Vertically Sheared Horizontal Flow with Mass Sources: A Canonical Balanced Model

Andrew J. Majda, Majid Mohammadian and *Yulong Xing*

Courant Institute of Mathematical Sciences
New York University

1: Vertically Sheared Horizontal Flow with Mass Sources: A Canonical Balanced Model, to appear in Geophysical and Astrophysical Fluid Dynamics (GAFD).
2: Multiscale equations for the hurricane embryo in a WTG environment, in preparation.

Introduction

Two common assumptions in the development of multi-scale models:

- horizontal weak temperature gradient (WTG) approximation for potential temperature

$$\Theta = \overline{\Theta}(z) + \epsilon\theta(\mathbf{x}_h, z, t), \qquad \epsilon \ll 1,$$

- low Froude number approximation for the horizontal flow U_h

$$\mathbf{U}_h = \epsilon \mathbf{u}_h \ , \qquad \epsilon \ll 1.$$

$\epsilon \cong \frac{1}{10}$ to $\frac{1}{7}$ are typical observed values for the lower/middle troposphere

Introduction

With the above background, the goal here is to study the following canonical balanced model.

Vertically sheared horizontal flow with mass (heat) sources (VSHFS):

$$\begin{aligned} \frac{D\mathbf{u}_h}{Dt} + f u_h^{\perp} &= -\nabla_h p + \mathbf{S}_{\mathbf{u}}, \\ div\mathbf{u}_h + w_z &= 0, \\ wN^2(z) &= S_{\theta}, \end{aligned} \tag{1.3}$$

Introduction

The equations in (1.3) arise in a variety of multiple spatial scale balanced dynamics for the tropics:

- on horizontal scales of order 1500 km and time scales of order 8 hours (see the BMESD model in Majda 2007b);
- on horizontal scales of order 10 km and time scales of order 15 minutes (Klein 2000; Klein and Majda 2006);
- with the beta plane approximation, $f = \beta y$, on horizontal scales of order 800 km and time scales on the order of 1 day (Sobel et al. 2001; Majda and Klein 2003);
- on seasonal planetary scales (see SPEWTG model in Majda and Klein 2006).

The Canonical Balanced Model

The forced Boussinesq equations takes the non-dimensional form:

$$
\begin{aligned}
\frac{D\mathbf{u}_h}{Dt} + (\mathrm{Ro})^{-1}\,\mathbf{u}_h^{\perp} &= -\nabla_h p + \mathbf{S}_{\mathbf{u}}, \\
\frac{Dw}{Dt} &= -\frac{\partial p}{\partial z} + \epsilon^{-1}\theta + S_w, \\
\frac{D\theta}{Dt} &= \epsilon^{-1}(-w + S_\theta), \\
div_h \mathbf{u}_h + w_z &= 0.
\end{aligned}
\tag{2.2}
$$

under the conditions:

1: WTG approximation,
2: low Froude number,
3: comparable horizontal and vertical velocity magnitudes,
4: large Rossby number $\mathrm{Ro}= \frac{LV}{f} \geq O(1)$,

Units: [**x**] = [z] = 10 *km*, [t] = 15 *minutes*, [❑] = 3 *K*,
[**u**] = [w] = 10 *m/s*, strong heating: 120 *K/hr*.

Key Unsolved Questions for Hurricane Embryo:

What preconditioning background environments (Shear, Vorticity, Temp, Moisture) Lead to Tropical Cyclogenesis?

H E - Stage

O(10 m/s) winds

Hot Towers O(10 km), 120 K/hr

(*) Mesovortices of T.C. O(100 km)

What is involved in creating (*)?

Hot Towers, Montgomery J P, 2004 -

Moist Thermodynamics, Bister - Emanuel

The Canonical Balanced Model

The derivation of the canonical model (1.3) is straightforward. They are

- the leading order ϵ^0 equations for horizontal momentum and mass conservation
- the leading order ϵ^{-1} equations for the potential temperature.

If the temperature perturbation θ is expanded as $\theta = \epsilon\theta_1$, then θ_1 is determined from the solution of (1.3) as given by

$$\theta_1 = \frac{Dw}{Dt} + \frac{\partial p}{\partial z} - S_w.$$

The Canonical Balanced Model

I) The canonical models in (1.3) has direct relevance for the troposphere with

horizontal scales of order 10 km and time scales of order 15 minutes;

Hot Tower Scales

II) It also apply on

horizontal scales of order 100 km and time scales of order 2.5 hours;

These time scales are relevant for the formation of mesovortices in the hurricane embryo.

The Canonical Balanced Model

To establish this fact, introduce the aspect ratio A = H/L , A ≤ 1 and the new rescaled variables

$$
\begin{aligned}
T &= At, \quad X = Ax_h, \\
w &= Aw_A, \\
(\mathrm{Ro})_A &= A\mathrm{Ro}, \\
AS_{\theta,A} &= S_\theta, \\
A\mathbf{S}_{\mathbf{u},A} &= \mathbf{S}_\mathbf{u}.
\end{aligned}
$$

Note: Horizontal Velocity, $u = u_A(X, T)$ still has units $[u] \cong 10\,m/s$

The Canonical Balanced Model

With these rescaling, the equations in (2.2) become

$$\begin{aligned}
\frac{D\mathbf{u}_h}{DT} + (\mathrm{Ro})_A^{-1}\mathbf{u}_h^\perp &= -\nabla_h p + S_{\mathbf{u},A}, \\
A^2\frac{Dw_A}{Dt} &= -\frac{\partial p}{\partial z} + \epsilon^{-1}\theta + S_w, \\
\frac{D\theta}{DT} &= \epsilon^{-1}(-w_A + S_{\theta,A}), \\
div_X \mathbf{u}_h + (w_A)_z &= 0.
\end{aligned} \tag{2.6}$$

The same derivation can be repeated now for any A with $A \ll 1$ to yield (1.3) as a canonical balanced model provided that $(\mathrm{Ro})^{-1}{}_A$ remains finite.

Choose $A = \varepsilon$ to have rotation important for 100 km spatial scales & 2.5 hrs.

Vertical Vorticity Dynamics

Vertically sheared horizontal flow with mass (heat) sources (VSHFS):

$$\begin{aligned} \frac{D\mathbf{u}_h}{Dt} + f u_h^{\perp} &= -\nabla_h p + \mathbf{S}_{\mathbf{u}}, \\ div\mathbf{u}_h + w_z &= 0, \\ wN^2(z) &= S_\theta, \end{aligned} \tag{1.3}$$

Vertical Vorticity Dynamics

Use the horizontal Helmholtz decomposition

$$\mathbf{u}_h = \nabla_h \Phi + \nabla_h^{\perp} \Psi + \mathbf{b}(z,t),$$

where Ψ is the stream function, Φ is the velocity potential, and $\mathbf{b}(z,t)$ is the specified background shear.

Vertical Vorticity Dynamics

By taking curl$_h$ of (1.3), we have the **Vertical Vorticity Dynamic Equation:**

$$\frac{D\omega}{Dt} = \underbrace{(\omega + f)\,(S_\theta)_z}_{\text{stretching}} + \underbrace{\left(\frac{\partial}{\partial z}\mathbf{u}_h^{\perp}\right) \cdot \nabla_h S_\theta}_{\text{tilt}} + curl_h \mathbf{S}_{\mathbf{u}}.$$

It can be decomposed, using Helmholtz decomposition, as:

$$\frac{D\omega}{Dt} = (\omega + f) w_z - \nabla_h^{\perp} w \cdot \frac{\partial \mathbf{b}}{\partial z} - \nabla_h^{\perp} w \cdot \frac{\partial \nabla_h \Phi}{\partial z} - \nabla_h^{\perp} w \cdot \frac{\partial \nabla_h^{\perp} \Psi}{\partial z} + curl_h \mathbf{S}_{\mathbf{u}}.$$

We investigate this equation in this presentation.