Reduced Modeling of the Magnetorotational Instability

Ben Jamroz

Advisor: Keith Julien
Department of Applied Mathematics
University of Colorado at Boulder

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Outline

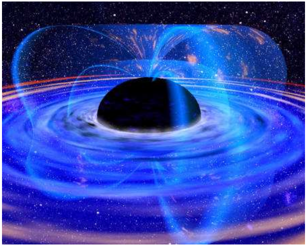
Accretion is a Fundamental Astrophysical Process

Accretion is the process by which a massive object collects surrounding matter by gravitation. Accretion disks are observed in many astrophysical processes (binary star systems, center of galaxies).

Keplerian balance in mom. eqn.

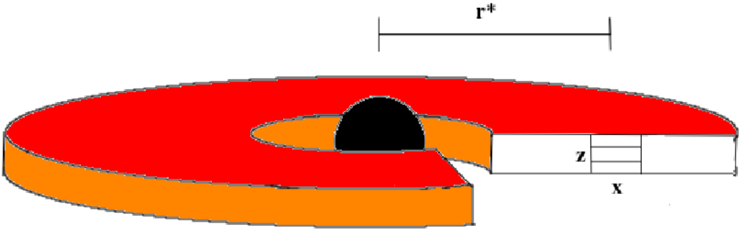
- $\Omega^2 = GM/r^3$

Angular momentum transport needed!



Geometry

- Astrophysical Disks: Differential Rotation
  - Keplerian balance: $\Omega^2 = \frac{GM}{r^3}$
- Laboratory: Taylor-Couette Geometry
  - Imposed $\Omega(r_1), \Omega(r_2)$
- Shearing Sheet - Simplified Geometry



Keplerian Disks are Hydrodynamically Stable

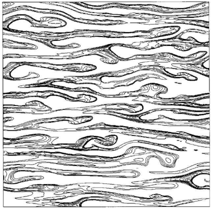
- Linear stability for the astrophysical regime
  - Rayleigh stable $\frac{d}{dr}(r^2\Omega) < 0$
- Finite amplitude disturbance can cause instability
- Turbulent mixing - Does it produce the needed angular momentum transport?
  - ν_{turb} large ("eddy") viscosity
  - Experiments - Hydrodynamic turbulence cannot transport angular momentum effectively in astrophysical disks (Ji et. al. Nature 2006)

Magnetorotational Instability

Shear instability in the presence of magnetic fields

- First discovered by Velikhov (1959) and Chandrasekhar (1960)
- Rediscovered by Balbus and Hawley (1991)
  - MRI generates the level of angular momentum transfer needed
- Operates in the Rayleigh stable regime $\frac{d}{dr}(r^2\Omega) < 0$
- Axisymmetric
- Linear instability

Magnetorotational Instability



- Balbus & Hawley's original simulation of the MRI
  - Contour plot of angular momentum perturbations (r, z)
- Transfer of angular momentum
- Large vertical gradient of the "fingering instability" motivates scalings

Previous Work

- Balbus and Hawley (1991)
  - Efficient transfer of angular momentum
  - Didn't include dissipative processes - viscous or ohmic
  - No Saturation
- Sano et. al. (1998)
  - First to show saturation of this instability
  - Compressible flow
  - Resistivity and viscosity included
  - Critical parameter, Elsasser number $\Lambda = v_A^2/\Omega\eta$
    - $\Lambda > 1$ Saturation
    - $\Lambda < 1$ No saturation
- Goodman and Xu (1994)
  - Nonaxisymmetric perturbations can saturate
- Lathrop group
  - First experimental observation of MRI (Sisan et. al. 2004)

Full Equations

Conservation of Momentum

$$\frac{D\boldsymbol{u}}{Dt} + 2\boldsymbol{\Omega} \times \boldsymbol{u} = -\frac{1}{\rho}\nabla p - \frac{1}{2\mu_0\rho}\nabla|\boldsymbol{B}|^2 + \frac{1}{\mu_0\rho}\boldsymbol{B}\cdot\nabla\boldsymbol{B} + \nu\nabla^2\boldsymbol{u},$$

Induction Equation

$$\frac{D\boldsymbol{B}}{Dt} = \boldsymbol{B}\cdot\nabla\boldsymbol{u} + \eta\nabla^2\boldsymbol{B}$$

Incompressibility, Solenoidal Condition

$$\nabla\cdot\boldsymbol{u} = 0, \nabla\cdot\boldsymbol{B} = 0$$

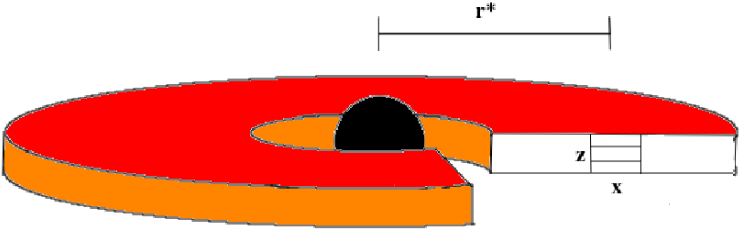
Reduced Modeling

Astrophysical Regime - Large Parameters

- $\nu \ll 1, \eta \ll 1$
  - $R_e = \frac{UL}{\nu}$, $R_m = \frac{UL}{\eta} \gg 1$
- Several different time scales:
  - Rotational time scale Ω^{-1}
  - Alfven time scale L/v_A^2
  - Diffusive time scale $L^2/\nu, L^2/\eta$
  - $\Omega^{-1} \ll L/v_A \ll L^2/\nu, L^2/\eta$
- To reach saturation we must integrate far in time

Shearing Sheet

- Shearing sheet approximation at r^* with local angular velocity $\Omega^*(r^*)\hat{z}$
- Local Cartesian coordinates $(r, \phi, z) \sim (x, y, z)$
- Straight channel: $-L^*/2 \leq x^* \leq L^*/2, -\infty < y^* < \infty, -\infty < z^* < \infty$
- Linear Shear: $U_0^* = (0, \sigma^* x^*, 0)$
- Constant Background Magnetic field: $B_0^* = (0, B_{tor}^*, B_{pol}^*)$



Formulation of the Model Problem

- Non-dimensionalize
- $v_A = B^*_{pol}/\sqrt{\mu_0 \rho^*} U^*$, Ω, ν, η are the dimensionless Alfven speed, rotation rate, kinematic viscosity and ohmic diffusivity
- $\nabla \cdot \boldsymbol{u} = 0$, $\nabla \cdot \boldsymbol{b} = 0$, axisymmetry allow the use of a stream function and flux funtion
  - $\boldsymbol{u} = (-\psi_z, v, \psi_x), \boldsymbol{b} = (-\phi_z, b, \phi_x)$

Nondimensionalized Equations

Axisymmetric perturbations of the form $\boldsymbol{u} = (u, v, w) = (-\psi_z, v, \psi_x)$, $\boldsymbol{b} = (-\phi_z, b, \phi_x)$ give the following equations

$$
\begin{aligned}
\nabla^2\psi_t + 2\Omega v_z + J(\psi, \nabla^2\psi) &= v_A^2\nabla^2\phi_z + v_A^2 J(\phi, \nabla^2\phi) + \nu\nabla^4\psi. \\
v_t - (2\Omega + \sigma)\psi_z + J(\psi, v) &= v_A^2 b_z + v_A^2 J(\phi, b) + \nu\nabla^2 v, \\
\phi_t + J(\psi, \phi) &= \psi_z + \eta\nabla^2\phi, \\
b_t + J(\psi, b) &= v_z - \sigma\phi_z + J(\phi, v) + \eta\nabla^2 b
\end{aligned}
$$

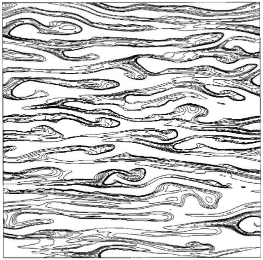
Here, $J(f, g) = f_x g_z - f_z g_x$.

Scaling Assumptions

- Traditional approach to nonlinear saturation: weakly nonlinear theory with $(\Lambda - \Lambda_c)/\Lambda_c \ll 1$.
  - Our approach: strongly nonlinear theory
- Shear is the dominant source of the energy for the MRI
  - rapid rotation, strong shear: $(\Omega, \sigma) = \delta^{-1}(\hat{\Omega}, \hat{\sigma})$
- MRI itself requires the presence of a (weaker) vertical magnetic field
  - magnetic field: $v_A \sim 1$, $v_A^* \equiv B_{pol}^*/\sqrt{\mu_0 \rho^*}U^*$
- Dissipative effects are weaker still but cannot be ignored since they are ultimately responsible for the saturation of the instability
  - weak dissipative processes: $(\nu, \eta) = \epsilon(\hat{\nu}, \hat{\eta})$
- Take $\epsilon \sim \delta \ll 1, \Lambda = \mathcal{O}(1)$ (Case A) or $\epsilon \ll \delta \ll 1, \Lambda \gg 1$ (Case B)
  - $Rm = |\sigma^*|L^{*2}/\eta^*$, $Pm = \nu^*/\eta^*$, $S = v_A^* L^*/\eta^*$
  - so $Rm \gg S \gg \max(1, Pm)$, while $\Lambda = O(1)$

Multiple Scales Expansion

- Motivated by Balbus & Hawley
- Large wavenumber in z - large variation
  - $\partial_z = \epsilon^{-1}\partial_z$
- Have large gradients in x direction
- Fast dynamic time scale, slow evolution to a statistically steady state
  - $\partial_t = \epsilon^{-1}\partial_t + \partial_T$
- **Set** ϵ by considering the size of our domain
  - L_z wavelength of fastest growing mode - Linear Dispersion
  - $L_X \sim \epsilon^{-1} L_z$



Asymptotic Expansion

- To solve the scaled equations we expand every variable

$$\psi(x, X, z, t, T) = \sum_{i,j} \epsilon^{\frac{i}{2}} \delta^{\frac{j}{2}} \psi_{ij}(x, X, z, t, T) + \ldots,$$

- Deduction: Leading order azimuthal fields v_{00}, b_{00} represent large-scale adjustment to background shear and toroidal field due to MRI
- Separate all variables into their mean and fluctuating components
  - $\psi_{ij}(x, X, z, t, T) = \overline{\psi}_{ij}(X, T) + \psi'_{ij}(x, X, z, t, T)$
  - $\overline{\psi}_{ij}(X, T) \equiv \lim_{\tau, V \to \infty} \frac{1}{\tau V} \int_{\tau, V} \psi_{ij}(x, X, z, t, T) dxdzdt$
- We now collect terms at each order in the evolution equations

Reduced Fluctuating Equations

At $O(\epsilon^{\frac{-1}{2}}\delta^{\frac{-1}{2}})$

$$\overline{\nabla}^2\psi'_{00t} + 2\widehat{\Omega}v'_{11z} + \left(\frac{\epsilon}{\delta}\right)^{\frac{1}{2}} J_x(\psi'_{00}, \overline{\nabla}^2\psi'_{00}) = v_A^2\overline{\nabla}^2\phi'_{00z} + v_A^2\left(\frac{\epsilon}{\delta}\right)^{\frac{1}{2}} J_x(\phi'_{00}, \overline{\nabla}^2\phi'_{00}) + \left(\frac{\epsilon}{\delta}\right)\widehat{\nu}\overline{\nabla}^4\psi'_{00}, \quad (1)$$

$$v'_{11t} - \left(2\widehat{\Omega} + \widehat{\sigma} + \partial_X\overline{v}_{00}\right)\partial_z\psi'_{00} + \left(\frac{\epsilon}{\delta}\right)^{\frac{1}{2}} J_x(\psi'_{00}, v'_{00}) = v_A^2\left(b'_{11z} - \partial_X\overline{b}_{00}\phi'_{00z} + \left(\frac{\epsilon}{\delta}\right)^{\frac{1}{2}} J_x(\phi'_{00}, b'_{00})\right) + \left(\frac{\epsilon}{\delta}\right)\widehat{\nu}\overline{\nabla}^2 v'_{11}, \quad (2)$$

$$\phi'_{00t} + \left(\frac{\epsilon}{\delta}\right)^{\frac{1}{2}} J_x(\psi'_{00}, \phi'_{00}) = \psi'_{00z} + \left(\frac{\epsilon}{\delta}\right)\widehat{\eta}\overline{\nabla}^2\phi'_{00}, \quad (3)$$

$$b'_{11t} - \partial_X\overline{b}_{00}\psi'_{00z} + \left(\frac{\epsilon}{\delta}\right)^{\frac{1}{2}} J_x(\psi'_{00}, b'_{11}) = v'_{11z} - \left(\widehat{\sigma} + \partial_X\overline{v}_{00}\right)\phi'_{00z} + \left(\frac{\epsilon}{\delta}\right)^{\frac{1}{2}} J_x(\phi'_{00}, v'_{11}), + \left(\frac{\epsilon}{\delta}\right)\widehat{\eta}\overline{\nabla}^2 b'_{11}. \quad (4)$$

Feedback onto the Shear

- For both sets of scalings ($\epsilon \ll \delta$ & $\epsilon \sim \delta$) there is feedback onto the imposed shear
- Readjustment of the local background state
  - $(\widehat{\sigma} + \partial_X \overline{v}_{00})$ terms
  - Tends towards solid body rotation, $|\widehat{\sigma} + \partial_X \overline{v}_{00}| < |\widehat{\sigma}|$
  - Angular momentum must have been transported
- At $\mathcal{O}(\epsilon\delta)$ we close this system and get relations for $\partial_X \overline{v}_{00}$ and $\partial_X \overline{b}_{00}$

$$\hat{\nu}\partial_X \overline{v}_{00} = -\overline{\psi_{00z} v_{11}} + v_A^2 \overline{\phi_{00z} b_{11}}$$

$$\hat{\eta}\partial_X \overline{b}_{00} = -\overline{\psi_{00z} b_{11}} + \overline{\phi_{00z} v_{11}}$$

- $\partial_X \overline{b}_{00}$ is the $O(1)$ correction to the background toroidal field B_{tor}

Single Mode Theory

- Goodman and Xu (1994) & Julien and Knobloch (2006)
  - Solutions
- $\psi'_{00} = e^{\lambda t}\cos(nz)\hat{\psi}'_{00}$, $v'_{11} = e^{\lambda t}\sin(nz)\hat{v}'_{11}$, $\phi_{00} = e^{\lambda t}\sin(nz)\hat{\phi}'_{00}$, $b_{11} = e^{\lambda t}\cos(nz)\hat{b}_{11}$
  - No variation in x
  - Nonlinear ODE $\rightarrow$ evolves to saturation
  - Does not work for a combination of such modes

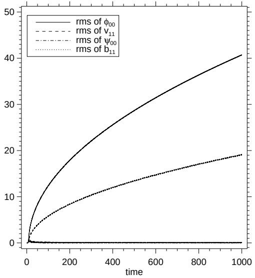
Modeling the Reduced Equations

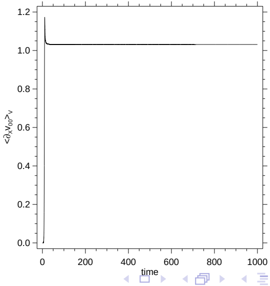
Numerical Method

- Time integration: Runge-Kutta scheme (Spalart et. al. 1991)
  - diffusion terms treated implicitly
  - all other explicitly
- Spectral
  - Periodic boundary conditions in z
  - Rigid (no slip), Stress Free, or Periodic in x

Case B: $\epsilon \ll \delta \ll 1, \Lambda \gg 1$

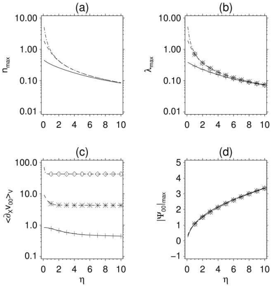
- Nonlinear and dissipative terms are subdominant
- Unbounded algebraic growth at leading order
- Saturation of $\overline{v}_{00}$
    - Decaying and growing terms $\longrightarrow$ steady product
- No quadratic nonlinearities - energy transferred through modification of the dispersion relation





Case A: $\epsilon \sim \delta, \Lambda = \mathcal{O}(1)$

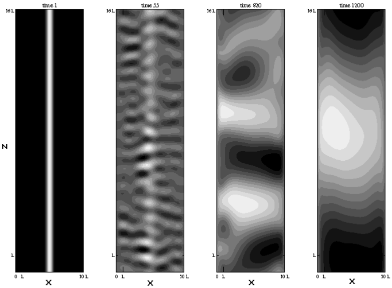
Single Mode results



- Good agreement with linear theory for a robust set of boundary conditions
- In particular the saturated value of $\partial_X \overline{v}_{00}$ matches the theory
- Can be of use in a parameterization model

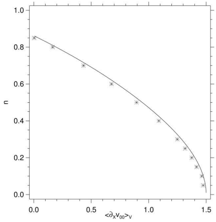
Case A: Channel Initial Condition

- ϕ'_{00} Gaussian in x, vertical magnetic field lines
- Small random perturbation in ψ'_{00}
- $v'_{11} = b'_{11} = 0$
- Critical wavenumber grows
- Saturated state takes up the largest allowed scale in z (coarsening)



Coarsening

- Begin from a random state with a fixed box length in z
- Fastest growing linear mode dominates early evolution
- Flow coarsens to fill computational domain
- The saturated value of $\partial_X \overline{v}_{00}$ matches the single mode theory for a mode with corresponding vertical wavenumber
    - Suggest using a parameterization model for large scale simulations
- Comparison with Case B



Summary & Future Work

- Derived reduced asymptotic models for the saturation of the MRI
- Found a back reaction on the imposed shear which allows for saturation
- Numerical results

- Future Work
  - Parameterization model
  - Non-axisymmetric
    - Non-axisymmetric saturation
    - Dynamo $\longrightarrow$ saturation