0}^{\infty} U_{III} d\psi_{III} \right] dz
\end{aligned}$$

Determination of Core Vorticity (Cont'd)

2. Downwind Momentum Flux Conservation.

Exact Relation:

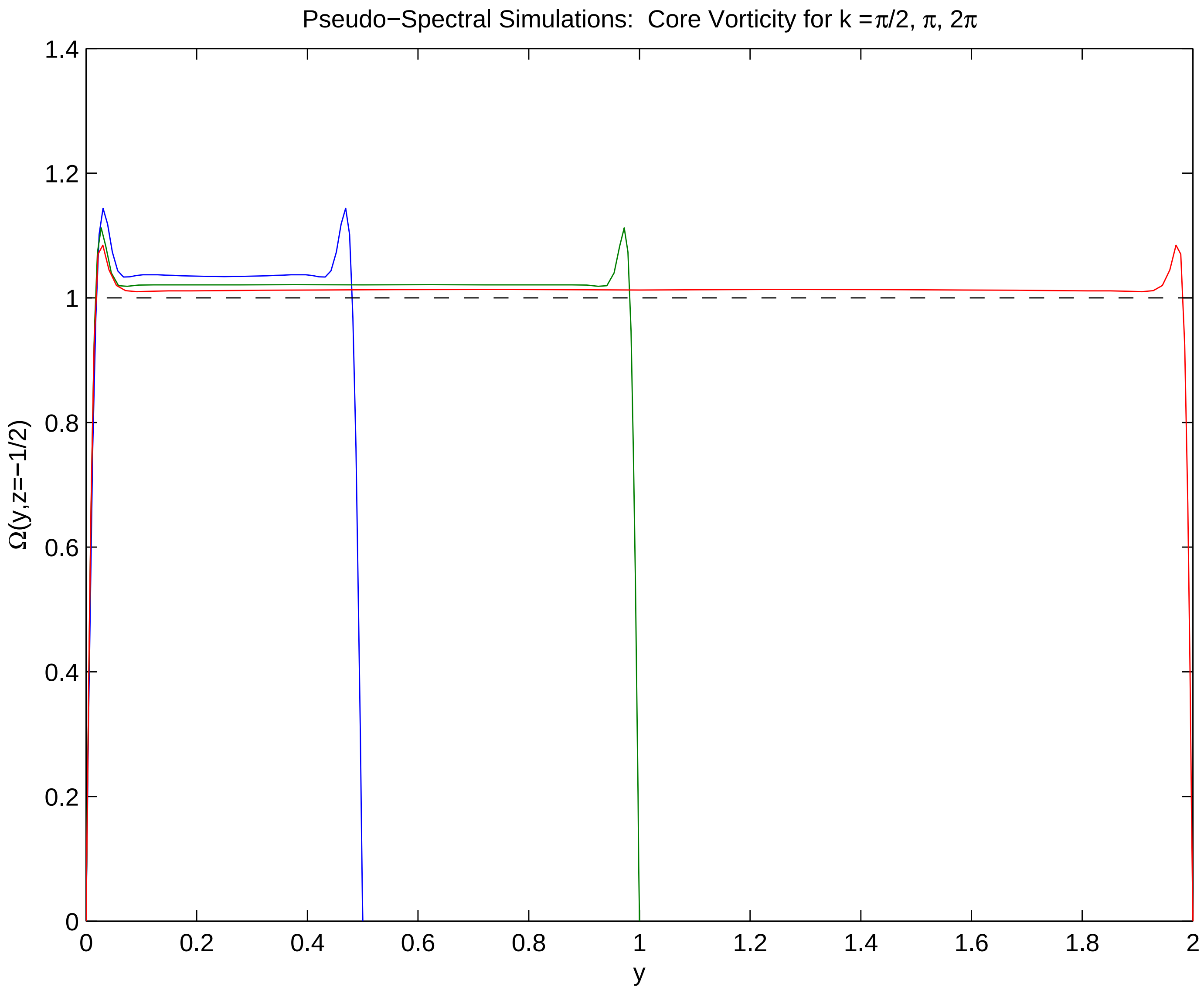
$$La\frac{\pi}{k} \;=\; \int_0^{\pi/k} \left[La\frac{\partial U}{\partial z} - w(U - \bar{U}) \right] dy$$

Asymptotic Approximation (True for *any* $\bar{\Omega}$):

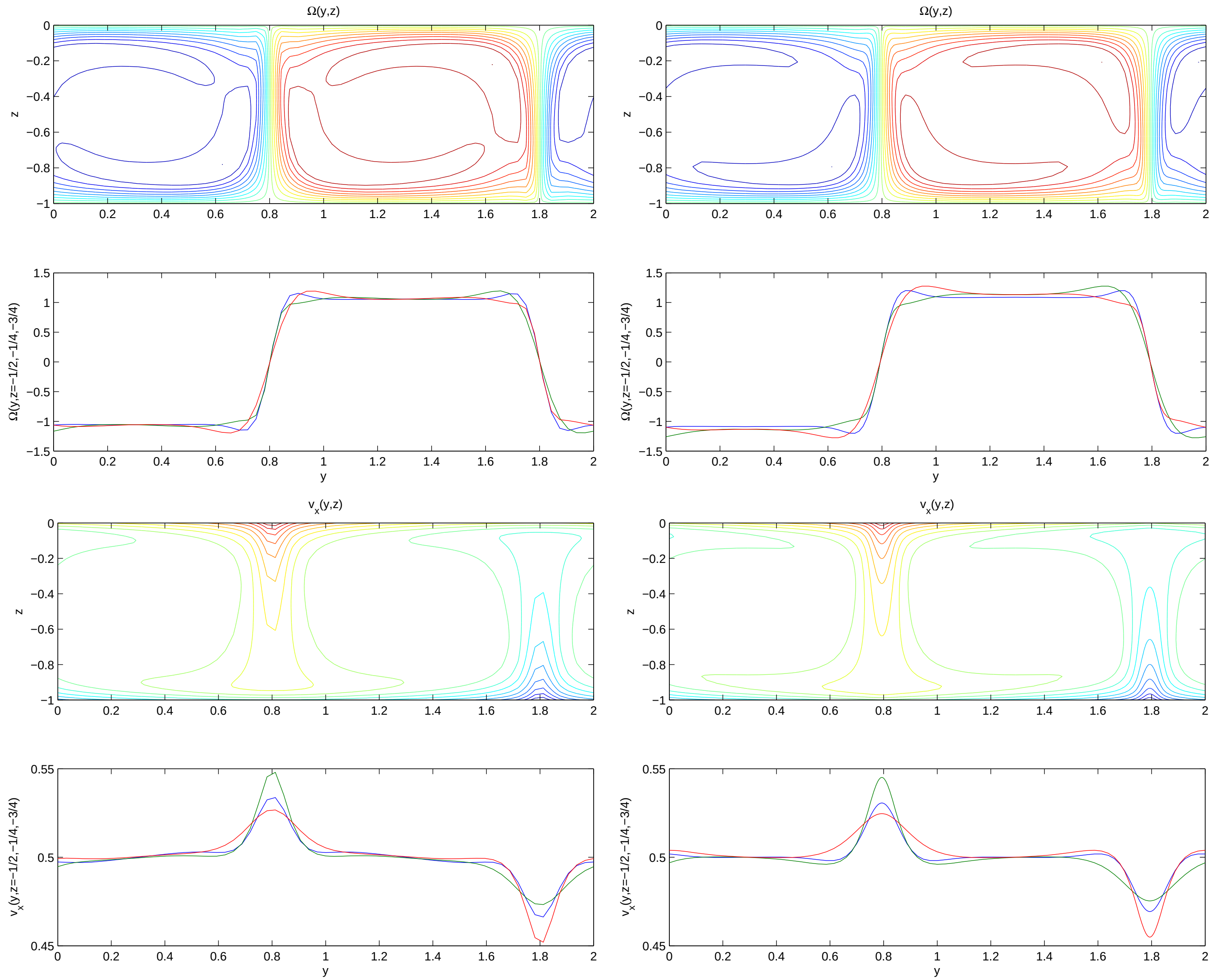
$$\frac{\pi}{2k} \;\sim\; -W(z) \int_0^{\infty} u_{III}(Y,z)\, dY \;=\; \int_0^{\infty} U_{III}(\psi_{III}, r)\, d\psi_{III}$$

$\Rightarrow \quad \bar{\mathbf{\Omega}} \sim 1$ (or -1), *independently of* k.

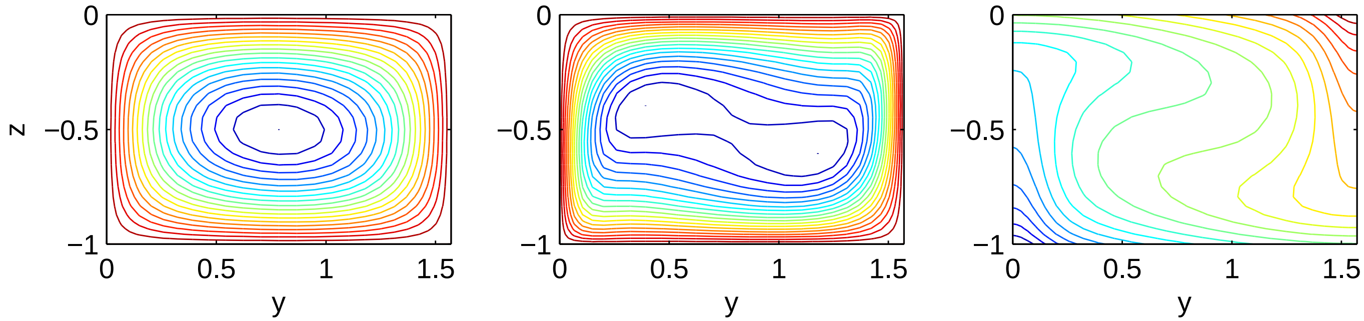
Numerical Simulations: Core Vorticity vs. k



Numerical Fields vs. Semi-Analytical Fields



Secondary Stability Analysis of Reduced PDEs



- Linearize about **fully nonlinear 2D** cellular flow, e.g.
  $U(X,y,z,T) = U_{2D}(y,z) + u(X,y,z,T)$
- Since coefficients in disturbance equations vary periodically in y, employ **Floquet theory**, e.g.

$$u(X,y,z,T) = \mathrm{e}^{\mathrm{i}\gamma y}\left[\sum_{n=-\infty}^{\infty} \widehat{u}_n(z)\mathrm{e}^{\mathrm{i}(nky)}\right]\mathrm{e}^{\mathrm{i}lX}\mathrm{e}^{\sigma t} + \text{c.c.}$$

- σ, l are the temporal growth rate, downwind wavenumber.
- k is the fundamental wavenumber of underlying 2D convective base flow.
- iγ is the Floquet exponent, with the real parameter γ providing the freedom to modify the crosswind wavenumber.

Secondary Stability Equations

Small-amplitude 3D disturbances to fully nonlinear 2D roll solutions satisfy:

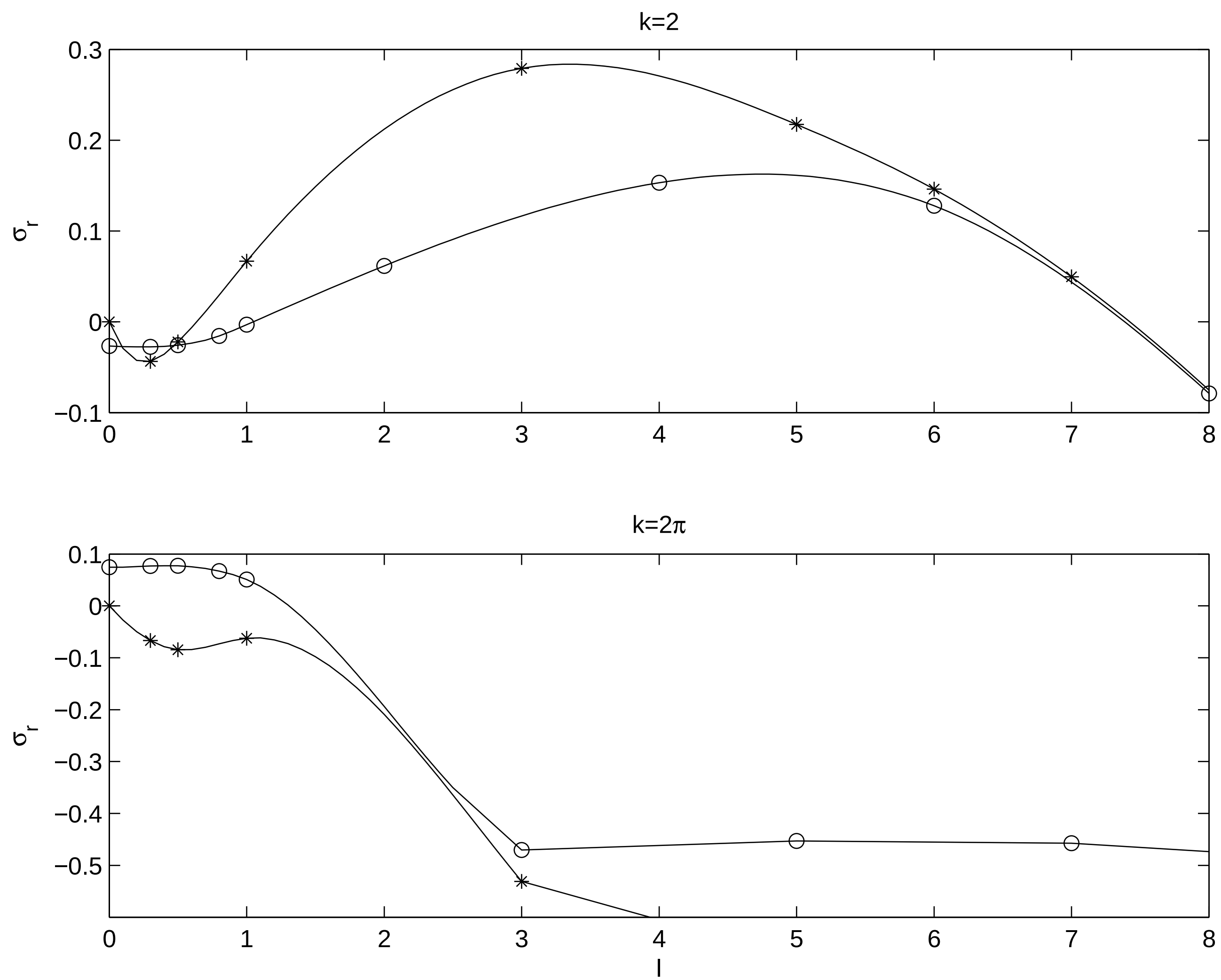
$$\partial_T u + \partial_z \psi_{2D} \partial_y u - \partial_y \psi_{2D} \partial_z u + \partial_y U_{2D} \partial_z \phi - \partial_z U_{2D} \partial_y \phi \;=\; -\partial_X p + La \nabla_\perp^2 u$$

$$\begin{aligned} \partial_T(\nabla_\perp^2 \phi) + U_s(z) \partial_X (\nabla_\perp^2 \phi) + \partial_z \psi_{2D} \partial_y (\nabla_\perp^2 \phi) & \\ - \partial_y \psi_{2D} \partial_z (\nabla_\perp^2 \phi) + \partial_y \Omega_{2D} \partial_z \phi - \partial_z \Omega_{2D} \partial_y \phi &= U_s'(z) \left[\partial_y u - \partial_X (\partial_z \phi)\right] \\ & \quad + La \nabla_\perp^4 \phi \end{aligned}$$

$$\begin{aligned} \nabla_\perp^2 p &= -4 \partial_y (\partial_z \psi_{2D}) \partial_y (\partial_z \phi) + 2 \partial_y^2 \psi_{2D} \partial_z^2 \phi + 2 \partial_z^2 \psi_{2D} \partial_y^2 \phi + U_s(z) \nabla_\perp^2 u \\ & \quad + U_s'(z) \left[\partial_z u + \partial_X (\partial_y \phi)\right] \end{aligned}$$

with $\partial_z u = \phi = \partial_z^2 \phi = \partial_z p = 0$ along $z = 0, -1$.

Reduced PDEs: Secondary Stability Results



Summary

- Derived reduced PDEs for anisotropic turbulent Langmuir circulation in strong vortex-force limit.

- Reduced PDEs capture dominant linear and secondary instabilities.

- Reduced PDEs offer several analytical and computational advantages:

  1. Filter rapid-distortion (i.e. fast) transients $\Rightarrow$ larger Δt (even in 2D).

  2. Filter fine x-scale variability $\Rightarrow$ larger Δx and Δt.

  3. Vortex stretching is sub-dominant process $\Rightarrow$ facilitates analysis (e.g. homogenization theory, upper-bound analysis).

  4. *Limiting process effectively suppresses "classical" shear-flow instability mechanisms.*

Ongoing Investigations and Future Work

1. Influence of density stratification – coupling with internal gravity waves.

2. Influence of background rotation (Coriolis accelerations) – both kinematic and dynamic effects.

3. Coupling b/w submesoscale convective turbulence and submesoscale and mesoscale eddies using homogenization theory.

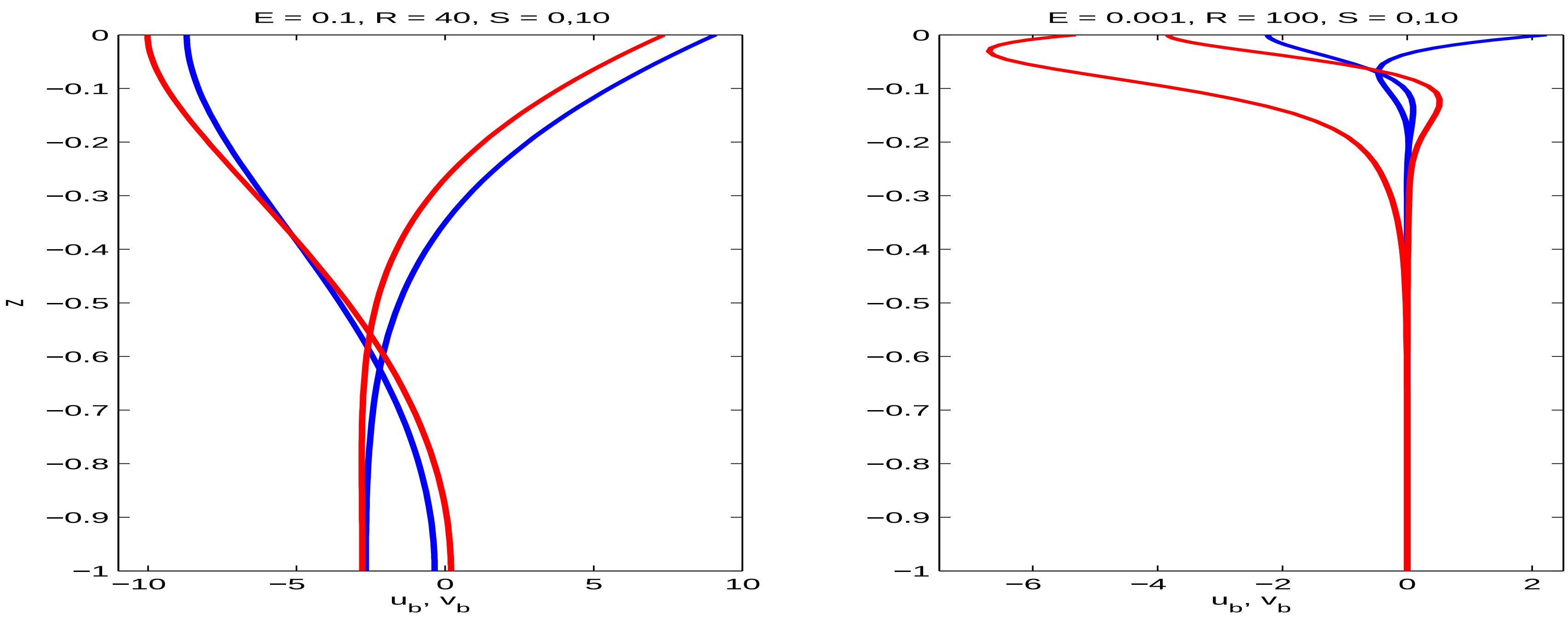
4. Application to other boundary-layer instability phenomena (e.g. Ekman rolls in the atmospheric boundary layer, classical shear flow rolls/streaks).

Langmuir Circulation–Internal Wave Interactions

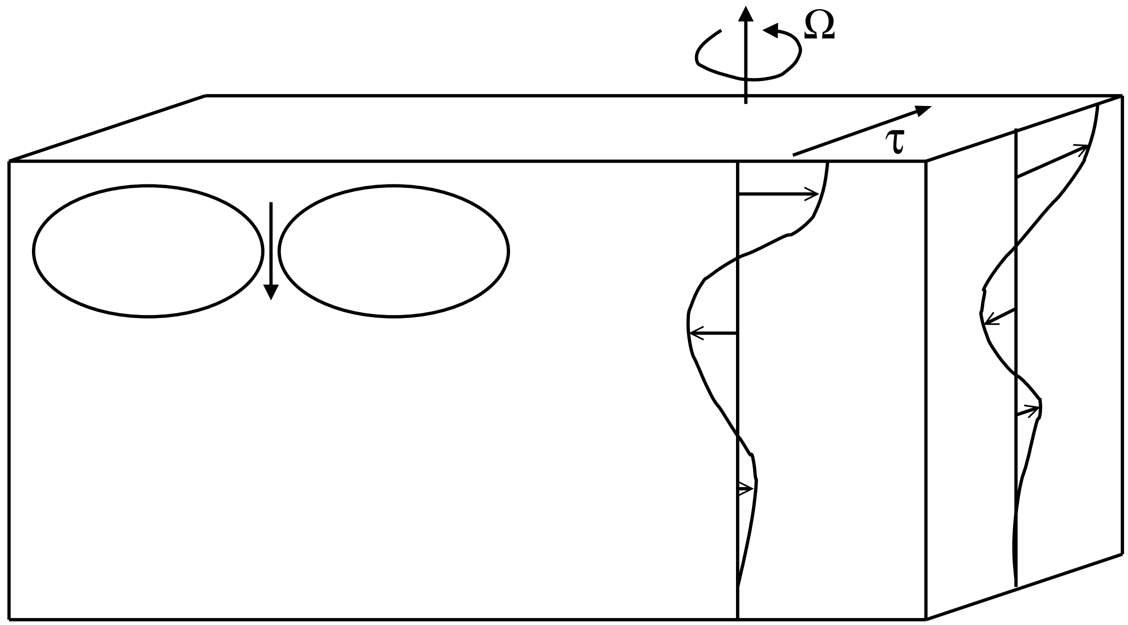
Rotating CL Equations

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \frac{1}{R_o}\left[\mathbf{f}\times\left(\mathbf{u}+\frac{1}{La_t^2}\mathbf{U}_s\right)\right] \;=\; -\nabla P + \frac{1}{La_t^2}(\mathbf{U}_s\times\boldsymbol{\omega}) + \frac{1}{R_*}\nabla^2\mathbf{u}$$

Inclusion of Stokes-drift-induced Coriolis force implies classical Ekman spiral is modified.



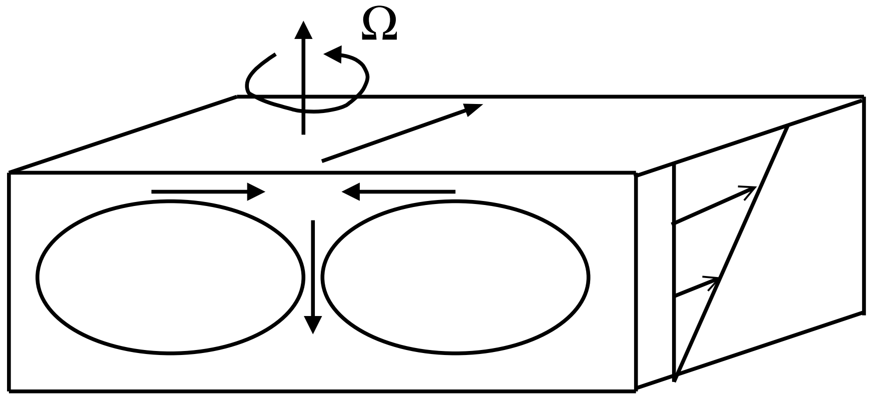
Ekman Instability Modes



1. **Mode I** – **Inviscid** instability arising from **inflectional shear** in component of Ekman spiral **perpendicular** to roll axis:
$$\widehat{R}_I = O(100),\ \widehat{k}_I = O(0.5).$$

2. **Mode II** – **"Viscous"** instability arising from shear in component of Ekman spiral **parallel** to roll axis: $\widehat{R}_c = 11.8$, $\widehat{k}_c = 0.32$.

Vortex-Force Driven Instabilities: LC and Mode II

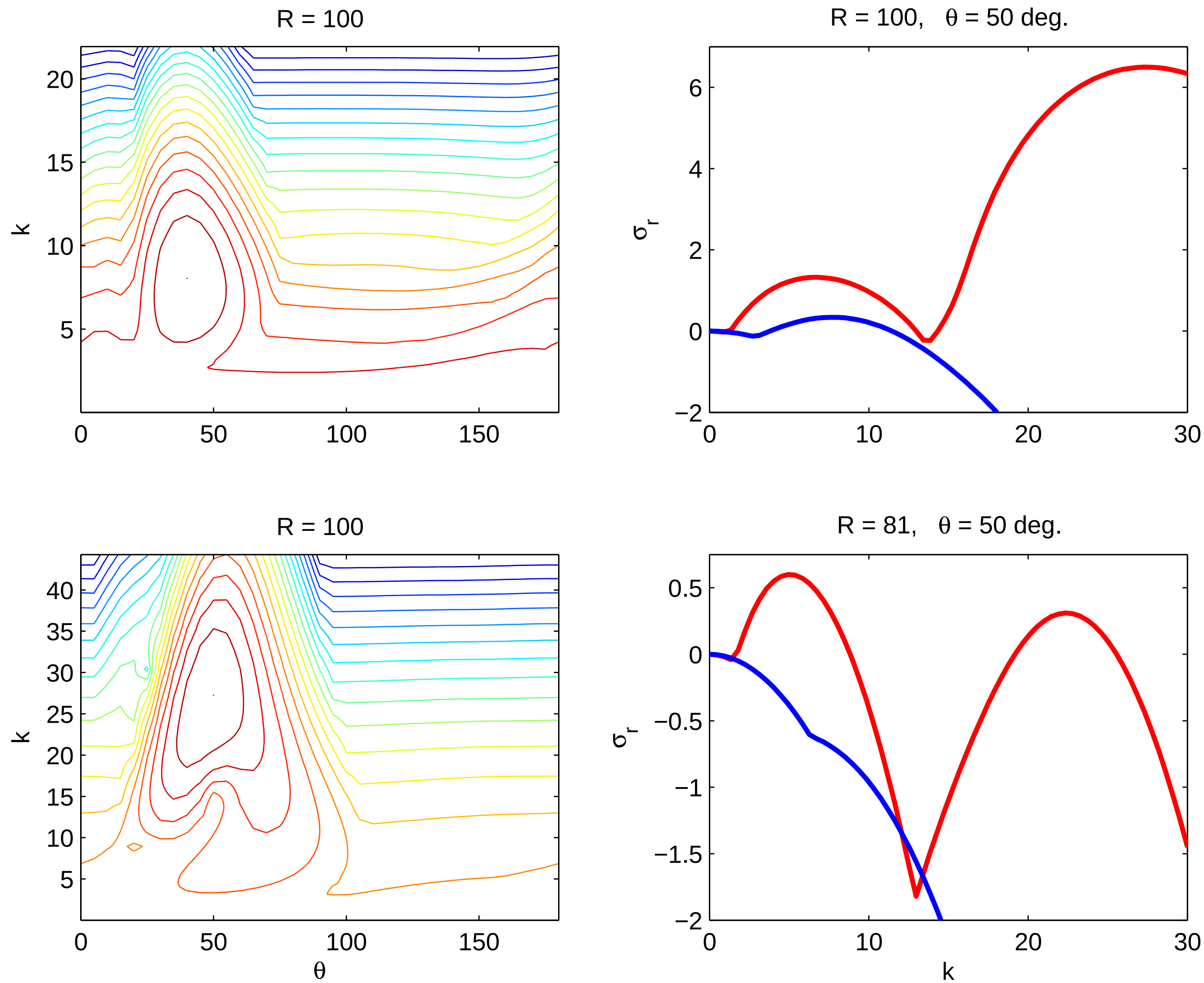


Common features:

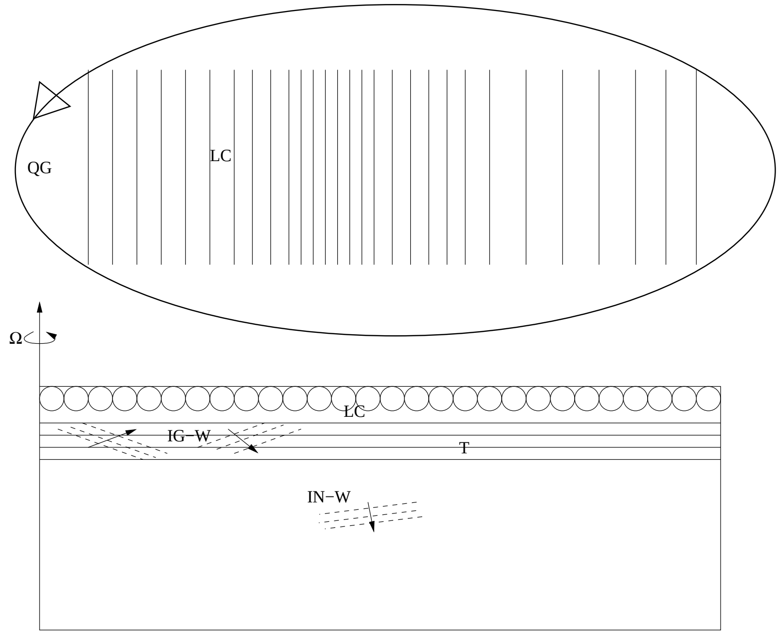
1. A component of shear parallel to the roll axes.

2. A vortex force that couples along-roll velocity perturbations to the cross-roll cellular flow.

Influence of Stokes Drift $(E = 0.001,\ R = 100)$



Multiscaling via Homogenization Theories



- Develop homogenization theory for asymptotic strongly-nonlinear 2D LC solutions, exploiting **linearization** of CL equations using known $\psi(y,z)$.

- Employ novel Lagrangian homogenization formalism (T. Hou) that does **not** require strict scale separation and that can treat a nonlinear, dynamic (turbulent) “cell problem.”