Adjoint Sensitivity Analysis for Two Layered Shallow Water Equations

Vani Cheruvu

Department of Scientific Computing
Florida State University
Collaborator: Dr. Joe Tribbia, NCAR

August 18, 2009

Organization of the talk

- Motivation and Literature Survey
- Direct Sensitivity Method
- Adjoint Sensitivity Method
- Adjoint Sensitivity Analysis for two layered shallow water equations
  - Shallow water approximation for two layered sea
  - Normal form and its adjoint
  - Standard Adjoint Formalism
  - Characteristic Adjoint Formalism
- Summary

Motivation and Literature Survey

- J.J. Stoker *Water Waves The Mathematical Theory with Applications* Pure and Applied Mathematics Vol. IV (1957).
- N.D. Katopodes *Two dimensional unsteady flow through a breached dam by the method of characteristics*, Ph.D. Thesis (1977).
- Marchuk *Adjoint Equations and Analysis of Complex Systems* (1995).
- B.F. Sanders *Control of shallow-water flow using the adjoint sensitivity method*, Ph.D. Thesis (1997).
- B. F. Sanders and N.D. Katopodes *Adjoint sensitivity analysis for shallow-water wave control* J of Engineering Mechanics (2000).
- P. Concus, G.H. Golub and Y. Sun *Object-oriented parallel algorithms for computing three-dimensional isopycnal flow* International Journal for Numerical Methods in Fluids (2002).
- A. Rousseau, R. Temam and J. Tribbia *Numerical simulations of the inviscid primitive equations in a limited domain* Advances in Mathematical Fluid Dynamics (2006).

Why an Adjoint Equation?

Diffusion equation defined in the domain $x \in (0, L)$ is

$$D\frac{\partial^2 C}{\partial x^2} + f = 0,\ C(0) = C(L) = 0$$

Objective function :

$$J = \int_0^L r(x)C(x)\,dx$$

where $r = \delta(x - x_T)$.

Adjoint Equation

$$D\frac{\partial^2 C^*}{\partial x^2} + r = 0$$

Objective function:

$$J = \int_0^L f(x)C^*(x)\,dx$$

where $f = M\delta(x - x_s)$.

Derivation

$$\int_0^L C^* \left[D \frac{\partial^2 C}{\partial x^2} + f \right] dx = D \left[C^* \frac{\partial C}{\partial x} \right]_0^L - D \left[\frac{\partial C^*}{\partial x} C \right]_0^L + \int_0^L \left[CD \frac{\partial^2 C^*}{\partial x^2} + C^* f \right] dx = 0$$

$$J = \int_0^L \left[CD \frac{\partial^2 C^*}{\partial x^2} + C^* f \right] dx + \int_0^L rCdx$$

$$J = \int_0^L C \left[D \frac{\partial^2 C^*}{\partial x^2} + r \right] dx + \int_0^L fC^* dx$$

Shallow Water Equations

$$\begin{aligned} \frac{\partial \eta}{\partial t} + \frac{\partial (Hu' + \bar{U}\eta)}{\partial x} &= 0 \\ \frac{\partial u'}{\partial t} + \frac{\partial (\bar{U}u' + g\eta)}{\partial x} &= 0 \end{aligned}$$

Objective Function

$$J = \int_0^T \int_0^L r(\eta, u'; x, t) dxdt$$

t= time; x = distance along the channel; η = flow perturbation; u' = discharge per unit width; r = user-defined measuring function

$$r = \frac{1}{2} \left[\eta(x_0, t) - \bar{\eta}(x_0)\right]^2 \delta(x - x_0)$$

Direct Sensitivity Method

- The basic problem is solved
- A reference value of the objective function J_0 is computed
- A flow variable is perturbed at the boundary at the selected perturbation time
- The associated objective function value is computed again
- This procedure is repeated for sufficiently many selections of time
- This yields the temporal evolution of the sensitivities

Adjoint Sensitivity Method

Derivation of 1D Adjoint Equations

- Adjoint variables : $\phi(x, t)$ and $\psi(x, t)$
- Multiply the continuity equation by $\phi(x, t)$ and the momentum equation by $\psi(x, t)$
- The sum of these two products is then integrated over space and time
- Flow control is desired
- Variation of the objective function with respect to η and u' is calculated
- Add these two variations
- Assign initial condition for the adjoint problem
- Assign boundary values for the adjoint variables

Sensitivity Equations

- Sensitivities required for subcritical flow control are given by $\frac{\delta J}{\delta u'(0,t)}$ and $\frac{\delta J}{\delta \eta(L,t)}$
- Sensitivities required for supercritical flow control are given by $\frac{\delta J}{\delta u'(0,t)}$ and $\frac{\delta J}{\delta \eta(0,t)}$
- To derive expressions for sensitivities in terms of the adjoint variables
- Initial condition for the basic problem $\eta(x,0) = \eta_0(x), u'(x,0) = u'_0(x); \delta\eta(x,0) = 0, \delta u'(x,0) = 0$
- Initial conditions in the adjoint problem $\phi(x, T) = 0 = \psi(x, T)$
- Selection of boundary conditions for the adjoint problem is motivated by the boundary conditions of the basic problem
- Expressions are then derived for sensitivities to these boundary values

Shallow water equations and its Adjoint

$$\begin{aligned}
\frac{\partial \eta}{\partial t} + \bar{U}\frac{\partial \eta}{\partial x} + \bar{V}\frac{\partial \eta}{\partial y} + H\frac{\partial u'}{\partial x} + H\frac{\partial v'}{\partial y} &= 0 \\
\frac{\partial u'}{\partial t} + \bar{U}\frac{\partial u'}{\partial x} + \bar{V}\frac{\partial u'}{\partial y} + g\frac{\partial \eta}{\partial x} &= 0 \\
\frac{\partial v'}{\partial t} + \bar{U}\frac{\partial v'}{\partial x} + \bar{V}\frac{\partial v'}{\partial y} + g\frac{\partial \eta}{\partial y} &= 0
\end{aligned}$$

Adjoint

$$\begin{aligned}
\frac{\partial \phi}{\partial \tau} - \bar{U}\frac{\partial \phi}{\partial x} - \bar{V}\frac{\partial \phi}{\partial y} - g\frac{\partial \psi_1}{\partial x} - g\frac{\partial \psi_2}{\partial y} + \frac{\partial r}{\partial \eta} &= 0 \\
\frac{\partial \psi_1}{\partial \tau} - \bar{U}\frac{\partial \psi_1}{\partial x} - \bar{V}\frac{\partial \psi_1}{\partial y} - H\frac{\partial \phi}{\partial x} + \frac{\partial r}{\partial u'} &= 0 \\
\frac{\partial \psi_2}{\partial \tau} - \bar{U}\frac{\partial \psi_2}{\partial x} - \bar{V}\frac{\partial \psi_2}{\partial y} - H\frac{\partial \phi}{\partial y} + \frac{\partial r}{\partial v'} &= 0
\end{aligned}$$

Nonreflective Outflow Boundary

Adjoint boundary conditions:

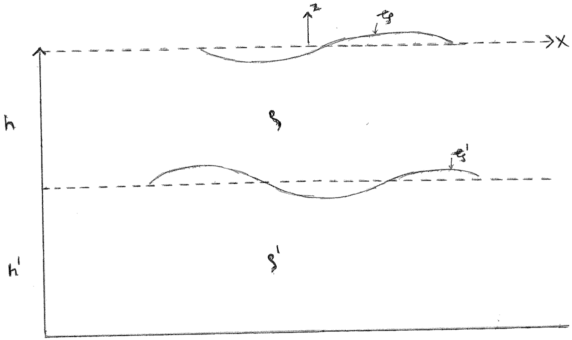
$\psi(0,t) = 0$ and

$$\phi(L,t) + 2\frac{u'(L,t)}{\eta(L,t)}\psi(L,t) = 0$$

Sensitivities for the subcritical flow then are:

$$\begin{aligned}
\frac{\delta J}{\delta u'(0,t_p)} &= \phi(0,t_p) \\
\frac{\delta J}{\delta \eta(L,t_p)} &= \left[\left(\frac{u'(L,t_p)}{\eta(L,t_p)}\right)^2 - g\eta(L,t_p)\right]\psi(L,t_p)
\end{aligned}$$

A two layered sea of constant depth



Two layered Shallow water model

The two layered shallow water model is given by,

$$\begin{aligned}\frac{\partial \eta_1}{\partial t} + \frac{\partial}{\partial x}\left[(\eta_1 - \eta_0)u_1\right] &= 0\\ \frac{\partial u_1}{\partial t} + \frac{\partial}{\partial x}\left[\frac{u_1^2}{2} + ga\eta_1 + gb\eta_2\right] &= 0\end{aligned}$$

for the lower layer and

$$\begin{aligned}\frac{\partial \eta_2}{\partial t} + \frac{\partial}{\partial x}\left[(\eta_1 - \eta_0)u_1 + (\eta_2 - \eta_1)u_2\right] &= 0\\ \frac{\partial u_2}{\partial t} + \frac{\partial}{\partial x}\left[\frac{u_2^2}{2} + gb\eta_2\right] &= 0\end{aligned}$$

for the upper layer where $a = \frac{(\rho_1 - \rho_2)}{\rho_0}$ and $b = \frac{\rho_2}{\rho_0}$ and $u_1 = u_1(x,t)$ and $u_2 = u_2(x,t)$ are the velocity components, $\eta_1 = \eta_1(x,t)$ and $\eta_2 = \eta_2(x,t)$ are perturbations of the height field for lower and upper layers respectively. The problem is defined in limited domain ie., $0 \leq x \leq L$.

Barotropic and Baroclinic Modes

The linearized two layered shallow water model in normal form is given by,

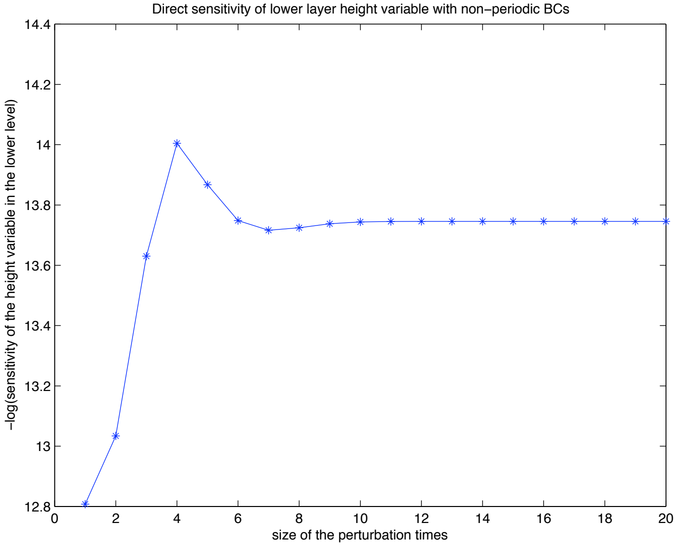
$$\begin{aligned}
\frac{\partial \zeta}{\partial t} + \left(\frac{h+h'}{h}\right)\frac{\partial U}{\partial x} &= 0 \\
\frac{\partial U}{\partial t} + gh\frac{\partial \zeta}{\partial x} &= 0 \\
\frac{\partial \zeta'}{\partial t} + \frac{\partial U'}{\partial x} &= 0 \\
\frac{\partial U'}{\partial t} + \left(\frac{g\epsilon h h'}{h+h'}\right)\frac{\partial \zeta'}{\partial x} &= 0
\end{aligned}$$

This is in barotropic and baroclinic modes. Here, $\epsilon = \frac{\rho'-\rho}{\rho'}$.
Initial conditions for the Direct Problem:

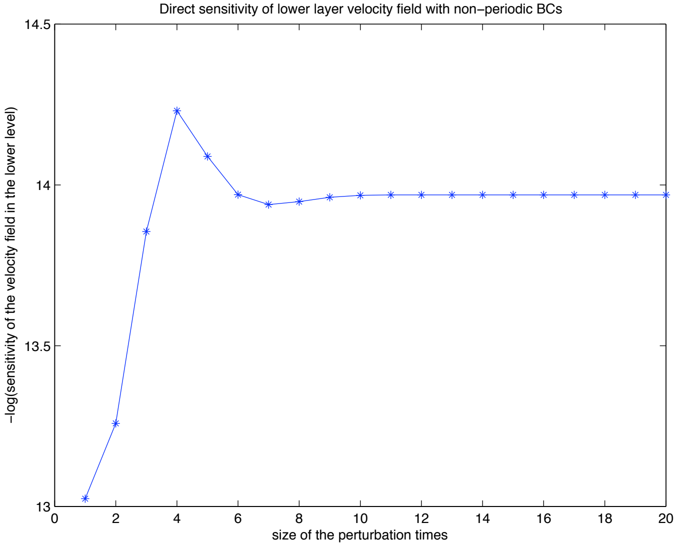
$$\zeta(x,0) = \zeta_0(x); \zeta'(x,0) = \zeta_0'(x); U(x,0) = U_0(x); U'(x,0) = U_0'(x)$$

$$\delta\zeta(x,0) = 0;\ \delta\zeta'(x,0) = 0;\ \delta U(x,0) = 0;\ \delta U'(x,0) = 0$$

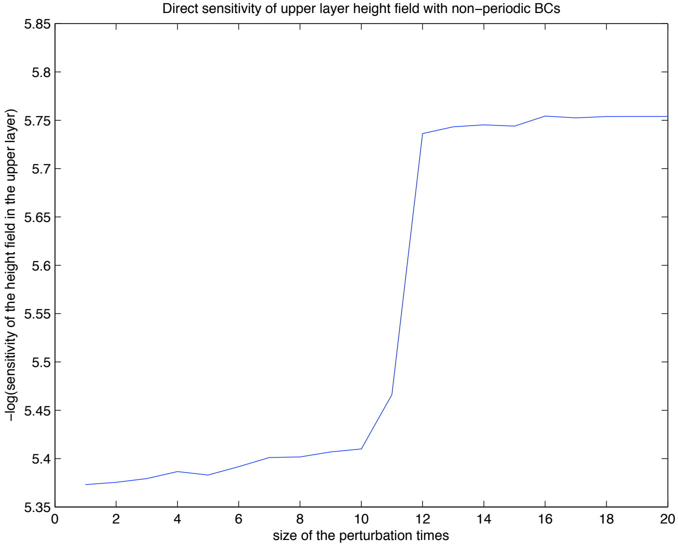
Sensitivities using Direct Sensitivity Method



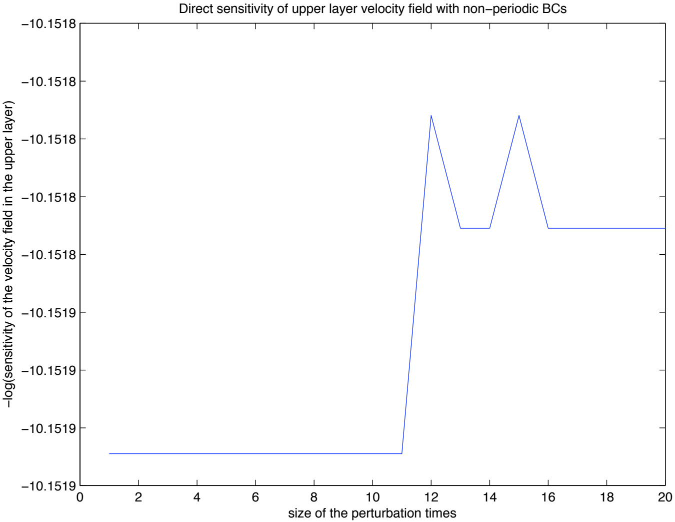
Sensitivities using Direct Sensitivity Method



Sensitivities using Direct Sensitivity Method



Sensitivities using Direct Sensitivity Method



The Adjoint

The adjoint for the normal form of two layered shallow water equations (in barotropic and baroclinic modes) is given by,

$$\begin{aligned}
\frac{\partial \phi}{\partial \tau} - gh\frac{\partial \psi}{\partial x} + \frac{\partial r}{\partial \zeta} &= 0 \\
\frac{\partial \psi}{\partial \tau} - \left(\frac{h+h'}{h}\right)\frac{\partial \phi}{\partial x} + \frac{\partial r}{\partial U} &= 0 \\
\frac{\partial \phi'}{\partial \tau} - gh'\frac{\partial \psi'}{\partial x} + \frac{\partial r}{\partial \zeta'} &= 0 \\
\frac{\partial \psi'}{\partial \tau} - \left(\frac{h+h'}{h'}\right) + \frac{\partial r}{\partial U'} &= 0
\end{aligned}$$

Initial conditions for the Adjoint Problem:

$$\phi(x, T) = 0;\ \phi'(x, T) = 0;\ \psi(x, T) = 0;\ \psi'(x, T) = 0$$

Standard Adjoint Problem Formalism

Boundary Conditions:

$$\phi(L,t)=\zeta(L,t);\ \psi(0,t)=U(0,t);\quad \phi'(L,t)=\zeta'(L,t);\ \psi'(0,t)=U'(0,t)$$

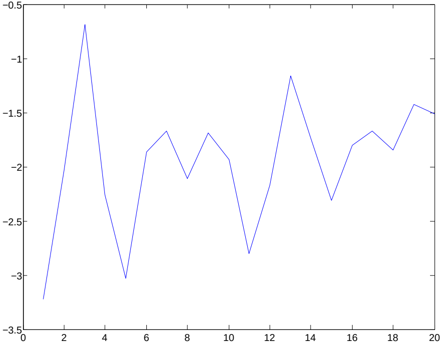
$$\delta J=\int_0^T\left[a\phi(L,t)\delta U+b\psi(L,t)\delta\zeta+c\phi'(L,t)\delta U'+d\psi'(L,t)\delta\zeta'\right]_0^L dt$$

Sensitivities:

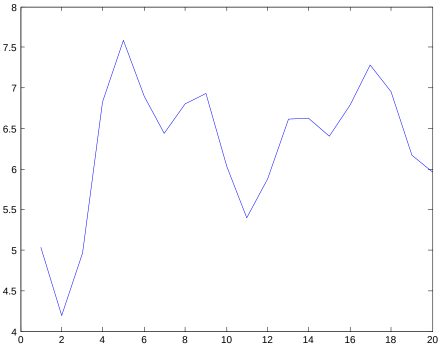
$$\frac{\delta J}{\delta U(0,t_p)}=-a\phi(0,t_p);\quad \frac{\delta J}{\delta U'(0,t_p)}=-c\phi'(0,t_p)$$

$$\frac{\delta J}{\delta\zeta(L,t_p)}=b\psi(L,t_p);\quad \frac{\delta J}{\delta\zeta'(L,t_p)}=d\psi'(L,t_p)$$

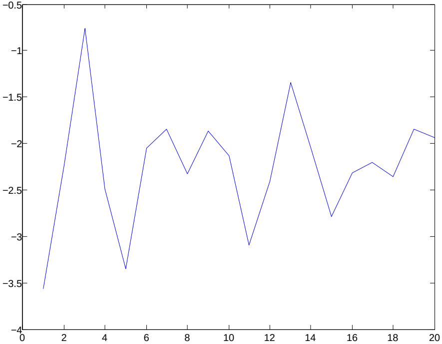
Sensitivities using Adjoint Sensitivity Method



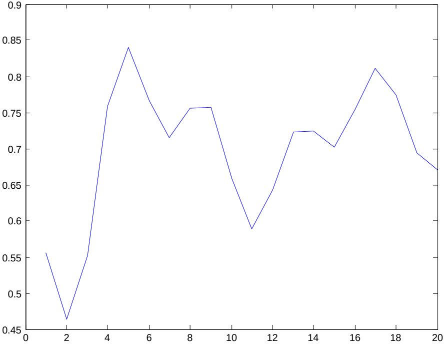
Sensitivities using Adjoint Sensitivity Method



Sensitivities using Adjoint Sensitivity Method



Sensitivities using Adjoint Sensitivity Method



Transformation of Direct Equations

$$\frac{\partial W}{\partial t} + A\frac{\partial W}{\partial x} = 0$$

where

$$W = \begin{pmatrix} \zeta \\ U \\ \zeta' \\ U' \end{pmatrix}, \quad A = \begin{bmatrix} 0 & a & 0 & 0 \\ b & 0 & 0 & 0 \\ 0 & 0 & 0 & c \\ 0 & 0 & d & 0 \end{bmatrix}$$

with $a = \frac{(h+h')}{h}$, $b = gh$, $c = 1$ and $d = \frac{g\epsilon hh'}{h+h'}$.

The eigenvalues of A are given by,
$\lambda_1 = -\sqrt{ab}$, $\lambda_2 = \sqrt{ab}$, $\lambda_3 = -\sqrt{cd}$, $\lambda_4 = \sqrt{cd}$.

Transformation of Direct Equations

The matrix of right and left eigenvectors are then given by,

$$RE = \begin{pmatrix} -\sqrt{\frac{a}{b}} & 1 & 0 & 0 \\ \sqrt{\frac{a}{b}} & 1 & 0 & 0 \\ 0 & 0 & -\sqrt{\frac{c}{d}} & 1 \\ 0 & 0 & \sqrt{\frac{c}{d}} & 1 \end{pmatrix}$$

$$LE = \frac{1}{2} \begin{pmatrix} -\sqrt{\frac{b}{a}} & \sqrt{\frac{b}{a}} & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -\sqrt{\frac{d}{c}} & \sqrt{\frac{d}{c}} \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Characteristics for Direct Equations

$$
\begin{aligned}
C^1 &: \quad \frac{d\zeta}{dt} - \frac{dU}{dt} = 0 \quad \text{along} \quad \frac{dx}{dt} = -\sqrt{ab} \\
C^2 &: \quad \frac{d\zeta}{dt} + \frac{dU}{dt} = 0 \quad \text{along} \quad \frac{dx}{dt} = \sqrt{ab} \\
C^3 &: \quad \frac{d\zeta'}{dt} - \frac{dU'}{dt} = 0 \quad \text{along} \quad \frac{dx}{dt} = -\sqrt{cd} \\
C^4 &: \quad \frac{d\zeta'}{dt} + \frac{dU'}{dt} = 0 \quad \text{along} \quad \frac{dx}{dt} = \sqrt{cd}
\end{aligned}
$$

Transformation of Adjoint Equations

$$\Phi_\tau + \tilde{A}\Phi_x + \tilde{R} = 0$$

where

$$\Phi = \begin{pmatrix} \phi \\ \psi \\ \phi' \\ \psi' \end{pmatrix}, \quad A = \begin{bmatrix} 0 & -b & 0 & 0 \\ -a & 0 & 0 & 0 \\ 0 & 0 & 0 & -d \\ 0 & 0 & -c & 0 \end{bmatrix}, \quad \tilde{R} = \begin{pmatrix} \frac{\partial r}{\partial \zeta} \\ \frac{\partial r}{\partial U} \\ \frac{\partial r}{\partial \zeta'} \\ \frac{\partial r}{\partial U'} \end{pmatrix}$$

with $a = \frac{(h+h')}{h}$, $b = gh$, $c = 1$ and $d = \frac{-g\epsilon hh'}{h+h'}$. The eigenvalues of $\tilde{A}$ are given as,
$\lambda_1 = -\sqrt{g(h+h')}$, $\lambda_2 = \sqrt{g(h+h')}$, $\lambda_3 = -\sqrt{\frac{g\epsilon hh'}{h+h'}}$, $\lambda_4 = \sqrt{\frac{g\epsilon hh'}{h+h'}}$.

Transformation of Adjoint Equations

The matrix of right and left eigenvectors are then given as,

$$RE = \begin{pmatrix} \sqrt{\frac{b}{a}} & 1 & 0 & 0 \\ -\sqrt{\frac{b}{a}} & 1 & 0 & 0 \\ 0 & 0 & \sqrt{\frac{d}{c}} & 1 \\ 0 & 0 & -\sqrt{\frac{d}{c}} & 1 \end{pmatrix}$$

and

$$LE = \frac{1}{2} \begin{pmatrix} \sqrt{\frac{a}{b}} & -\sqrt{\frac{a}{b}} & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & \sqrt{\frac{c}{d}} & -\sqrt{\frac{c}{d}} \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Characteristics for Adjoint Equations

$$
\begin{aligned}
C^1 &: \quad \frac{d\phi}{d\tau} - \frac{d\psi}{d\tau} + \frac{\partial r}{\partial \zeta} - \frac{\partial r}{\partial U} = 0 \quad \text{along} \quad \frac{dx}{d\tau} = -\sqrt{ab} \\
C^2 &: \quad \frac{d\phi}{d\tau} + \frac{d\psi}{d\tau} + \frac{\partial r}{\partial \zeta} + \frac{\partial r}{\partial U} = 0 \quad \text{along} \quad \frac{dx}{d\tau} = \sqrt{ab} \\
C^3 &: \quad \frac{d\phi'}{d\tau} - \frac{d\psi'}{d\tau} + \frac{\partial r}{\partial \zeta'} + \frac{\partial r}{\partial U'} = 0 \quad \text{along} \quad \frac{dx}{d\tau} = -\sqrt{cd} \\
C^4 &: \quad \frac{d\phi'}{d\tau} + \frac{d\psi'}{d\tau} + \frac{\partial r}{\partial \zeta'} + \frac{\partial r}{\partial U'} = 0 \quad \text{along} \quad \frac{dx}{d\tau} = \sqrt{cd}
\end{aligned}
$$

Characteristic Adjoint Formalism

To obtain sensitivity expressions for U and U' at upstream and downstream, we substitute the above characteristics in δJ, we get

$$\frac{\delta J}{\delta U(0,t_p)} = -\frac{h+h'}{h}\phi(0,t_p) + gh\psi(0,t_p)$$

$$\frac{\delta J}{\delta U'(0,t_p)} = -\phi'(0,t_p) + \frac{g\epsilon hh'}{h+h'}\psi'(0,t_p)$$

$$\frac{\delta J}{\delta U(L,t_p)} = \frac{h+h'}{h}\phi(L,t_p) + gh\psi(L,t_p)$$

$$\frac{\delta J}{\delta U'(L,t_p)} = \phi'(L,t_p) + \frac{g\epsilon hh'}{h+h'}\psi'(L,t_p)$$

Characteristic Adjoint Formalism

Similarly by substituting the above characteristics, we can also obtain the sensitivity expressions for height field at both the upstream and downstream as,

$$\frac{\delta J}{\delta\zeta(0,t_p)} = \frac{h+h'}{h}\phi(0,t_p) - gh\psi(0,t_p)$$

$$\frac{\delta J}{\delta\zeta'(0,t_p)} = \phi'(0,t_p) - \frac{g\epsilon h' h}{h+h'}\psi'(0,t_p)$$

and

$$\frac{\delta J}{\delta\zeta(L,t_p)} = \frac{h+h'}{h}\phi(L,t_p) + gh\psi(L,t_p)$$

$$\frac{\delta J}{\delta\zeta'(L,t_p)} = \phi'(L,t_p) + \frac{g\epsilon hh'}{h+h'}\psi'(L,t_p)$$

Summary

Characteristic adjoint sensitivity analysis for transient motion in a two-layered sea

- To determine how a specific condition is dependent on many parameters which influence it
- Adjoint sensitivity method is more efficient than Direct sensitivity method
- Fast and accurate tool for computing the sensitivities
- Characteristic adjoint sensitivity method gives better results than Standard adjoint sensitivity method
- Boundary conditions derived for linear approximation can be used for the nonlinear problem
- Numerical work in progress