Investigating Ocean Deep Convection Using Multi-Scale Asymptotics

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Work Performed With and Under the Direction of
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Outline

- Review
  - Ocean Deep Convection (ODC): Observations
  - Derivation of UNH-QGE
- Adding Multiple Scales
- Several Equation Sets
- Future Work

ODC: Observations

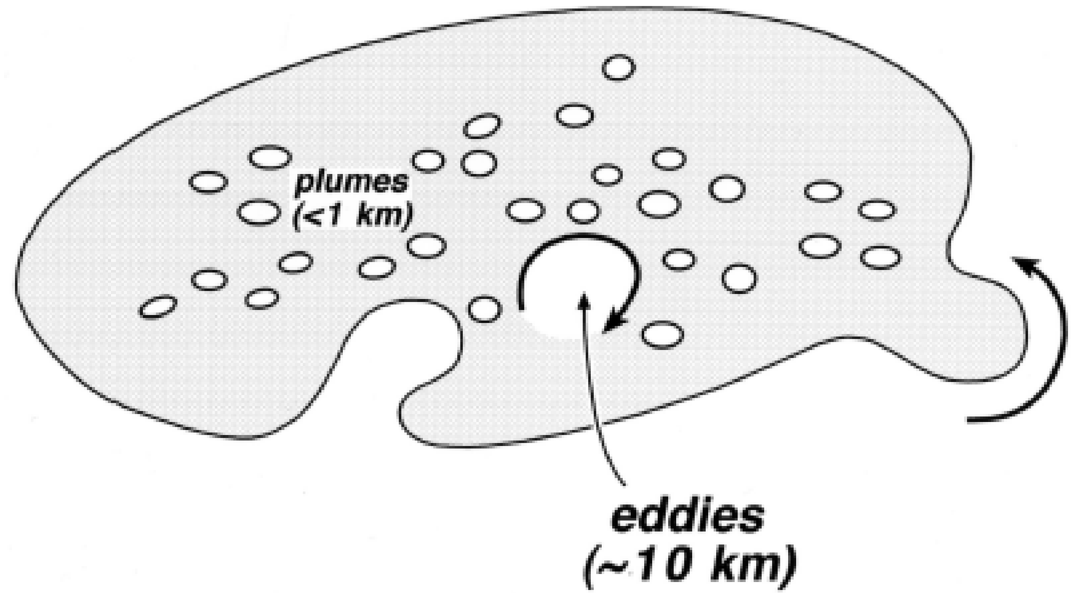

Figure 4. Lateral scales of the key phenomena in the water mass transformation process: the mixed patch on the preconditioned scale created by plumes together with eddies that orchestrate the exchange of fluid and properties between the mixed patch and the stratified fluid of the periphery. The fluid being mixed is shaded; the stratified fluid is unshaded.

ODC: Observations

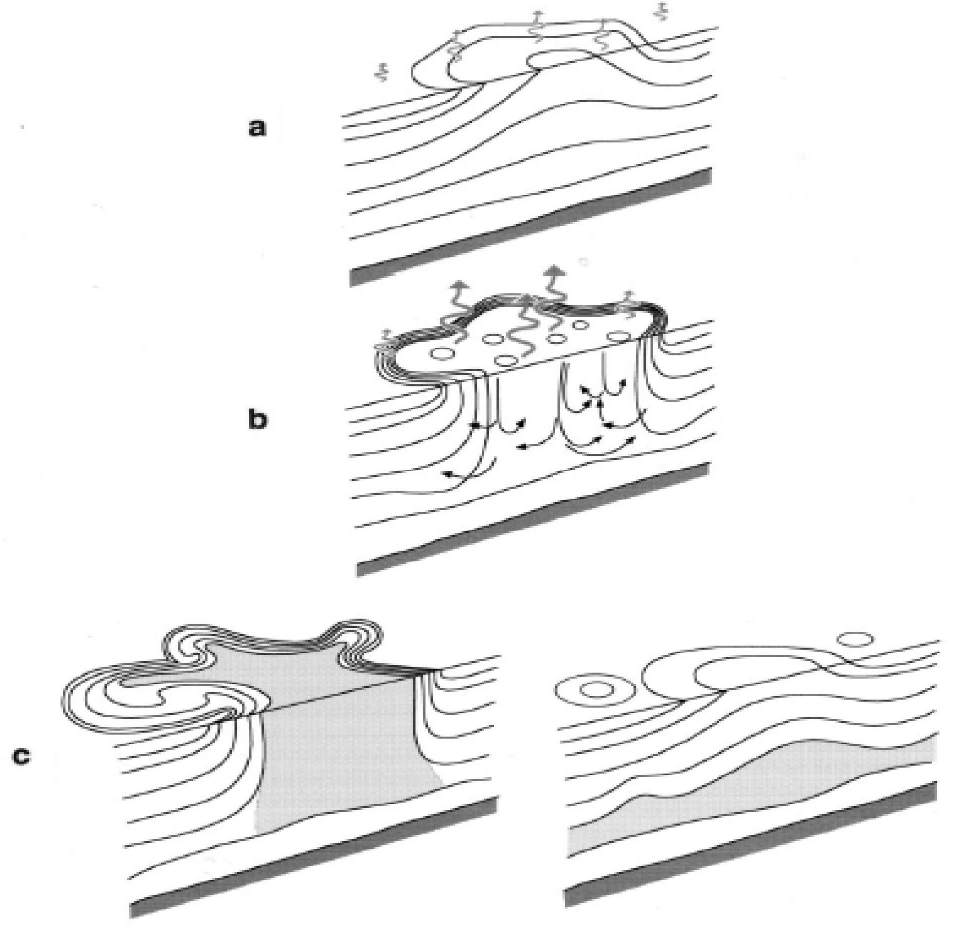

Figure 3. Schematic diagram of the three phases of open-ocean deep convection: (a) preconditioning, (b) deep convection, and (c) lateral exchange and spreading. Buoyancy flux through the sea surface is represented by curly arrows, and the underlying stratification/outcrops is shown by continuous lines. The volume of fluid mixed by convection is shaded.

ODC: Observations

- Horizontal Plume Scale < 1km
- Eddy Scale 10 km
- Gyre Scale 100 km
- Plume Depth 2-5 km

Upright Non-Hydrostatic Quasi-Geostrophic Equations

Start with the Rotating Boussinesq Equations. Nondimensionalize using different horizontal and vertical scales, letting the aspect ratio (tall) be a small parameter that scales like the Rossby number. (As in Keith Julien's talk)

$$A_z = L / H \sim \epsilon$$

The UNH-QGE

$$\partial_t w + J[\psi, w] + D\psi = \bar{\Gamma}\theta + \nabla_\perp^2 w$$

$$\partial_t \nabla_\perp^2 \psi + J[\psi, \nabla_\perp^2 \psi] - Dw = \nabla_\perp^4 \psi$$

$$\partial_t \theta + J[\psi, \theta] + wD\bar{\theta} = \sigma^{-1} \nabla_\perp^2 \theta$$

$$\partial_T \bar{\theta} + D\left(\overline{w\theta}\right) = \sigma^{-1} D^2 \bar{\theta}$$

Adding Multiple Scales

The UNH-QGE describe the plume scale, but aren't well suited to examining the dynamics of the ocean gyres where deep convection takes place. In order to capture the evolution of the gyre and the baroclinic eddies that form along its edge, we add another horizontal scale.

Adding Multiple Scales

$$\nabla \rightarrow \nabla_{\perp} + A_{\perp} \bar{\nabla}_{\perp} + A_z \hat{z} D$$

$$\partial_t \rightarrow \partial_t + A_T \partial_T$$

$$\bar{f} = \lim_{t, V \rightarrow \infty} \frac{1}{t V} \int f \, dt \, dV$$

$$f(X, Y, Z, T, x, y, t) = \bar{f}(X, Y, Z, T) + f'(X, Y, Z, T, x, y, t)$$

$$\overline{f'} = 0$$

$$\overline{f g} = \bar{f} \bar{g} + \overline{f' g'}$$

Adding Multiple Scales: The Mean Equations

$$A_T \partial_T \bar{\boldsymbol{u}} + A_\perp \bar{\nabla}_\perp \cdot (\overline{\boldsymbol{u}\boldsymbol{u}}) + A_z D(\overline{w\boldsymbol{u}}) + \mathrm{Ro}^{-1} \hat{z} \times \bar{\boldsymbol{u}} =$$
$$-P(A_\perp \bar{\nabla}_\perp + A_z \hat{z} D)\bar{p} + \Gamma \bar{\theta} \hat{z} + \mathrm{Re}^{-1}(A_\perp \bar{\nabla}_\perp + \hat{z} A_z D)^2 \bar{\boldsymbol{u}}$$

$$A_T \partial_T \bar{\theta} + A_\perp \bar{\nabla}_\perp \cdot (\overline{\theta \boldsymbol{u}}) + A_z D(\overline{w\theta}) = \mathrm{Pe}^{-1}(A_\perp \bar{\nabla}_\perp + \hat{z} A_z D)^2 \bar{\theta}$$

$$A_\perp \bar{\nabla}_\perp \cdot \bar{\boldsymbol{u}} + A_z D \bar{w} = 0$$

Adding Multiple Scales: The Distinguished Limit

The choice of relationships between the various small parameters is called a distinguished limit. We make this choice consistently with observations of the phenomena we wish to describe, but also for the purpose of arriving at a closed system of equations. In order for the mean equations to be in geostrophic and hydrostatic balance, and to have feedbacks to and from the fluctuating equations, we choose as follows.

$$\mathrm{Ro} \equiv \epsilon,\ A_T \sim A_\perp \sim \epsilon^2,\ A_z \sim \epsilon,\ P \sim \epsilon^{-3},\ \Gamma \sim \epsilon^{-2}\bar{\Gamma}$$

Adding Multiple Scales: Asymptotic Series

To arrive at a closed system for the mean and fluctuating velocities and potential temperature, it is not sufficient to simply examine the leading order balances (as we have seen in the derivation of both traditional QG and of the UNH-QGE). We must also look at corrections to leading order balances by using asymptotic series.

$$\bar{\boldsymbol{u}} = \bar{\boldsymbol{u}}_0 + \epsilon\, \bar{\boldsymbol{u}}_1 + O(\epsilon^2) \qquad \boldsymbol{u}' = \boldsymbol{u}_{\boldsymbol{0}}' + \epsilon\, \boldsymbol{u}_{\boldsymbol{1}}' + O(\epsilon^2)$$

$$\bar{\theta} = \bar{\theta}_0 + \epsilon\, \bar{\theta}_1 + O(\epsilon^2) \qquad \theta' = \theta_0' + \epsilon\, \theta_1' + O(\epsilon^2)$$

Adding Multiple Scales: The Final Result

Using the idea of an asymptotic series, balancing the equations order by order, and imposing solvability conditions now allow us to derive equations for the mean and fluctuating flows (not shown). The result links the Planetary Geostrophic Equations describing the mean flow to the UNH-QGE describing the fluctuations.

Adding Multiple Scales: The Final Result

Fluctuating equations:

$$\partial_t w' + \bar{\boldsymbol{u}} \cdot \nabla_\perp w' + J[\psi', w'] + D\psi' = \bar{\Gamma}\sigma^{-1}\theta' + \nabla_\perp^2 w'$$

$$\partial_t \nabla_\perp^2 \psi' + \bar{\boldsymbol{u}} \cdot \nabla_\perp (\nabla_\perp^2 \psi') + J[\psi', \nabla_\perp^2 \psi'] - D w' = \nabla_\perp^4 \psi'$$

$$\partial_t \theta' + \bar{\boldsymbol{u}} \cdot \nabla_\perp \theta' + J[\psi', \theta'] + w' D\bar{\theta} = \sigma^{-1}\nabla_\perp^2 \theta'$$

Mean Equations:

$$\partial_T \bar{\theta} + \bar{\boldsymbol{u}} \cdot \bar{\nabla}_\perp \bar{\theta} + D(\overline{w'\theta'}) = \sigma^{-1} D^2 \bar{\theta}$$

$$\hat{z} \times \bar{\boldsymbol{u}} = -\bar{\nabla}_\perp \bar{p}$$

$$D\bar{p} = \bar{\Gamma} D\bar{\theta}$$

$$\bar{w} = 0$$

Several Equation Sets: PGE + QGE

Fluctuating Equations:

$$\partial_t q' + \bar{\boldsymbol{u}} \cdot \nabla_\perp q' + J[\psi', q'] = 0$$

$$q' = \nabla_\perp^2 \psi' - \boldsymbol{f}(Y) D \left(\frac{\theta'}{D\bar{\theta}} \right)$$

$$p' = \boldsymbol{f}\, \psi', \quad \boldsymbol{u}' = -\nabla \times (\psi' \hat{z}), \quad \bar{\Gamma}\, \theta' = \boldsymbol{f}\, D \psi'$$

Mean Equations

$$\partial_T \bar{\theta} + \bar{\boldsymbol{u}} \cdot \bar{\nabla}_\perp \bar{\theta} + \bar{w} D \bar{\theta} + \bar{\nabla}_\perp \cdot (\overline{\boldsymbol{u}'\theta'}) =$$

$$D\left((D\bar{\theta})^{-1} (\overline{\theta' \boldsymbol{u}'}) \cdot \bar{\nabla}_\perp \bar{\theta} + D\bar{\theta} \right)$$

$$\bar{\Gamma}\, \bar{\theta} = D \bar{p}$$

$$\boldsymbol{f} \times \bar{\boldsymbol{u}} = -\bar{\nabla}_\perp \bar{p}$$

$$\bar{\nabla}_\perp \cdot \bar{\boldsymbol{u}} + D \bar{w} = 0$$

Several Equation Sets: QGB + QGE

Fluctuating Equations:
The UNH-QGE

Mean Equations:

$$\partial_T \bar{\omega} + \bar{\boldsymbol{u}} \cdot \bar{\nabla}_\perp \bar{\omega} = -\hat{z} \cdot \left[\bar{\nabla}_\perp \times \bar{\nabla}_\perp \int_0^1 \left(\overline{\boldsymbol{u}'\boldsymbol{u}'} \right) dz \right]$$

$$\partial_T \bar{\theta} + \bar{\boldsymbol{u}} \cdot \bar{\nabla}_\perp \bar{\theta} + D \left(\overline{w'\theta'} \right) = D^2 \bar{\theta}$$

$$\bar{\omega} = \hat{z} \cdot \left[\bar{\nabla}_\perp \times \bar{\boldsymbol{u}} \right] = -\bar{\nabla}_\perp^2 \bar{p}$$

$$D \bar{p} = 0$$

Future Work

- Do some computations
- Add the effects of wind and topography
- Do something about the equator (PGE break down at the equator)