What we can learn about Snowball Earth using simple math

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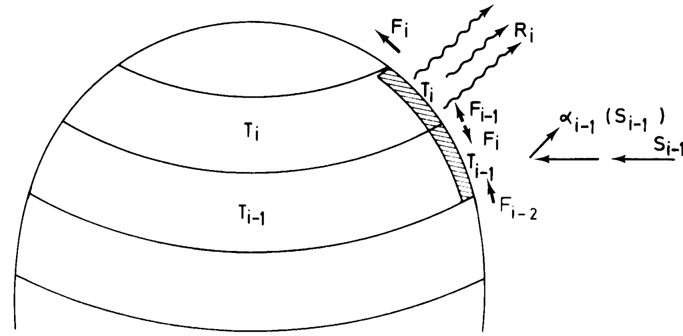
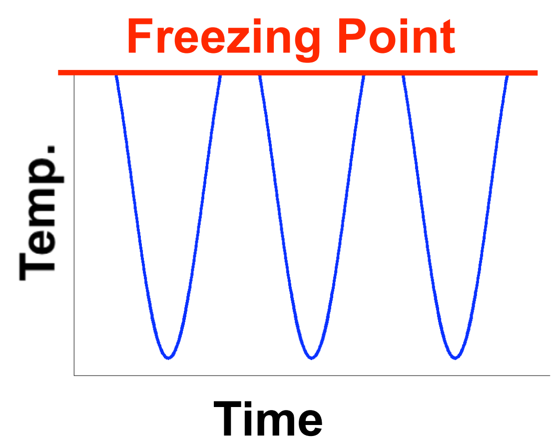
Outline

I. What the rocks tell us



II. Energy Balance Models: Simple Global Climate Models

III. Ice Growth and Flow



IV. Internal and Surface Ice Temperature Solution

II. What the rocks tell us

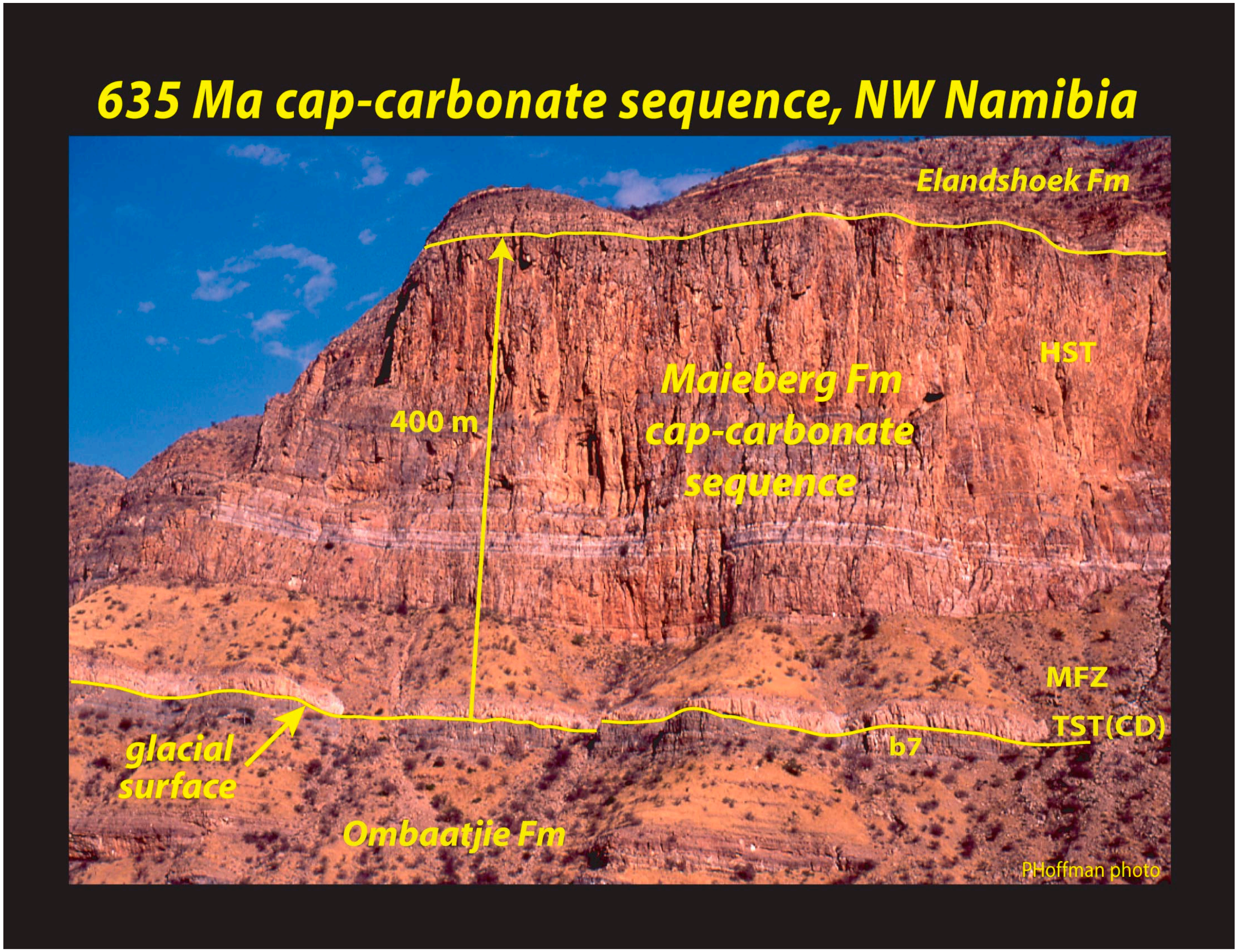
Behold, I show you a mystery:
we shall not all sleep,
but we shall all be changed, in a moment,
in the twinkling of an eye,
at the last trump:
for the trumpet shall sound,
and the ***rocks*** shall be ***analyzed***,
and we shall tell of a ***Snowball***!

The mad Snowballers

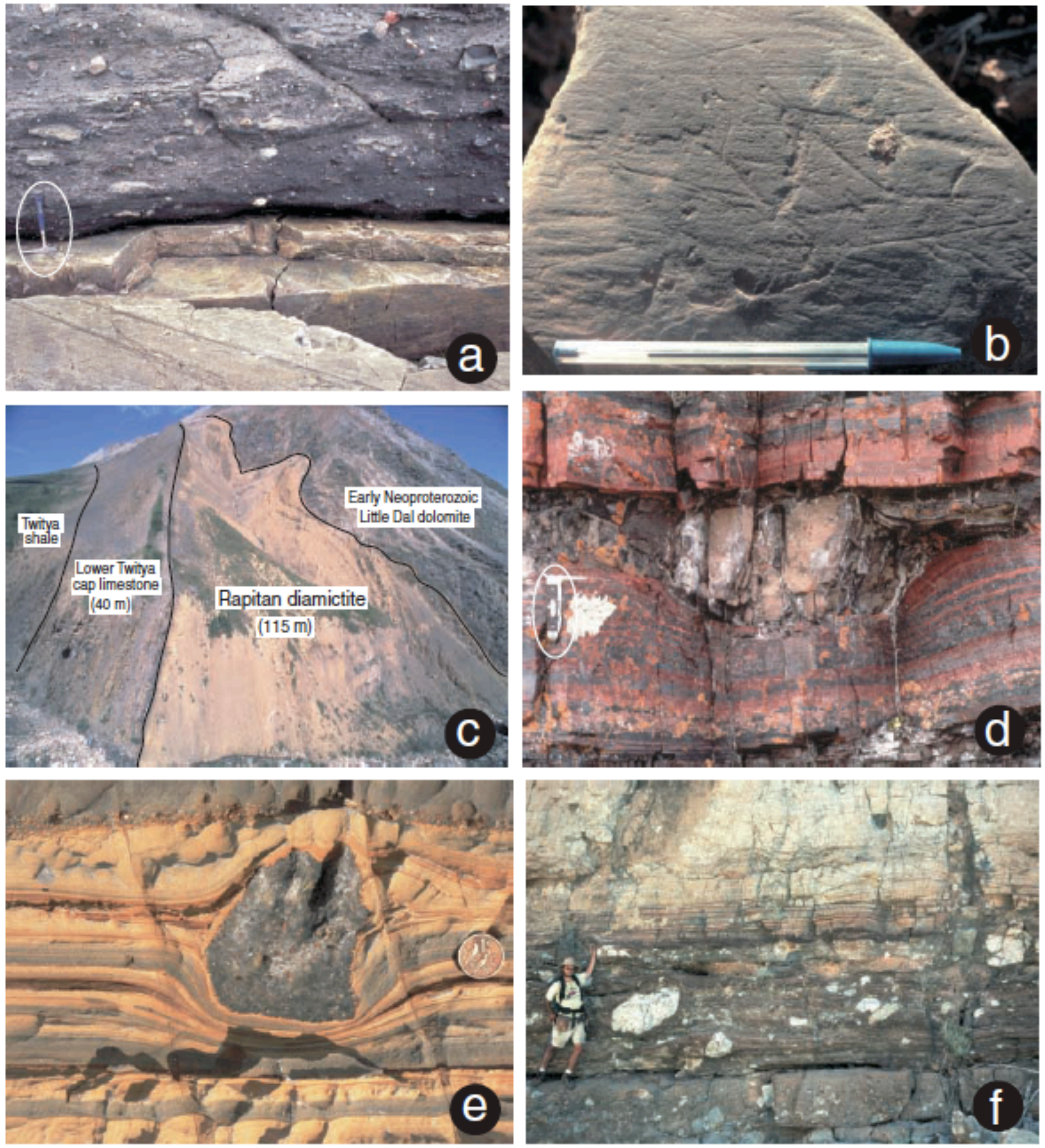
Hoffman, P.F., Kaufman, A.J., Halverson, G.P. and Schrag, D.P., 1998. A Neoproterozoic snowball Earth.

Kirschvink, J.L., 1992. Late Proterozoic low-latitude glaciation: the snowball Earth.

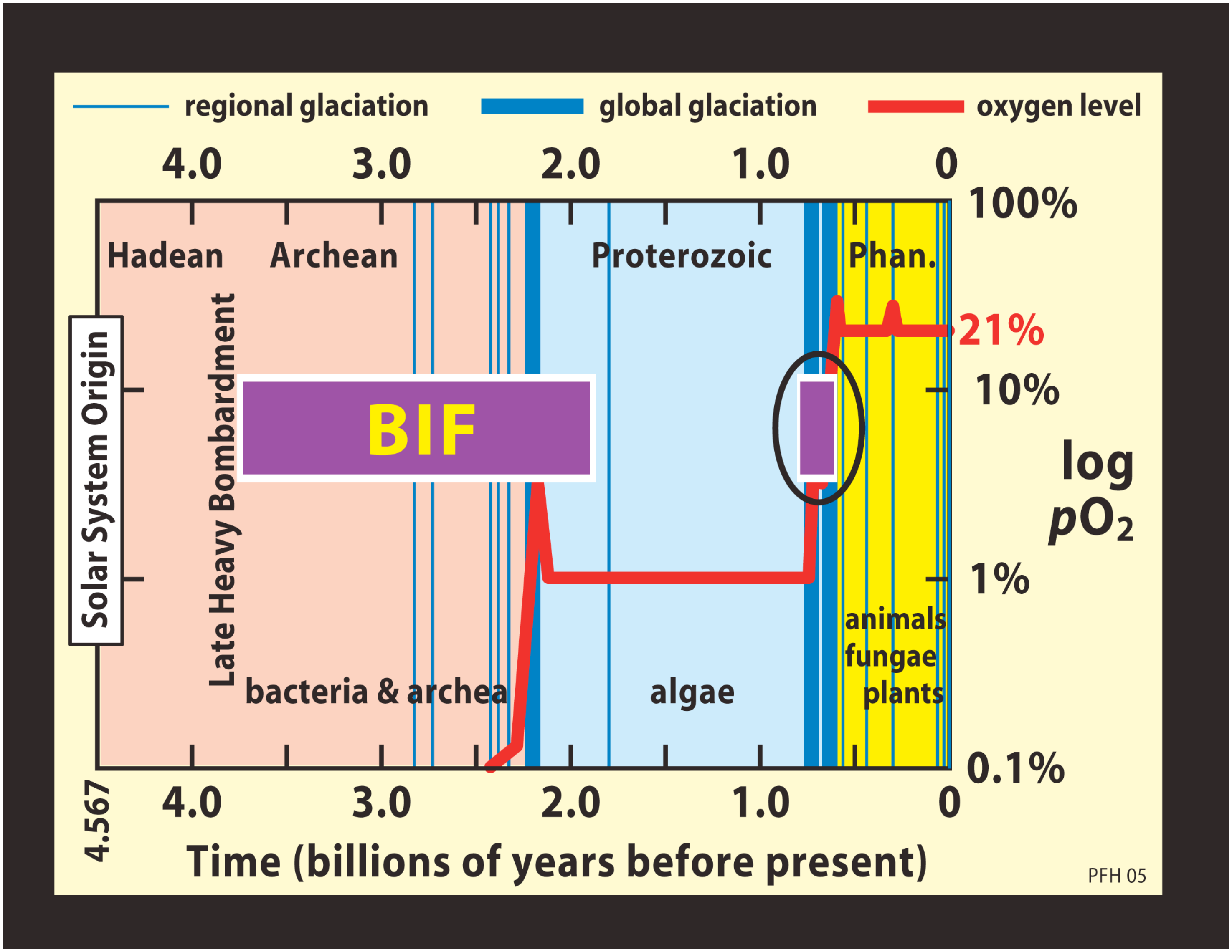
Glacial rocks are overlain with rocks that suggest an extremely hot and moist climate.



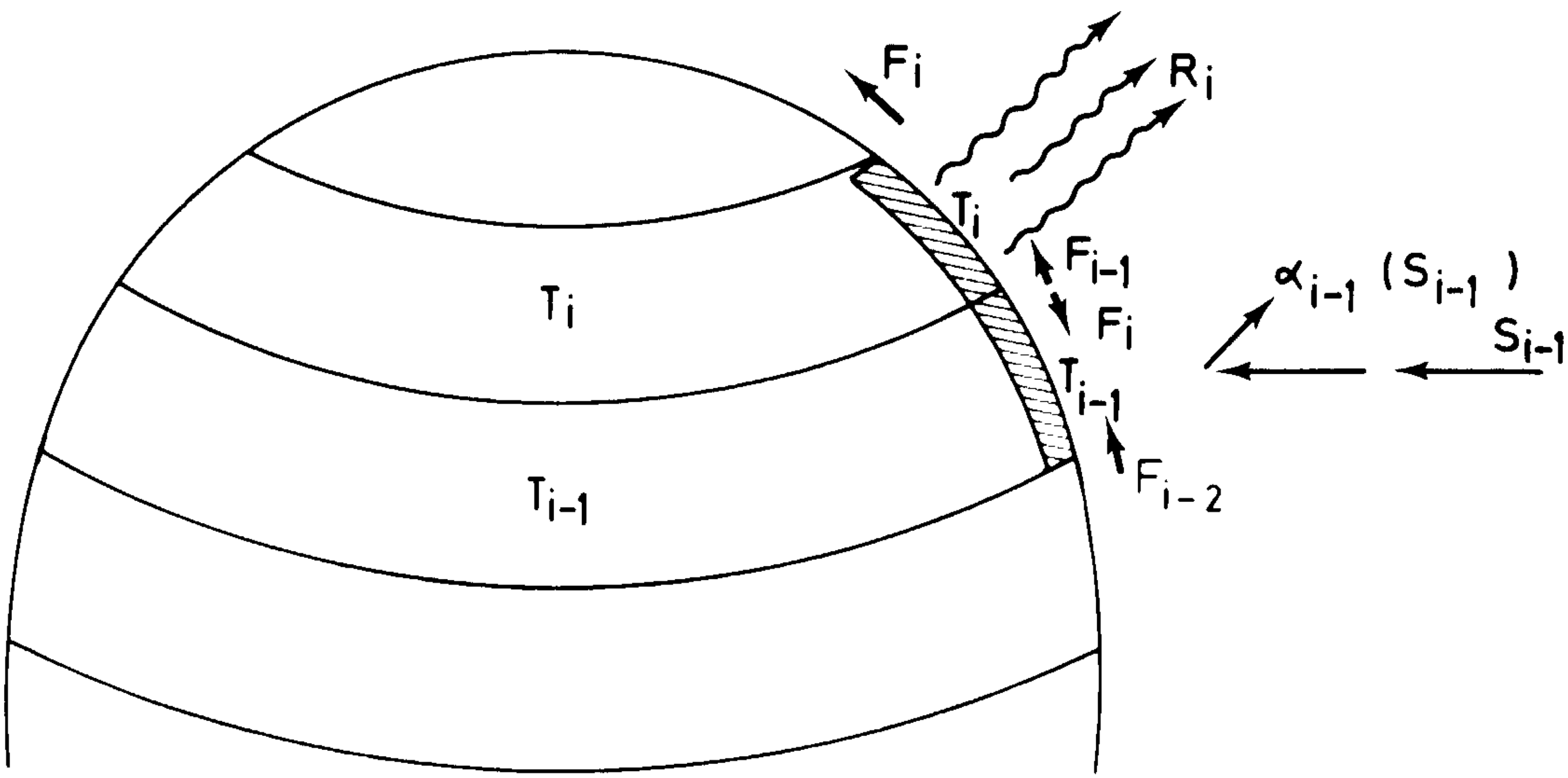
Rocks show evidence of glaciers at sea level at the equator about 600 million years ago.



The occurrence of banded iron formations may suggest the entire ocean was covered in ice.



II. Energy Balance Models: Simple Global Climate Models

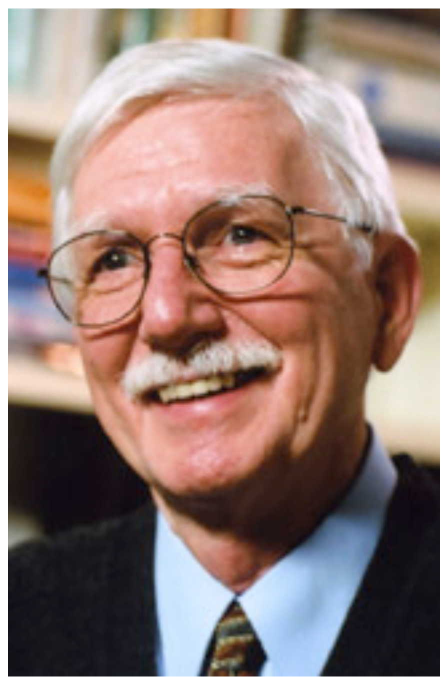


Early Climate Theorists

Budyko, M.I., 1969. The effect of solar radiation variations on the climate of the Earth.

Sellers, W.D., 1969. A global climatic model based on the energy balance of the Earth-atmosphere system.

North, G. R, 1975: Theory of energy-balance climate models.



The nonlinear mathematics of an energy balance model.

Solar Radiation

Atmospheric and oceanic heat transport.

$$\frac{Q}{4}S(x)(1-\alpha(x)) = A + BT(x) + \nabla \cdot \vec{F}.$$

Albedo: reflected light increases
if T(x) drops below freezing!

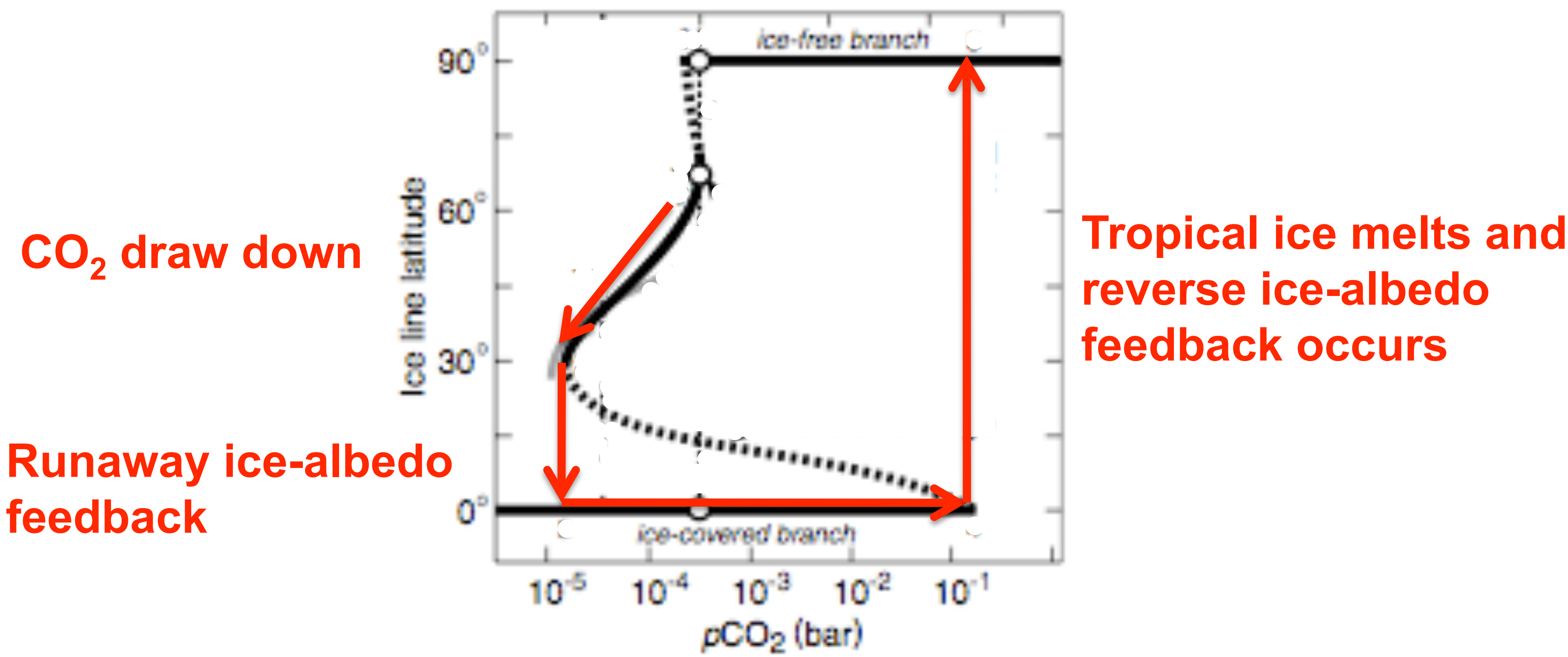
Linearization of Outgoing
Terrestrial Radiation

Assume that atmospheric and oceanic
movements smooth the temperature profile

$$\nabla \cdot \vec{F} = C(T - \overline{T}),$$

$$\nabla \cdot \vec{F} = -D\frac{d}{dx}(1-x^2)\frac{dT}{dx}.$$

Energy Balance Models produces a bifurcation diagram that might explain the geological observations.



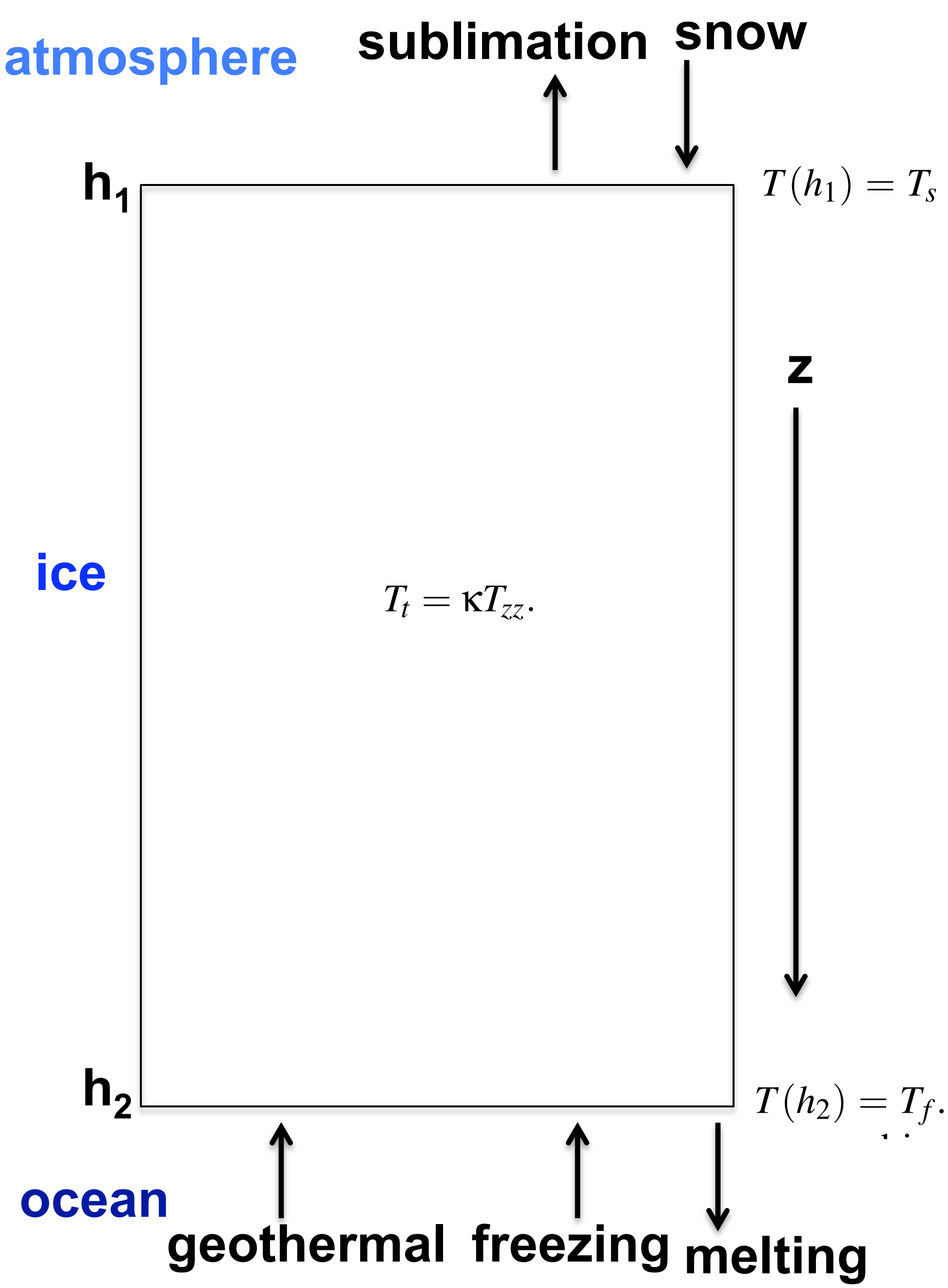
Very low weathering allows CO_2 to build up to ~10% of atmosphere over 1-10 million years

III. Ice Growth and Flow

Herr Stefan invented the famous Stefan condition while studying sea ice!

Stefan, J., 1889: Uber die Theorie der Eisbildung, insbesondere uber die Eisbildung im Polarmeere.

Solving for ice thickness neglecting horizontal flow



Boundary Conditions

$$\frac{dh_1}{dt} = w.$$

Net Sublimation Rate

Stefan Condition

freeze/melt

geothermal heat flux

$$-\rho_i L_i \frac{dh_2}{dt} = -\rho_i C_{pi} \kappa \frac{dT}{dz}\bigg|_{h_2^-} + F_{geo}.$$

heat diffusion at boundary

Finding an ODE for ice thickness

$$S = \frac{L_i}{C_{pi}(T_f - T_s)}.$$

S is the ratio of the latent heat of solidification to the sensible heat required to cool the newly formed solid to the atmospheric temperature.

For ΔT=30-40 K, S=3-5. Solidification rate is small, so assume temperature field is linear.

$$\left.\frac{dT}{dz}\right|_{h_2^-} = \frac{T_f - T_s}{h_2 - h_1} = \frac{\Delta T}{h},$$

The upper and lower boundary conditions become an ODE for ice depth.

$$\frac{dh_1}{dt} = w.$$

$$-\rho_i L_i \frac{dh_2}{dt} = -\rho_i C_{pi} \kappa \left.\frac{dT}{dz}\right|_{h_2^-} + F_{geo}.$$

$$\boxed{\frac{dh}{dt} = \frac{C_{pi}\kappa \Delta T}{L_i h} - w - \frac{F_{geo}}{\rho_i L_i}.}$$

Solving for steady-state ice thickness

$$\boxed{\frac{dh}{dt} = \frac{C_{pi}\kappa\Delta T}{L_i h} - w - \frac{F_{geo}}{\rho_i L_i}.}$$

±1-10 mm/yr **-1 mm/yr**

I. Net sublimation or small net precipitation

$$w \geq -\frac{F_{geo}}{\rho_i L_i}.$$

$$\boxed{h = \frac{k\Delta T}{\rho_i L_i w + F_{geo}},}$$

Assume ΔT=40 K:
w=0 ➔ h=1700 m
w=1 cm/yr ➔ h=100m

II. Large net precipitation

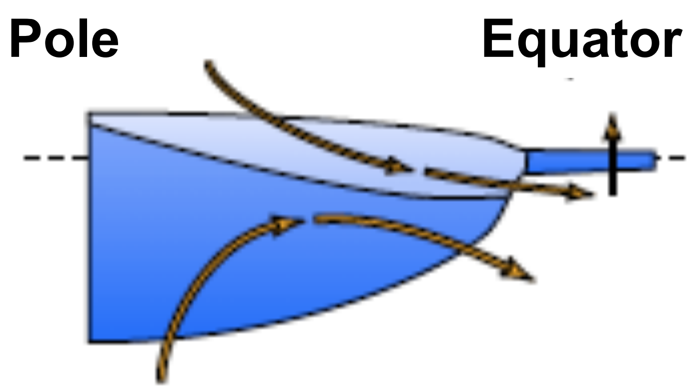
$$w < -\frac{F_{geo}}{\rho_i L_i}$$

h grows with time!

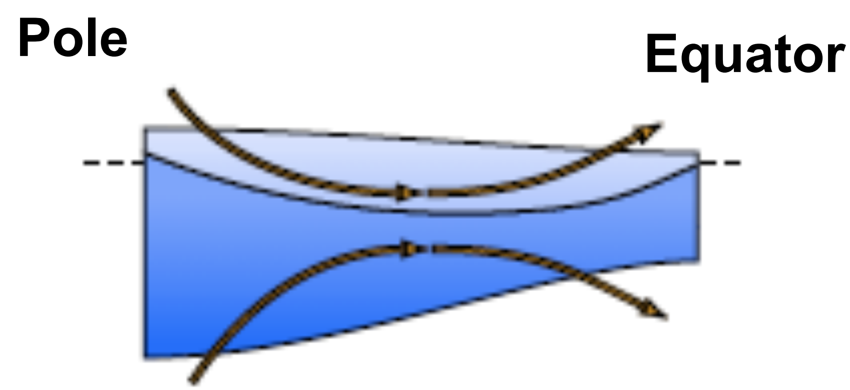
1 cm/yr for 1 million years is 10 km of ice!

Flow of sea glaciers will remove accumulation issue.

Sea Ice Elevator
[Hoffman and Schrag, 1999]

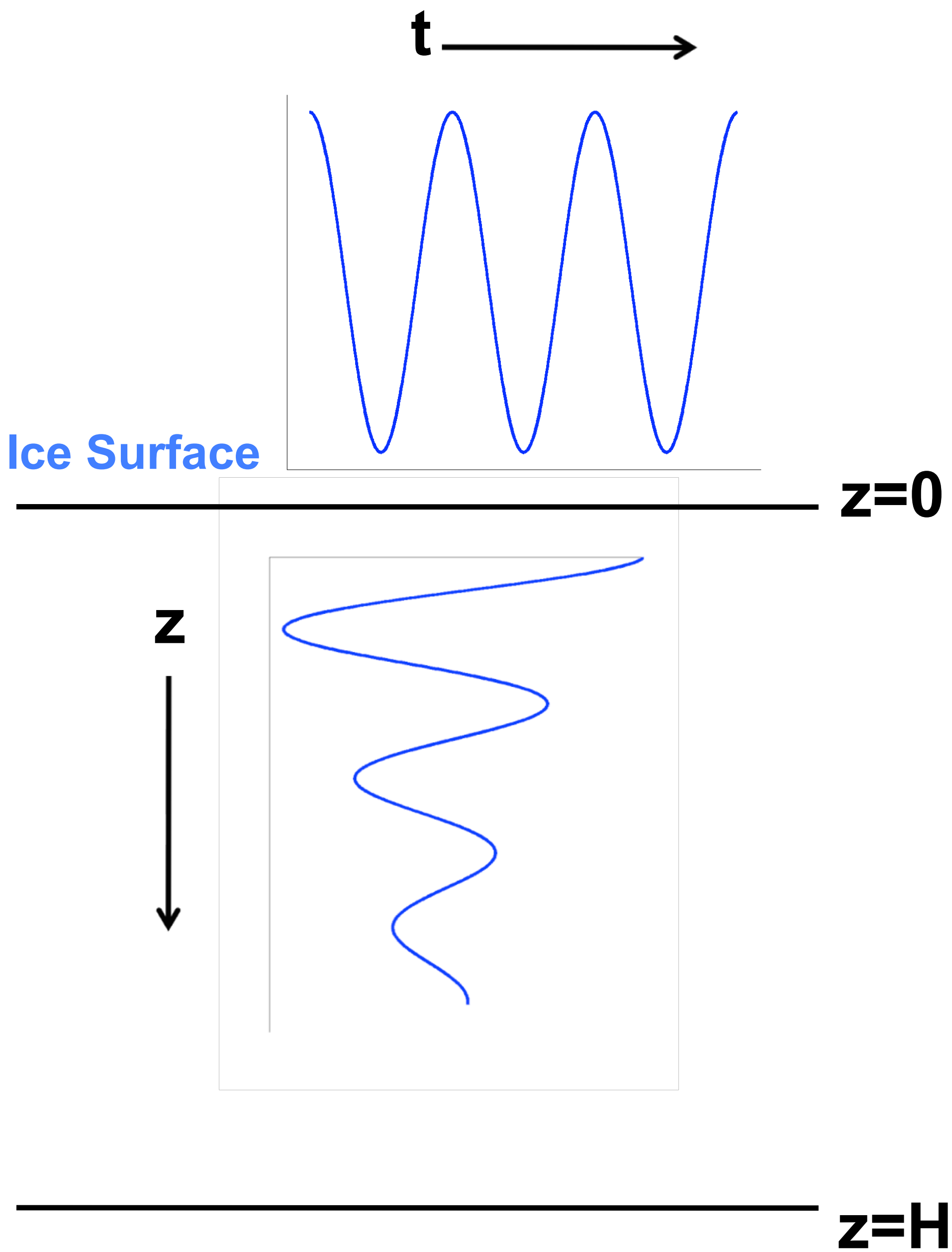


Ice flow from higher latitudes
[Goodman and Pierrehumbert, 2003]

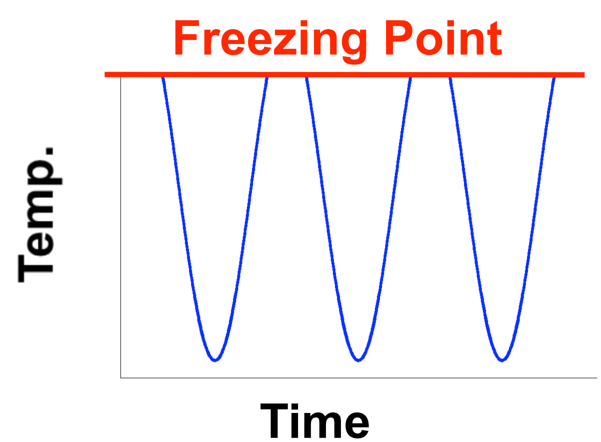


Figures from Goodman (2006)

IV. Internal and Surface Ice Temperature Solution

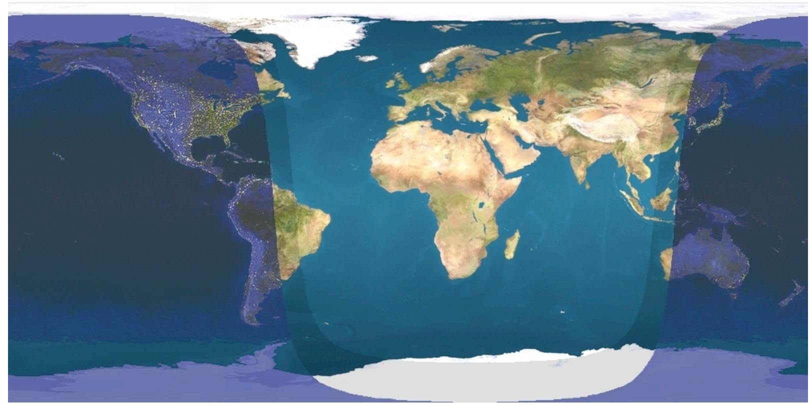


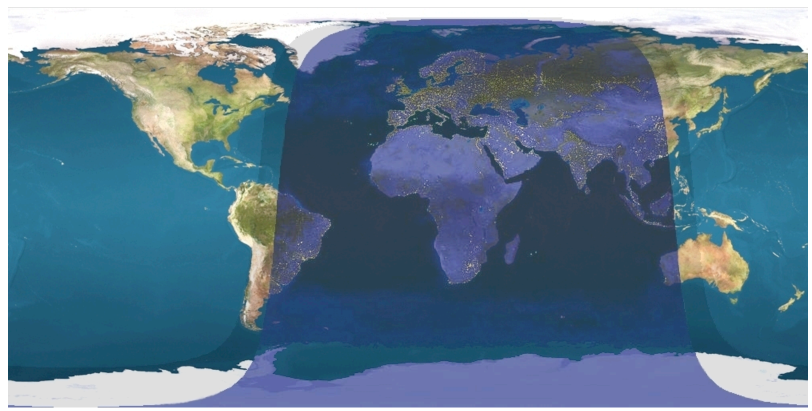
We care because melting occurs at noon



Monsieur Fourier solved for the temperature within the Earth when it is forced with a varying temp. BC at the surf.

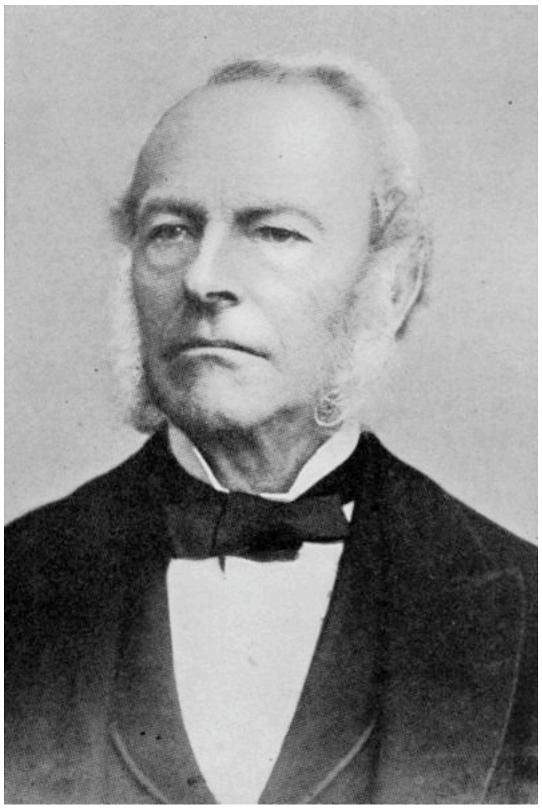
Fourier, J., 1826: Théorie du mouvement de la chaleur dans les corps solides.





Mister Stokes found an exact solution to the Navier-Stokes equations of the same mathematical form.

Stokes, G., 1851: On the effect of the internal friction of fluids on the motion of pendulums.



Stokes' Second Problem

Viscous Fluid

Transversely Jiggling Plate

The heat equation between two infinite plane-parallel boundaries with a time-varying Dirichlet BC

Equation:

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial z^2},$$

Boundary Conditions:

$$T(z=0,t) = T_m + T_v \cos(\omega t),$$

$$T(z=H,t) = T_b,$$

Ansatz:

$$T(z,t) = T_m + \frac{T_b - T_m}{H} z + T_v \Re\left(e^{i(kz-\omega t)}\right),$$

The heat equation between two infinite plane-parallel boundaries with a time-varying Dirichlet BC

The solution:

$$T(z,t) = T_m + \frac{T_b - T_m}{H} z + T_v \Re\left(\frac{1}{\beta - 1}\left(\beta e^{-\frac{z}{z^*}} e^{i\left(\frac{z}{z^*} - \omega t\right)} - e^{\frac{z}{z^*}} e^{-i\left(\frac{z}{z^*} + \omega t\right)}\right)\right),$$

With: $\beta \equiv e^{\frac{2H}{z^*}(1-i)}$

Penetration depth: $z^* \equiv \left(\frac{2\kappa}{\omega}\right)^{\frac{1}{2}} = \left(\frac{\kappa P}{\pi}\right)^{\frac{1}{2}}$

Considering the solutions through thick and thin.

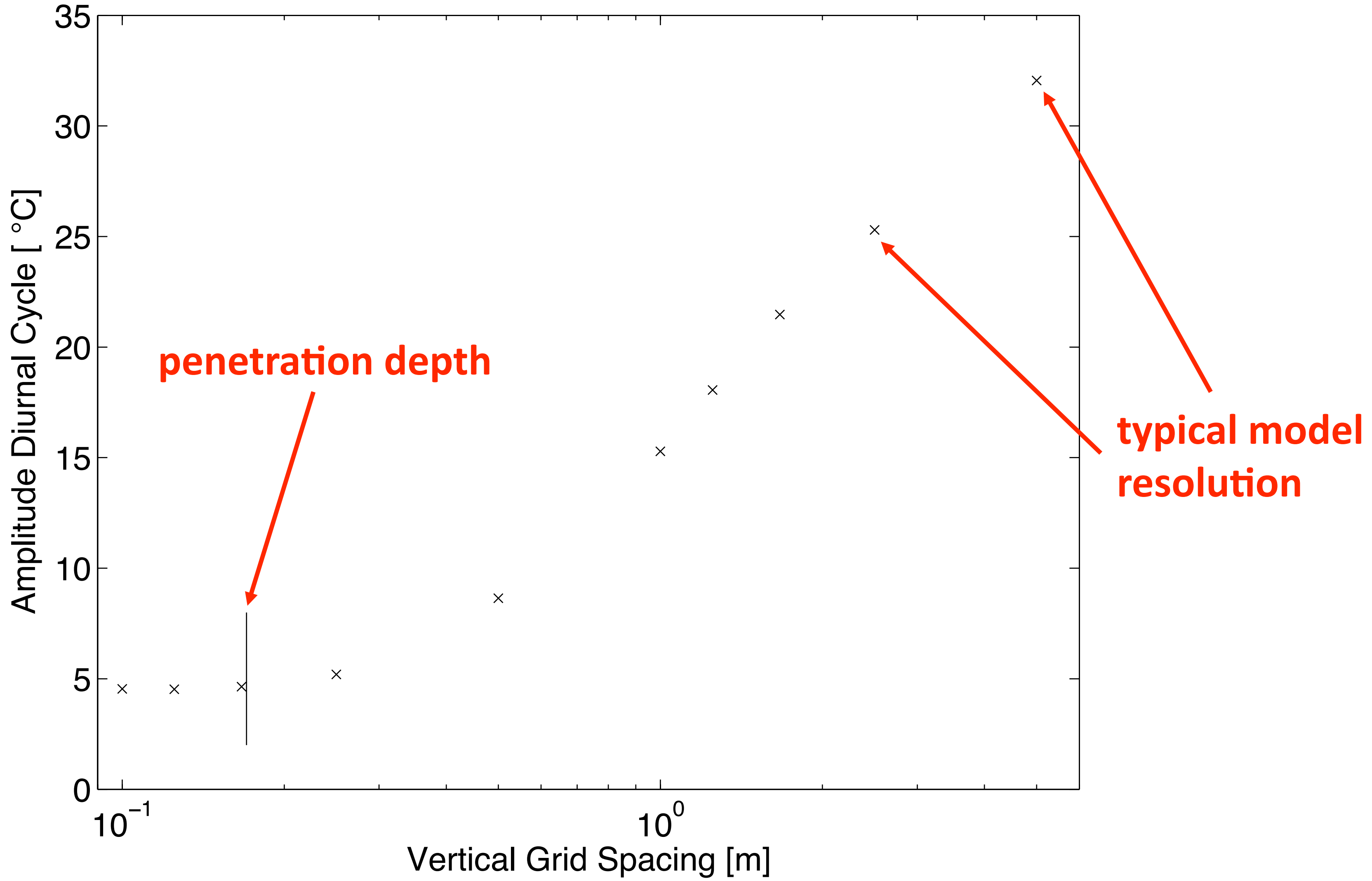
For thick ice, get exponentially-damped waves propagating into ice interior

$$H \gg z^* \qquad T(z,t) \approx T_m + \frac{T_b - T_m}{H} z + T_v e^{-\frac{z}{z^*}} \cos(\frac{z}{z^*} - \omega t),$$

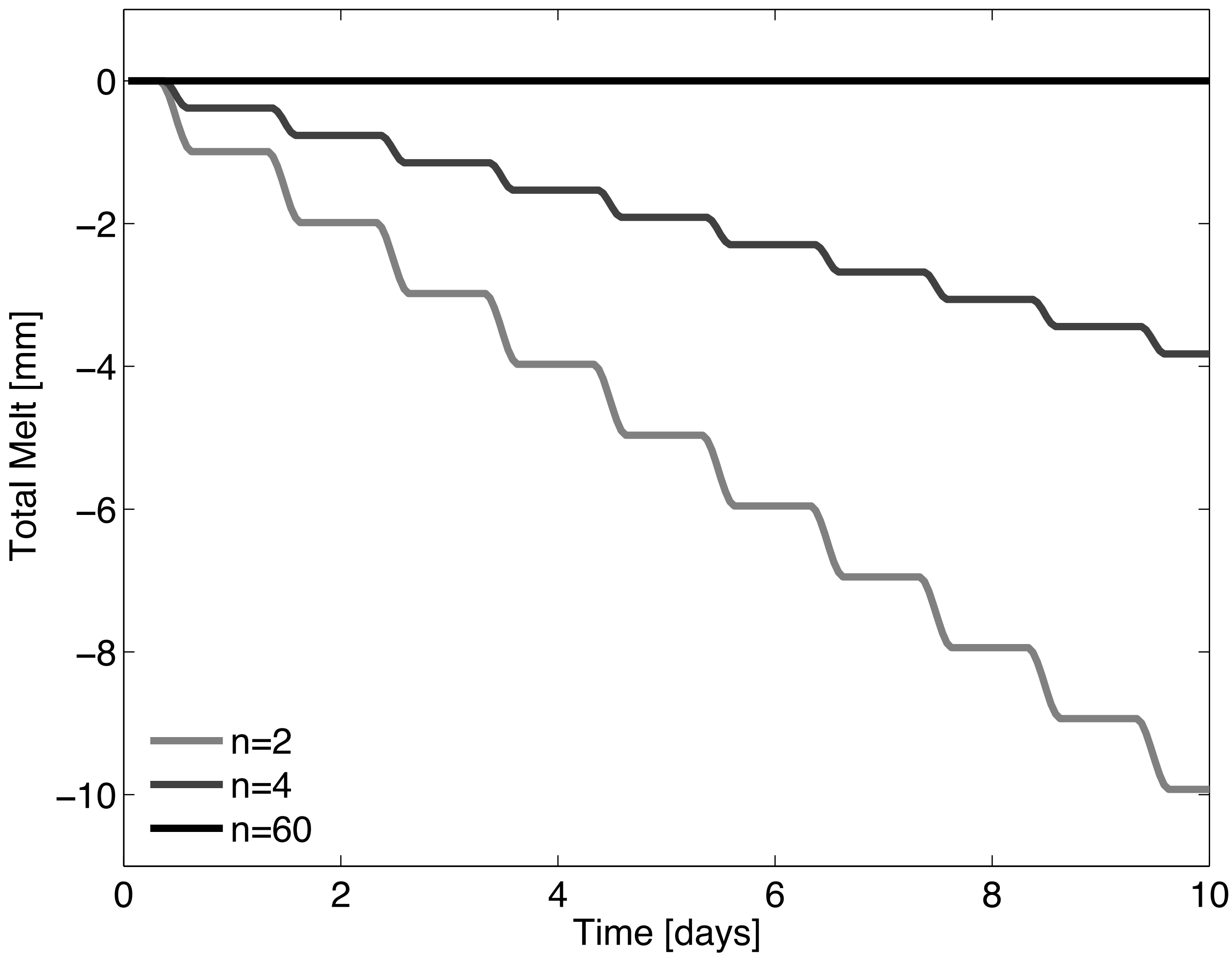
For thin ice, boundary layer extends through ice

$$z, H \ll z^* \qquad T(z,t) \approx T_m + \frac{T_b - T_m}{H} z + T_v \left(1 - \frac{z}{H}\right) \cos(\omega t).$$

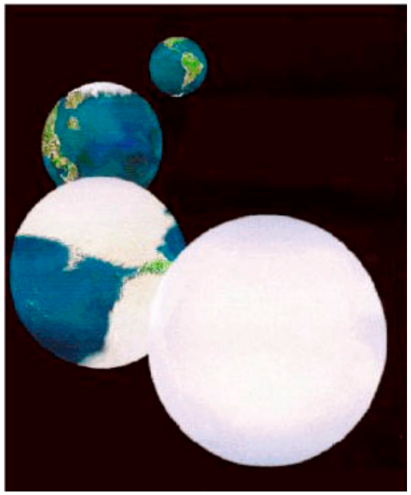
A sea ice scheme must resolve the penetration depth in order to produce a reasonable surf. temp. diurnal cycle



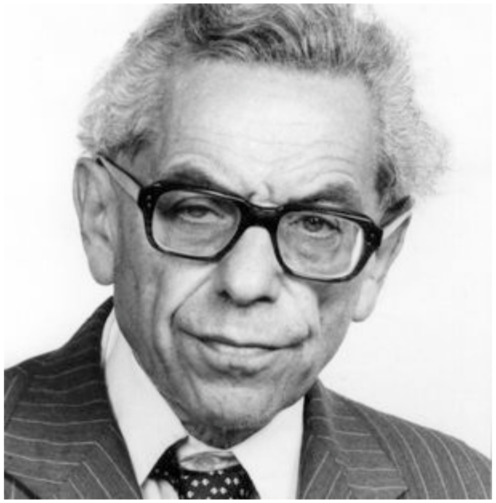
Low sea ice vertical resolution can lead to false Snowball Earth deglaciations in global climate models!



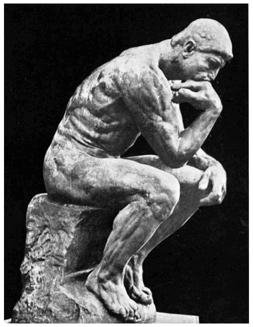
Summary and Conclusions



Strange and mysterious "Snowball Earth" events happened about 600, 700, and 2200 million years ago.



Math people can help rock people understand the story the rocks tell.



Using math and thinking allows us to understand things about the Snowball that we cannot by simply running global climate models.