The Error Estimate of The Near-Resonant Approximation of 3D Rotating Navier-Stokes Equations

Qiang Deng
Mathematics Department
University of Wisconsin-Madison

July 14, 2008

Anatoli Babin, Alex Mahalov and Basil Nicolaenko

- Regularity and integrability of 3D Euler and Navier-Stokes equations for rotating fluids, *Asymptotic Analysis* 15(1997);
- Global regularity of 3D rotating Navier-Stokes equations for resonant domains, *Applied Mathematics Letters* 13, No. 4(2000).

Full equations:

3D Rotating Navier-Stokes Equations

$$\partial_t \vec{U} + (\vec{U}\cdot\nabla)\vec{U} - \nu\triangle\vec{U} + \Omega\vec{e}_3\times\vec{U} = -\nabla p + \vec{F}, \quad \nabla\cdot\vec{U} = 0,$$

$$\vec{U}(t,x)|_{t=0} = \vec{U}(0,x) = \vec{U}(0)$$

Fourier series expansions for velocity fields,

$$\vec{U} = \sum_n exp(i(n_1x_1/a_1 + n_2x_2/a_2 + n_3x_3/a_3))\vec{U}_n = \sum_n exp(i\check{n}\cdot x)\vec{U}_n$$

$$\check{n} = (n_1/a_1, n_2/a_2, n_3/a_3),$$

a_1, a_2 and a_3 are the aspect ratios of space periods.

Change of variables:

Set

$$\vec{U} = exp(-\Omega PJPt)\vec{u} = E(-\Omega t)\vec{u}$$

P is the projection operator on the space of divergence free vector fields, and J is the rotation matrix.

$$J = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$E_k(\Omega t) = exp(\Omega PJPt)_k = \cos(\xi_k \Omega t) I + \frac{1}{|\check{k}|}\sin(\xi_k \Omega t) R_k,$$

$$\xi_k = \check{k}_3/|\check{k}|,$$

$$R_k = \begin{pmatrix} 0 & -\check{k}_3 & \check{k}_2 \\ \check{k}_3 & 0 & -\check{k}_1 \\ -\check{k}_2 & \check{k}_1 & 0 \end{pmatrix}, P_k = I - \frac{1}{|\check{k}|^2}\begin{pmatrix} \check{k}_1^2 & \check{k}_1\check{k}_2 & \check{k}_1\check{k}_3 \\ \check{k}_2\check{k}_1 & \check{k}_2^2 & \check{k}_2\check{k}_3 \\ \check{k}_3\check{k}_1 & \check{k}_3\check{k}_2 & \check{k}_3^2 \end{pmatrix}$$

Simplified equations:

$$\begin{cases} \partial_t \vec{u} = \vec{B}(\Omega t, \vec{u}, \vec{u}) - \nu A\vec{u} + exp(\Omega St)\vec{F} \\ \vec{B}(\Omega t, \vec{u}, \vec{u}) = exp(\Omega St)P\{exp(-\Omega St)\vec{u} \times curl(exp(-\Omega St)\vec{u})\} \\ A = -P\Delta, \quad S = PJP. \end{cases}$$

$$\partial_t \vec{u}_n = \sum_{k+m=n} \vec{B}_n(\Omega t, \vec{u}_k, \vec{u}_m) - \nu|\check{n}|^2\vec{u}_n + E_n(\Omega t)\vec{F}_n$$

$$\vec{B}_n(\Omega t, \vec{u}_k, \vec{u}_m) = i \sum_{k+m=n} P_n E_n(\Omega t)(E_k(-\Omega t)\vec{u}_k \times (\check{m} \times E_m(-\Omega t)\vec{u}_m))$$

Decomposition:

$$\vec{B}(\Omega t, \vec{u}, \vec{u}) = \vec{B}^0(\vec{u}, \vec{u}) + \vec{B}^+(\Omega t, \vec{u}, \vec{u})$$

$\vec{B}^0(\vec{u}, \vec{u})$ contains all resonant(Ωt-independent) terms,

$\vec{B}^+(\Omega t, \vec{u}, \vec{u})$ contains all non-resonant (Ωt-dependent) terms.

$$\vec{B}_n(\Omega t, \vec{u}, \vec{u}) = \sum_{j, k+m=n, D_j(k,m,n)= \text{ or } \neq 0} exp(i\Omega t D_j(k, m, n))\vec{Q}_{kmn,j}(\vec{u}_k, \vec{u}_m)$$

$\vec{Q}_{kmn,j}(\vec{u}_k, \vec{u}_m)$ is a bilinear form in $\vec{u}_k, \vec{u}_m$, and

$$|\vec{Q}_{kmn,j}(\vec{u}_k, \vec{u}_m)| \leq C_{B^+}|\check{m}||\vec{u}_k||\vec{u}_m|.$$

Resonant, Non-resonant, Near-resonant

$$D_j = \pm\xi_k \pm \xi_m \pm \xi_n, \quad n = k + m, \quad j = 1, ..., 8.$$

$$\xi_k = \check{k}_3/|\check{k}|,$$

$$D_j = 0, \quad D_j \neq 0, \quad D_j \leq \epsilon.$$

Babin, Mahalov and Nicolaenko's results:

$$\partial_t \vec{w} + \nu A \vec{w} = \vec{B}^0(\vec{w}, \vec{w}) + E(\Omega t)\vec{F}$$

$$\vec{w}|_{t=0} = \vec{U}(0).$$

Let $T(M_{\sigma F}, \nu, a_1, a_2, a_3)$ be the local(small) existence time for the solution $\vec{U}$,

$$\|\vec{u} - \vec{w}\|_\alpha \leq C_0 R^{\alpha+2} C_1(R) C_2(M_\sigma)/\Omega + C_3 M_\sigma^2 R^{\alpha+1-\sigma} + C_4 M_\sigma R^{\alpha-\sigma},$$

$$\forall t \in [0, T_\sigma]$$

Assume

$$\sup_t \int_T^{T+1} \|\vec{F}\|^2_{\sigma-1} dt \leq M^2_{\sigma F},$$

$$\|\vec{u}\|_\sigma \leq M_\sigma(M_{\sigma F}, \nu, a_1, a_2, a_3), \quad on \quad [0, T_\sigma]$$

then

$$\|\vec{u} - \pi_R \vec{u}\|_\alpha \leq M_\sigma R^{\alpha-\sigma}$$

$\pi_R \vec{u}$ is the projection of $\vec{u}$ onto the Fourier modes with $|\check{n}| \leq R$.

$$\frac{1}{|D_j(k, m, n|)} \leq C_1(R), \quad when \quad |\check{k}|, |\check{m}|, |\check{n}| \leq R.$$

$$C_1(R) \sim R^{12}$$

$$P(\vec{k}, \vec{m}, \vec{n}) = a_3^4 |\check{k}|^4 |\check{m}|^4 |\check{n}|^4 \cdot D_1 \cdot D_2 \cdot D_3 \cdot D_4$$

$$= a_3^4 |\check{k}|^4 |\check{m}|^4 |\check{n}|^4 \cdot$$
$$(\xi_k + \xi_m + \xi_n)(\xi_k - \xi_m + \xi_n)(-\xi_n + \xi_m + \xi_k)(-\xi_n - \xi_m + \xi_k)$$

$$= (k_3^2 |\check{m}|^2 |\check{n}|^2 + m_3^2 |\check{k}|^2 |\check{n}|^2 - n_3^2 |\check{k}|^2 |\check{m}|^2)^2 - 4 k_3^2 m_3^2 |\check{k}|^2 |\check{m}|^2 |\check{n}|^4$$

$P(\vec{k}, \vec{m}, \vec{n})$ is a polynomial of $(\vec{k}, \vec{m}, \vec{n})$, so P has fixed number zeros. Since $(\vec{k}, \vec{m}, \vec{n})$ are integer vectors, there exist a $C_0 > 0$ such that

$$P(\vec{k}, \vec{m}, \vec{n}) \geq C, \quad \forall \vec{k}, \vec{m}, \vec{n} \in Z^3$$

Since

$$|\xi_k| = |\check{k}_3|/|\check{k}| \leq 1,$$

so

$$|D_j| \leq 3.$$

When $|\check{k}|, |\check{m}|, |\check{n}| \leq R$,

$$C \leq P(\vec{k}, \vec{m}, \vec{n}) \leq a_3^4 R^{12} 3^3 |D_j|, \quad \forall j = 1, 2, 3, 4,$$

$$\frac{1}{|D_j|} \leq a_3^4 R^{12} 3^3 / C \equiv C R^{12}.$$

Near-Resonant Approximation:

$$\partial_t \vec{w}_\epsilon + \nu A \vec{w}_\epsilon = \vec{B}^\epsilon(\Omega t, \vec{w}_\epsilon, \vec{w}_\epsilon) + E(\Omega t)\vec{F}$$

$$\vec{w}_\epsilon\big|_{t=0} = \vec{U}(0).$$

$$\vec{B}^\epsilon_n(\Omega t, \vec{u}, \vec{u}) = \sum_{j,k+m=n,D_j(k,m,n)\leq\epsilon} exp(i\Omega t D_j(k,m,n))\vec{Q}_{kmn,j}(\vec{u}_k, \vec{u}_m)$$

$$\|\vec{u} - \vec{w}_\epsilon\|_\alpha \leq \frac{C_0 R^{\alpha+2} C_2(M_\sigma)}{\max\{\epsilon, R^{-12}\}\Omega} + C_3 M_\sigma^2 R^{\alpha+1-\sigma} + C_4 M_\sigma R^{\alpha-\sigma},$$

$$\forall t \in [0, T_\sigma]$$

For a long time interval,

$$\Omega \to +\infty.$$

Recent work:

The vertical equation:

$$\epsilon \frac{Dw}{Dt} = -\frac{\partial p}{\partial z} - \frac{\rho' g}{\rho_0}.$$

$$0 \leq \epsilon \leq 1$$

$$\textit{Hydrostatic} \quad \leftrightarrow \quad \textit{Boussinesq}$$

Dispersion relationship, eigenvectors, energy conservation, potential vorticity conservation...