“minimal complexity”

Tipping points in a simple model of Arctic sea ice

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Supported by the NSF IGMS program

Minimum Arctic Sea-ice Extent

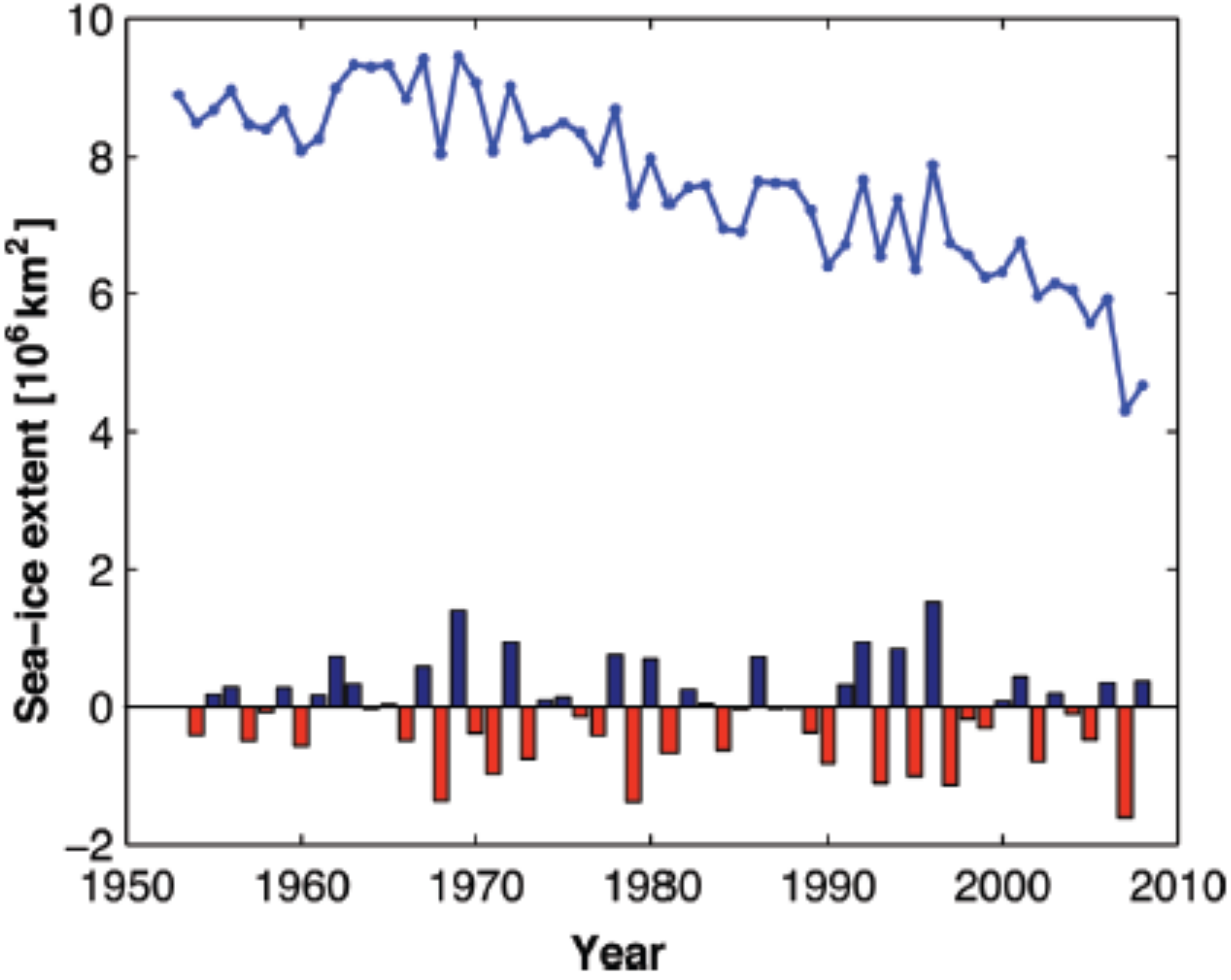
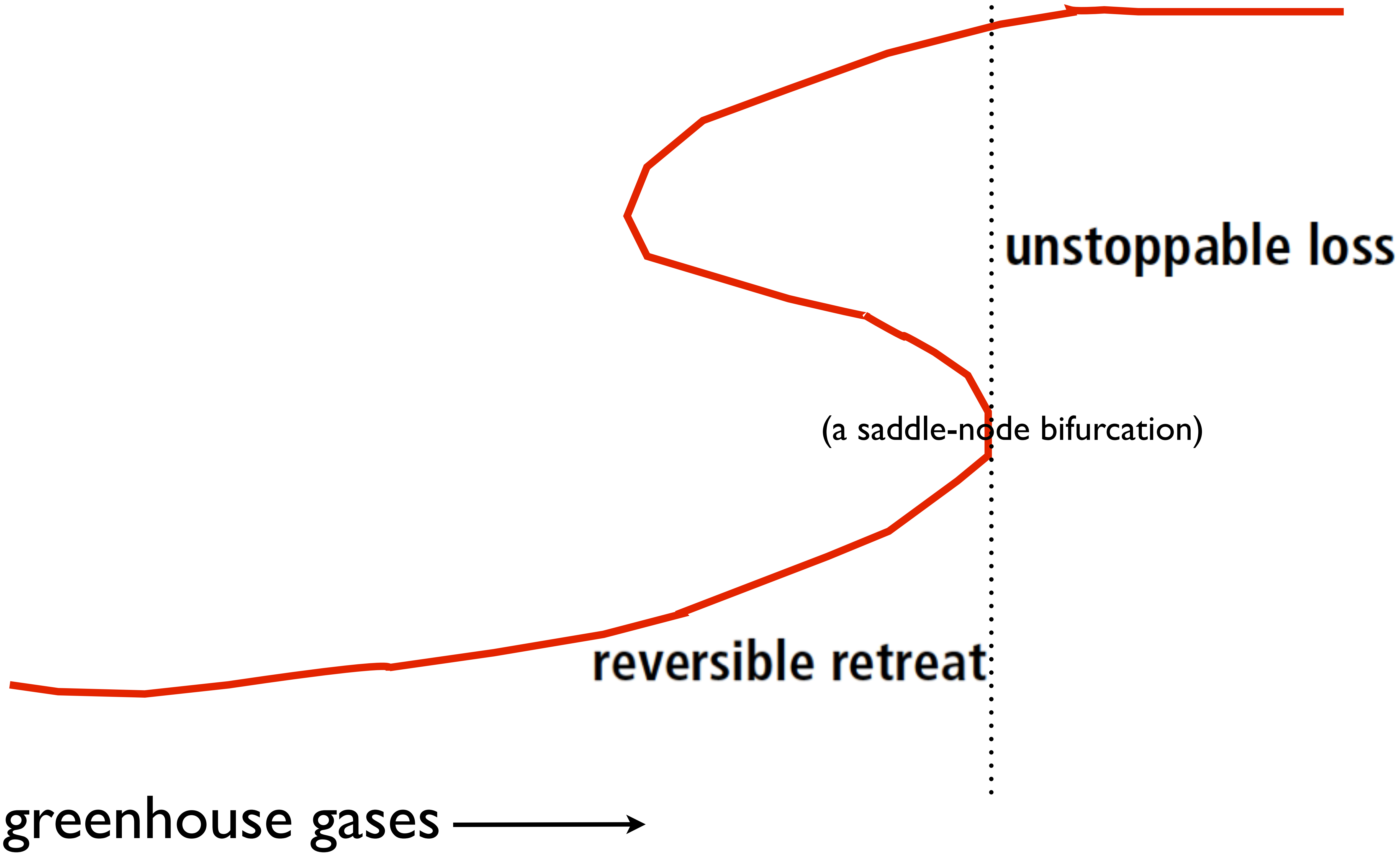


Figure from: **The future of ice sheets and sea ice: Between reversible retreat and unstoppable loss**

Dirk Notz[1]

PNAS | **December 8, 2009** | vol. 106 | no. 49 | **20595**

Arctic sea-ice "tipping point"



minimal complexity model: positive ice-albedo feedback *vs.* stabilizing sea ice thermodynamics

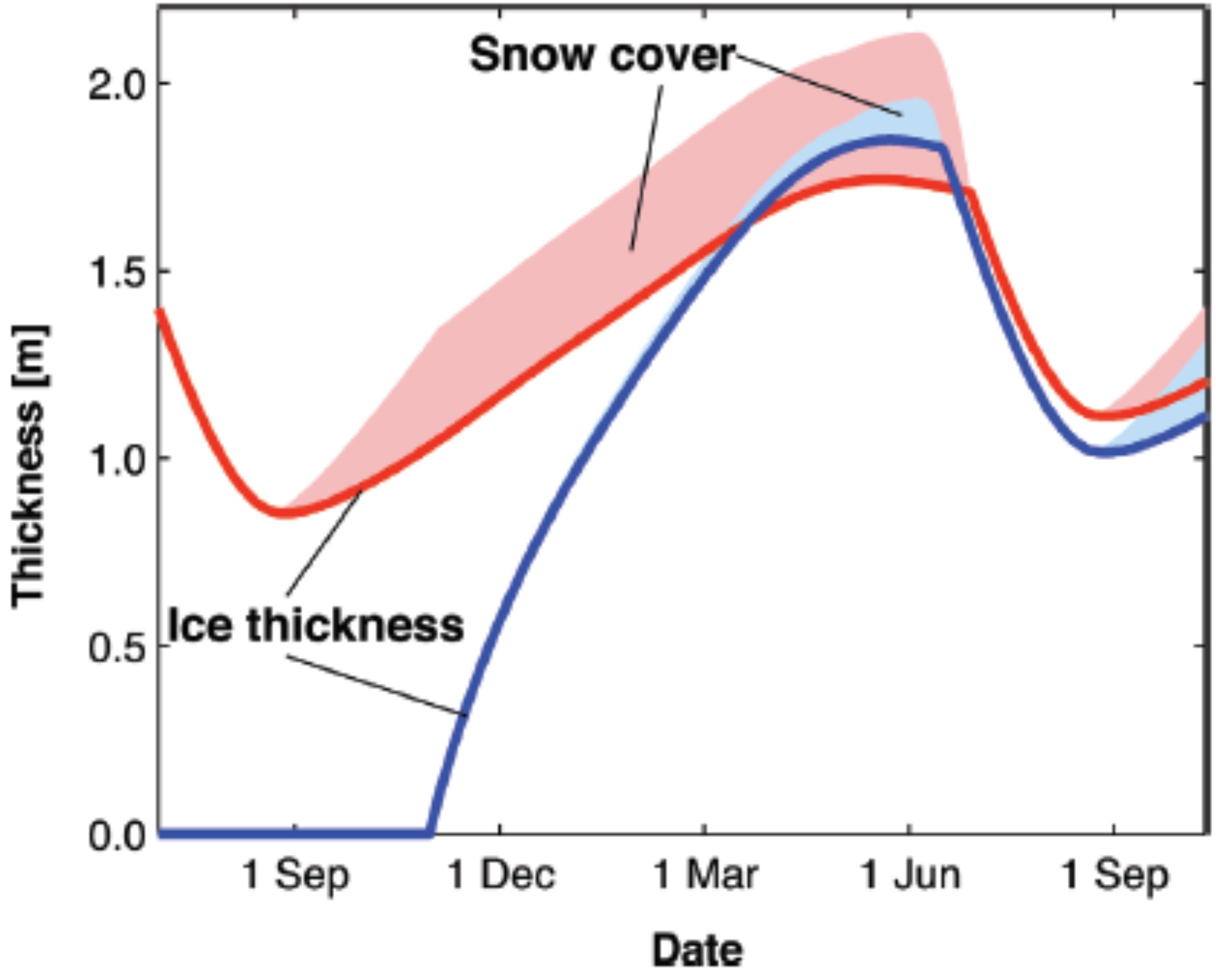


Figure from:

The future of ice sheets and sea ice: Between reversible retreat and unstoppable loss

Dirk Notz[1]

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The 0-d model

(after Eisenman & Wettlaufer, PNAS 2009)

State variable *E(t)*: energy per unit surface area

(relative to Arctic ocean mixed layer at the freezing point)

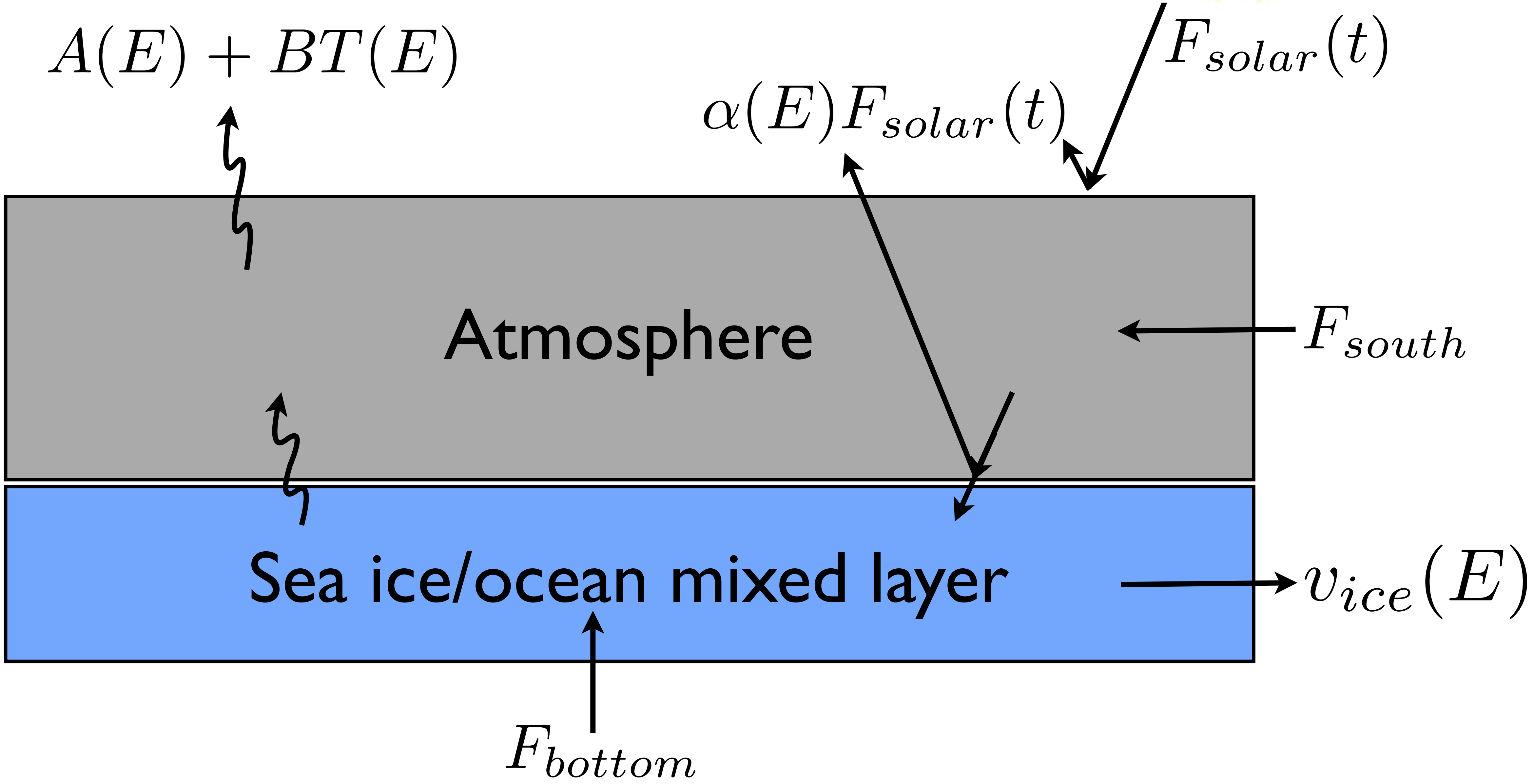
$$E(t) = \begin{cases} -L_i h_i(t) & \text{if } E < 0 \quad (i.e.\ E \propto \text{ice thickness } h_i) \\ C_s T(t) & \text{if } E \geq 0 \quad (i.e.\ E \propto \text{mixed layer temp. } T) \end{cases}$$

$L_i =$ **latent heat of fusion of ice**

$C_s =$ **ocean heat capacity per unit surface area**

The 0-d model

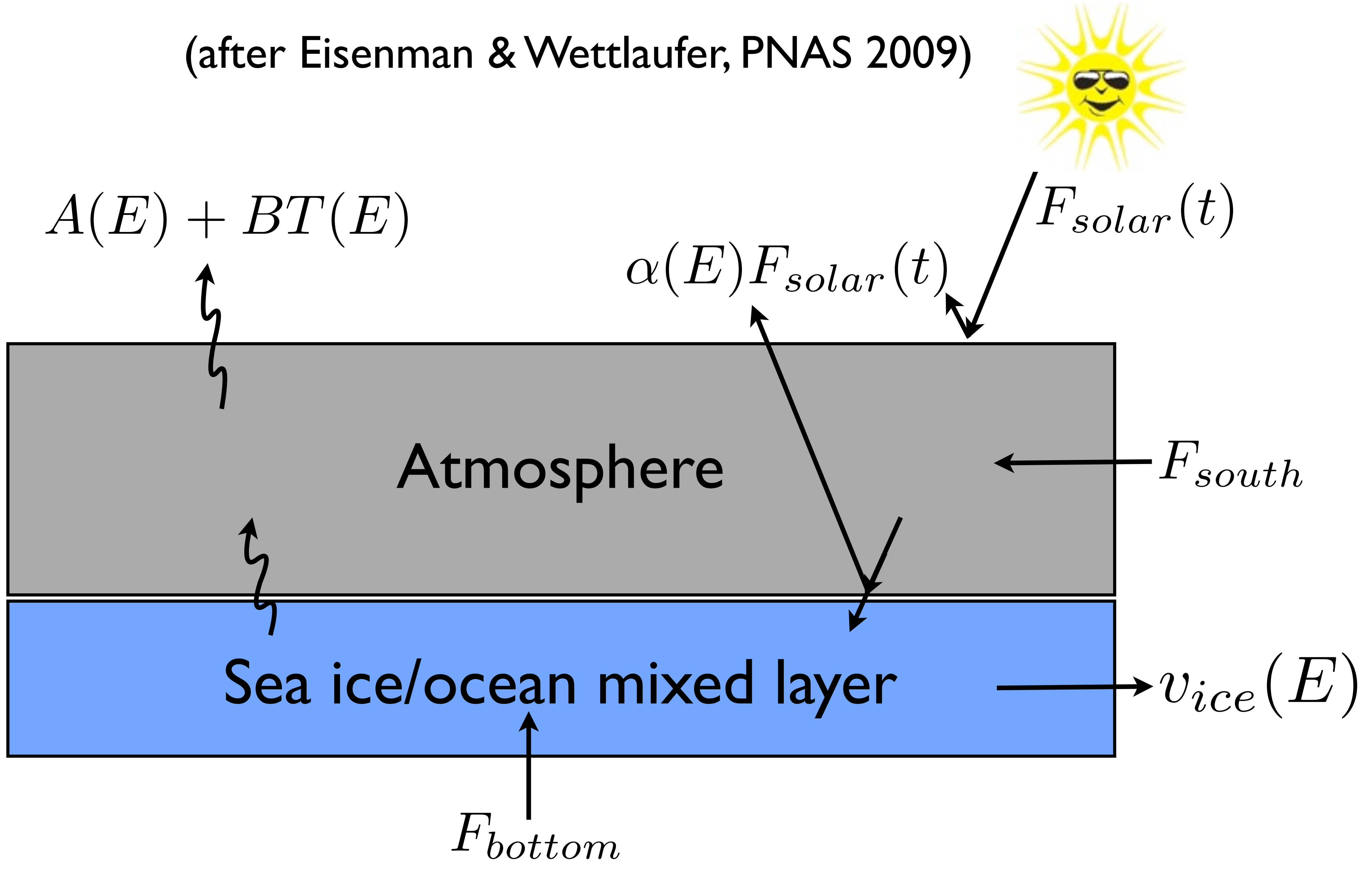
(after Eisenman & Wettlaufer, PNAS 2009)



$$\frac{dE}{dt} = [1 - \alpha(E)] F_{solar}(t) + F_{bottom} + F_{south} + v_{ice}(E) - [A(E) + BT(E)]$$

The 0-d model

(after Eisenman & Wettlaufer, PNAS 2009)



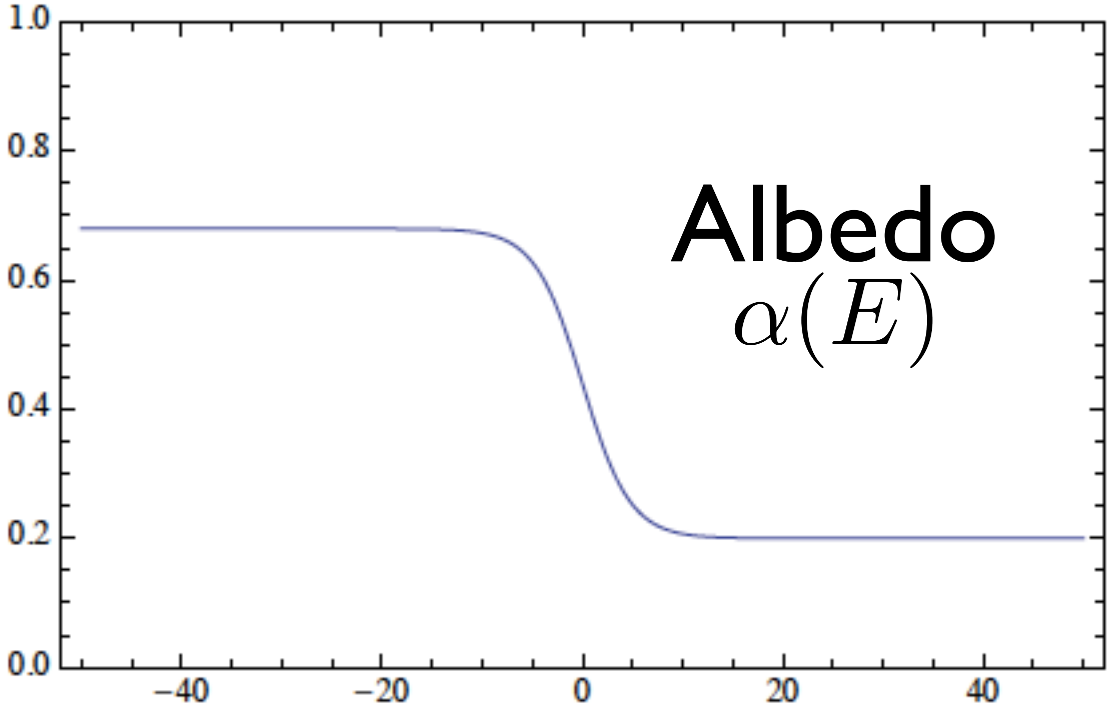
$$\frac{dE}{dt} = [1 - \alpha(E)]F_{solar}(t) + F_{bottom} + F_{south} + v_{ice}(E) - [A(E) + BT(E)]$$

constants

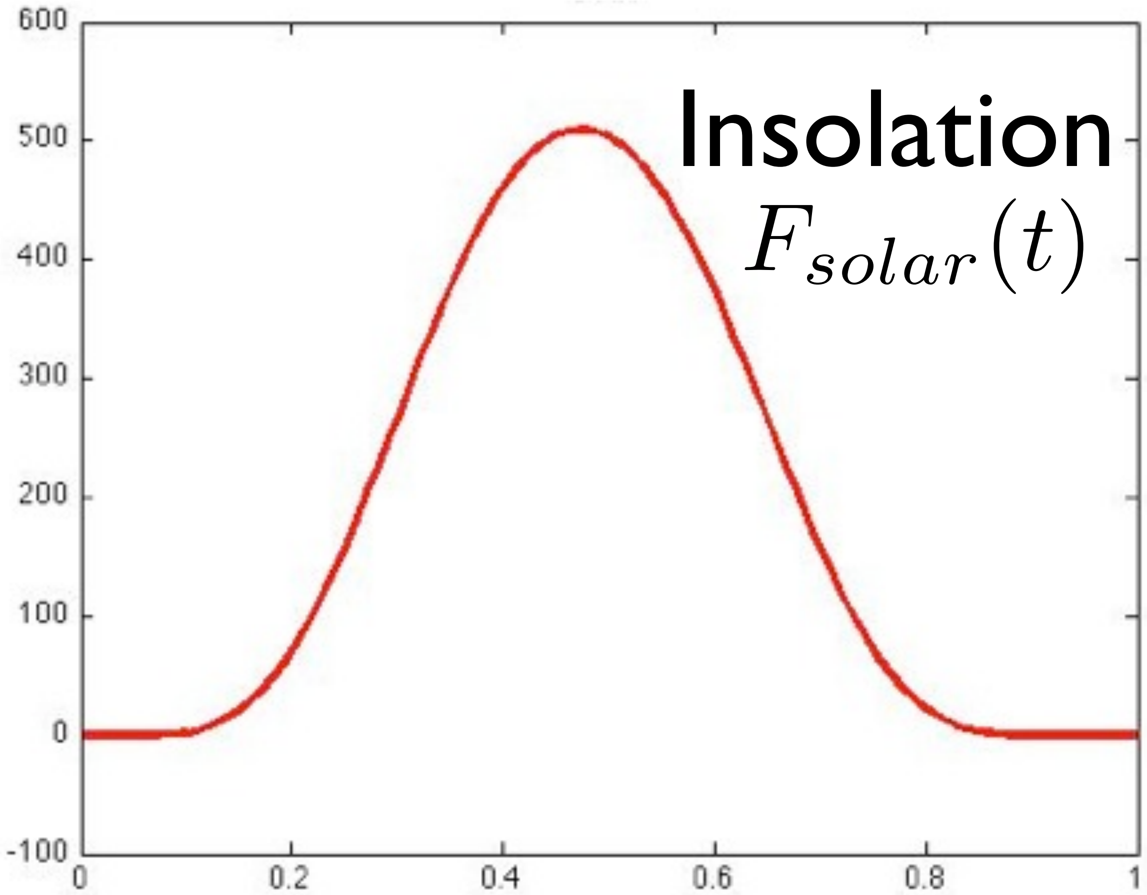
~E (for E<0, otherwise 0)

The 0-d model

(after Eisenman & Wettlaufer, PNAS 2009)



Incoming Solar Radiation:
Positive Ice Albedo feedback



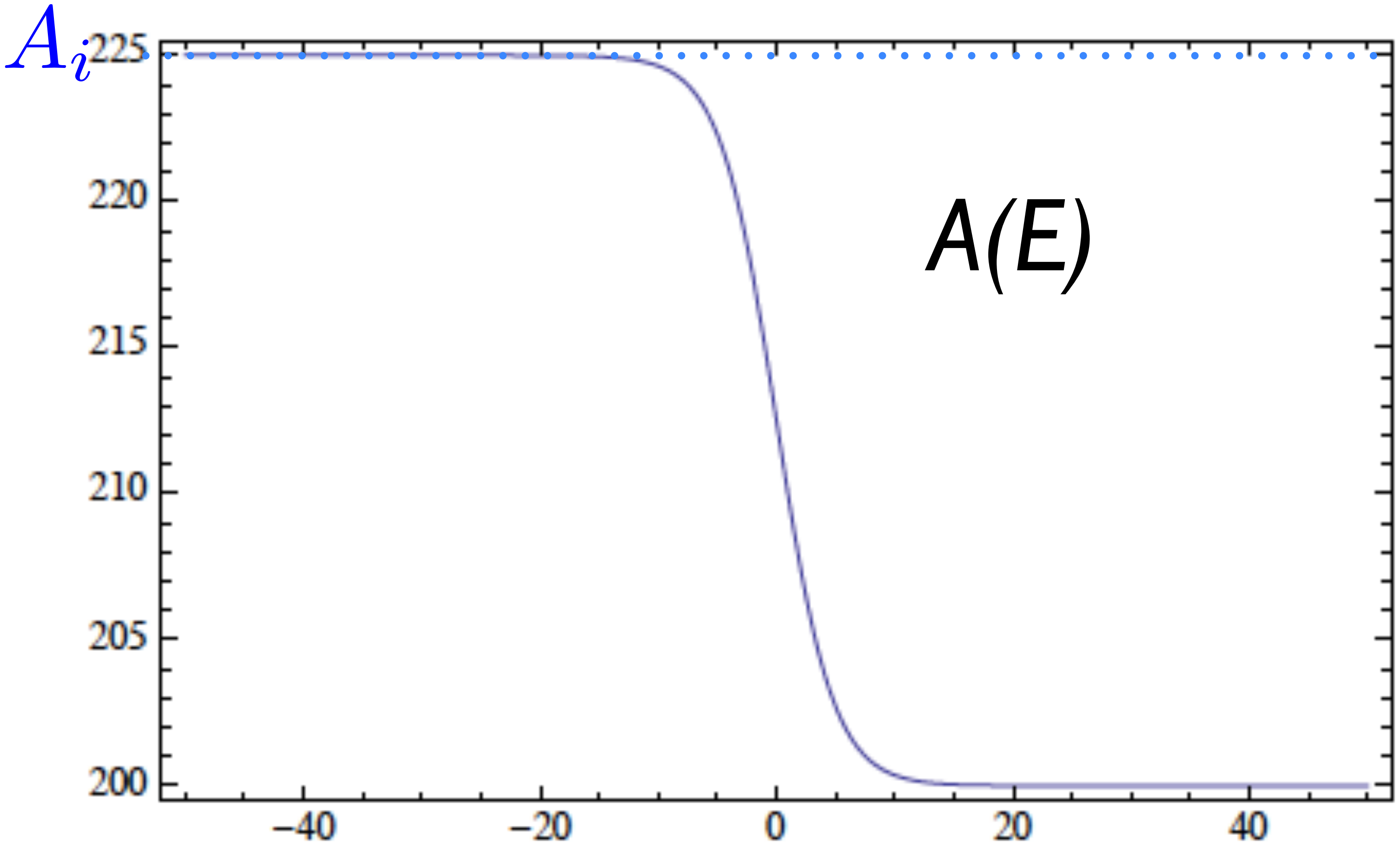
$$\frac{dE}{dt} = \boxed{[1 - \alpha(E)]F_{solar}(t)} + F_{bottom} + F_{south} + v_{ice}(E) - [A(E) + BT(E)]$$

The 0-d model

(after Eisenman & Wettlaufer, PNAS 2009)

Outgoing Long-Wave Radiation (Top of the atmosphere):

Ingredients: "σT^4" linearized about 273 K (T=0)



"Long-wave cloud feedback"

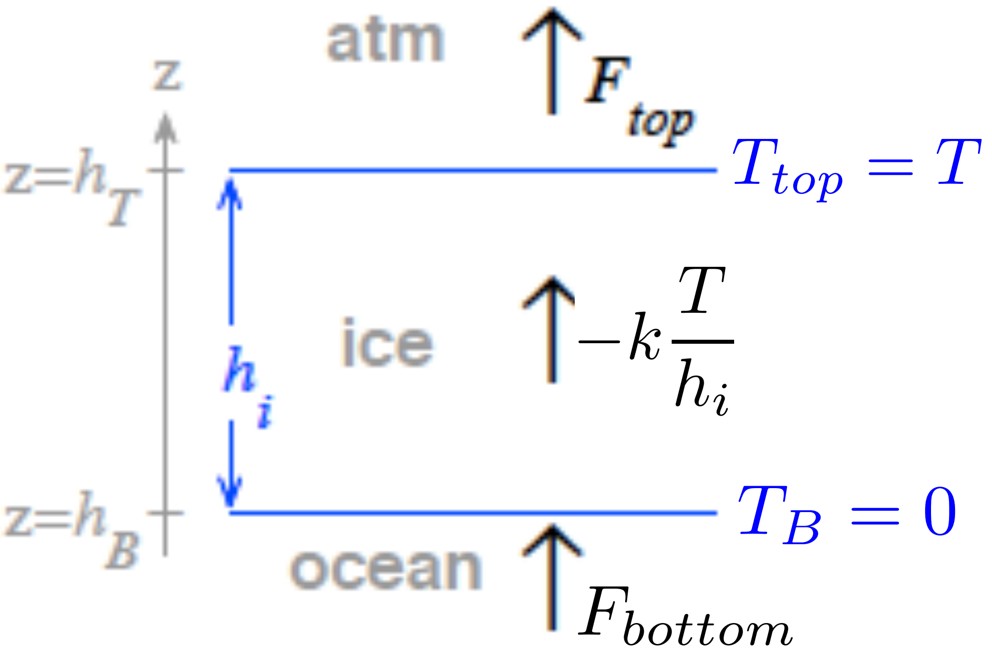
$$A_i = A_0 - \Delta A_{ghg}$$

$$\frac{dE}{dt} = [1 - \alpha(E)]F_{solar}(t) + F_{bottom} + F_{south} + v_{ice}(E) - [A(E) + BT(E)]$$

The 0-d model

(after Eisenman & Wettlaufer, PNAS 2009)

Sea-ice thermodynamics (*E*<*0*)



$$F_{top} = \begin{cases} -k\frac{T}{h_i} & \text{if } T < 0 \ (F_{top} > 0) \\ L_i\frac{dh_T}{dt} & \text{if } T = 0 \ (F_{top} \leq 0) \end{cases}$$

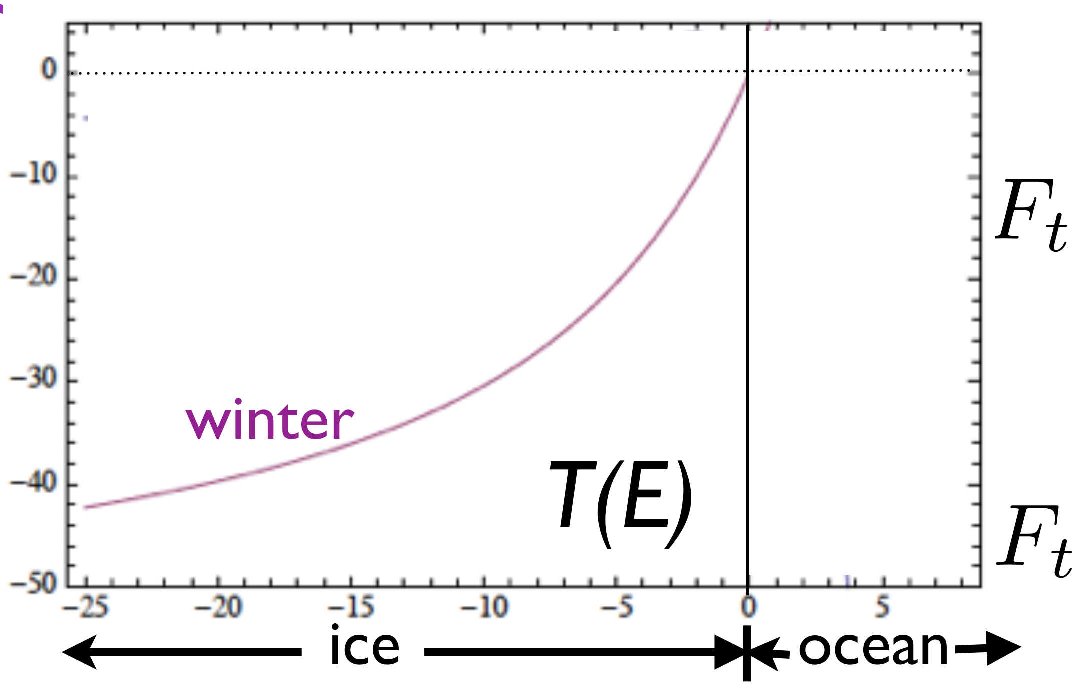
$$F_{top} = F^{out}_{surface} - F^{in}_{surface}$$
$$= [A + BT - F_{south}] - [1 - \alpha]F_{solar}$$

$$\frac{dE}{dt} = [1 - \alpha(E)]F_{solar}(t) + F_{bottom} + F_{south} + v_{ice}(E) - [A(E) + \boxed{BT(E)}]$$

The 0-d model

(after Eisenman & Wettlaufer, PNAS 2009)

$$E(t) = \begin{cases} -L_i \boxed{h_i(t)} & \text{if } E < 0 \quad (i.e.\ E \propto \text{ice thickness } h_i) \\ C_s T(t) & \text{if } E \geq 0 \quad (i.e.\ E \propto \text{mixed layer temp. } T) \end{cases}$$



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$$\begin{aligned} F_{top} &= F^{out}_{surface} - F^{in}_{surface} \\ &= [A + BT - F_{south}] - [1 - \alpha]F_{solar} \end{aligned}$$

$$\frac{dE}{dt} = [1 - \alpha(E)]F_{solar}(t) + F_{bottom} + F_{south} + v_{ice}(E) - [A(E) + \boxed{BT(E)}]$$

The 0-d model

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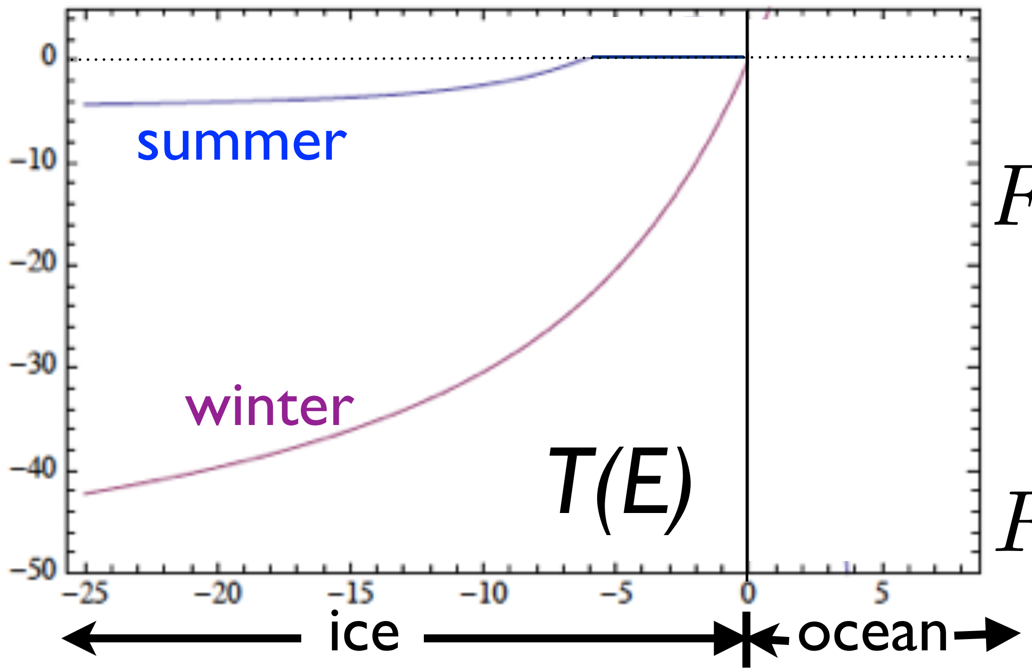


winter sea ice grows

$$F_{top} = \begin{cases} -k\frac{T}{h_i} & \text{if } T < 0 \ (F_{top} > 0) \\ L_i \frac{dh_T}{dt} & \text{if } \underline{T = 0} \ (F_{top} \leq 0) \end{cases}$$

summer sea ice ablates

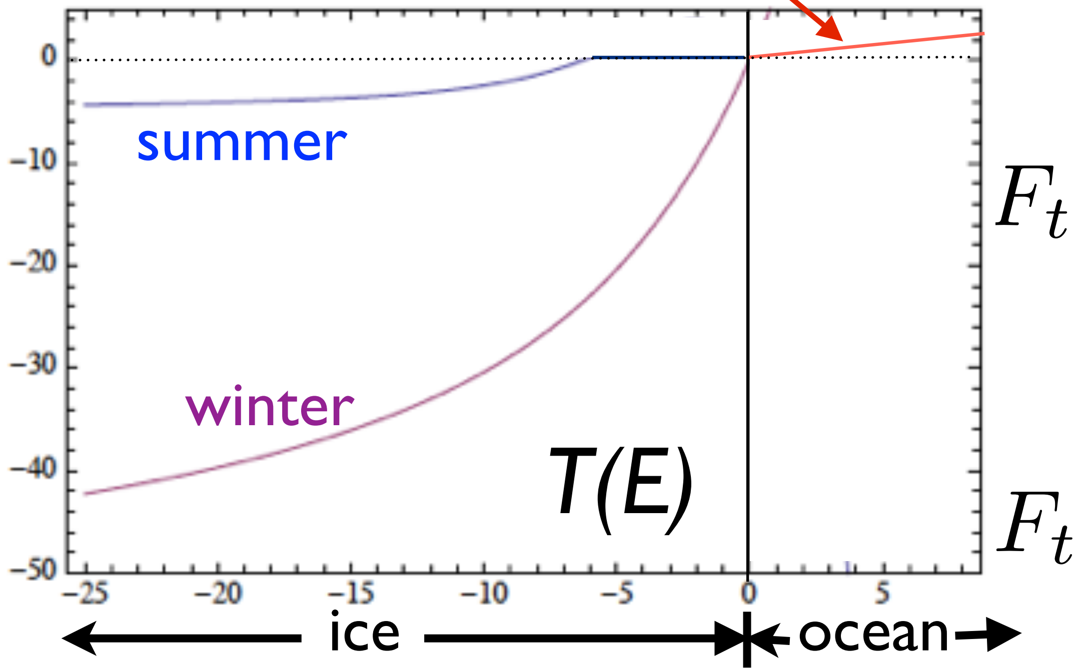
$$\begin{aligned} F_{top} &= F^{out}_{surface} - F^{in}_{surface} \\ &= [A + BT - F_{south}] - [1 - \alpha]F_{solar} \end{aligned}$$

$$\frac{dE}{dt} = [1 - \alpha(E)]F_{solar}(t) + F_{bottom} + F_{south} + v_{ice}(E) - [A(E) + \boxed{B\underline{T(E)}]}$$

The 0-d model

(after Eisenman & Wettlaufer, PNAS 2009)

$$E(t) = \begin{cases} -L_i h_i(t) & \text{if } E < 0 \quad (i.e.\ E \propto \text{ice thickness } h_i) \\ \underline{C_s T(t)} & \text{if } E \geq 0 \quad (i.e.\ E \propto \text{mixed layer temp. } T) \end{cases}$$



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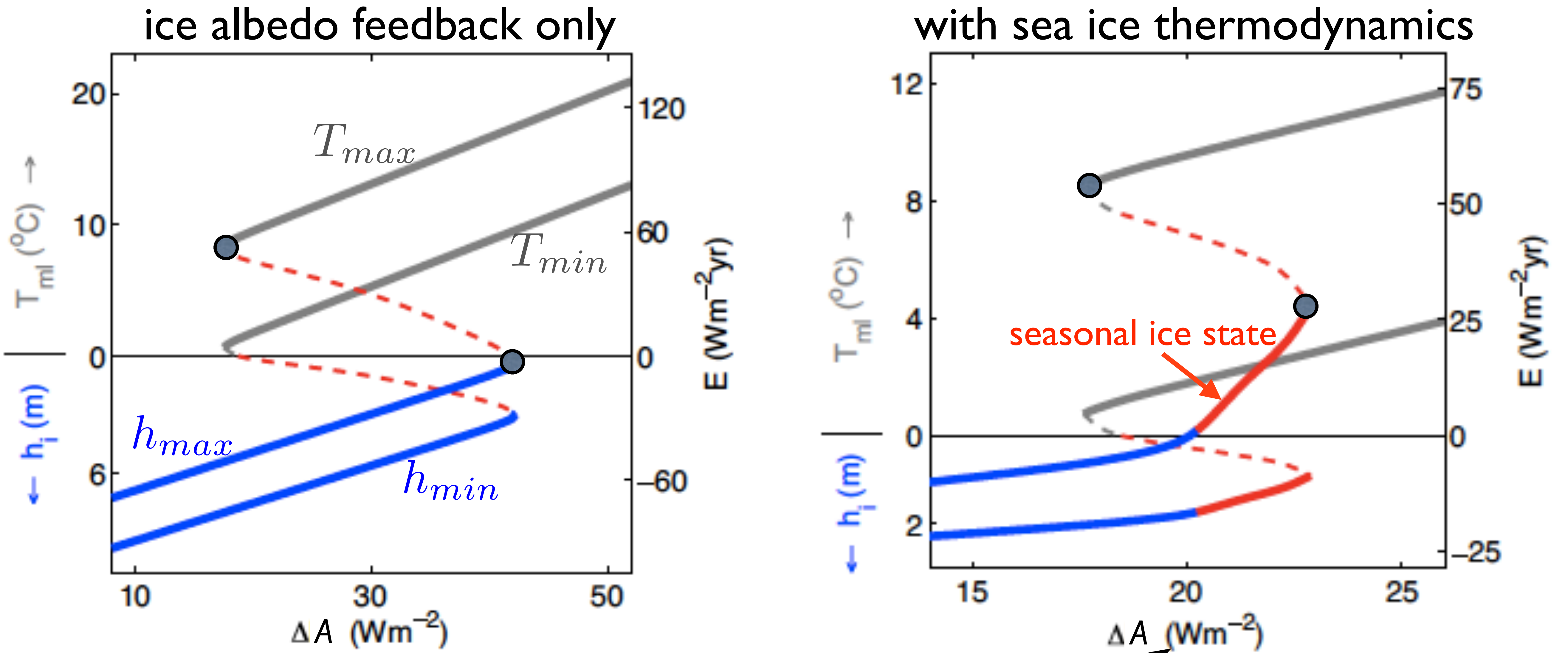
$$\frac{dE}{dt} = [1 - \alpha(E)]F_{solar}(t) + F_{bottom} + F_{south} + v_{ice}(E) - [A(E) + B\underline{T(E)}]$$

EW09 results:

the role of sea ice thermodynamics: no summer tipping point?

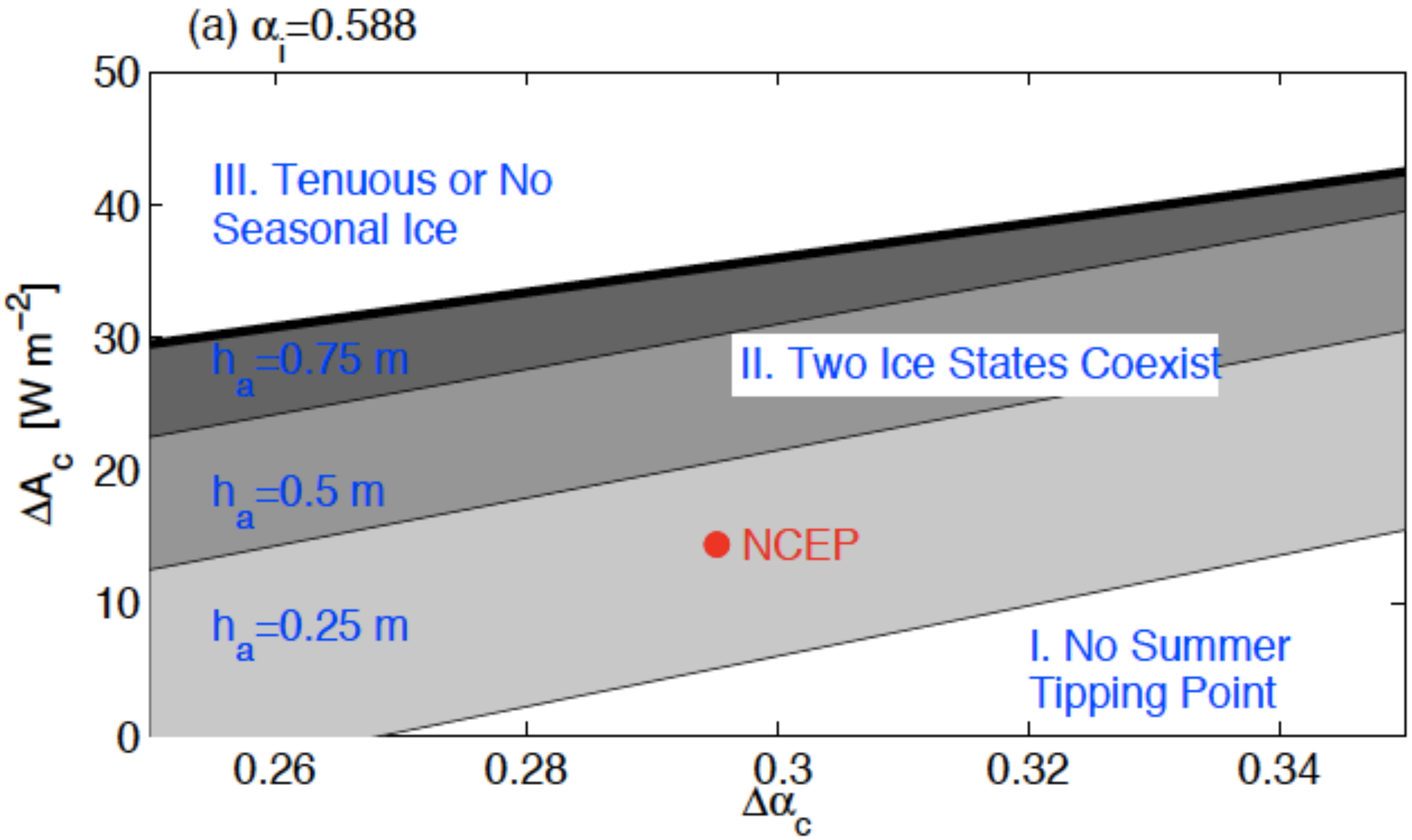


bifurcation diagrams from Eisenman & Wettlaufer, PNAS 2009

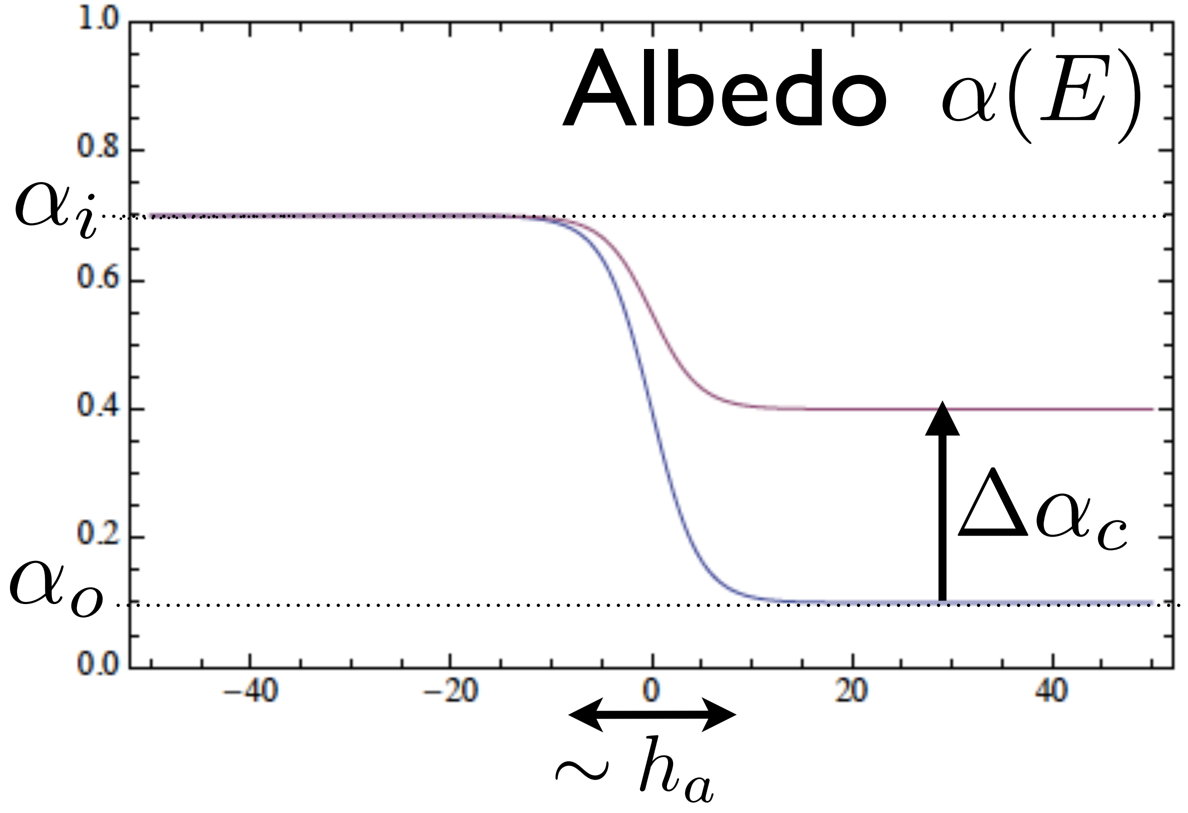
$A = A_0 - \Delta A$

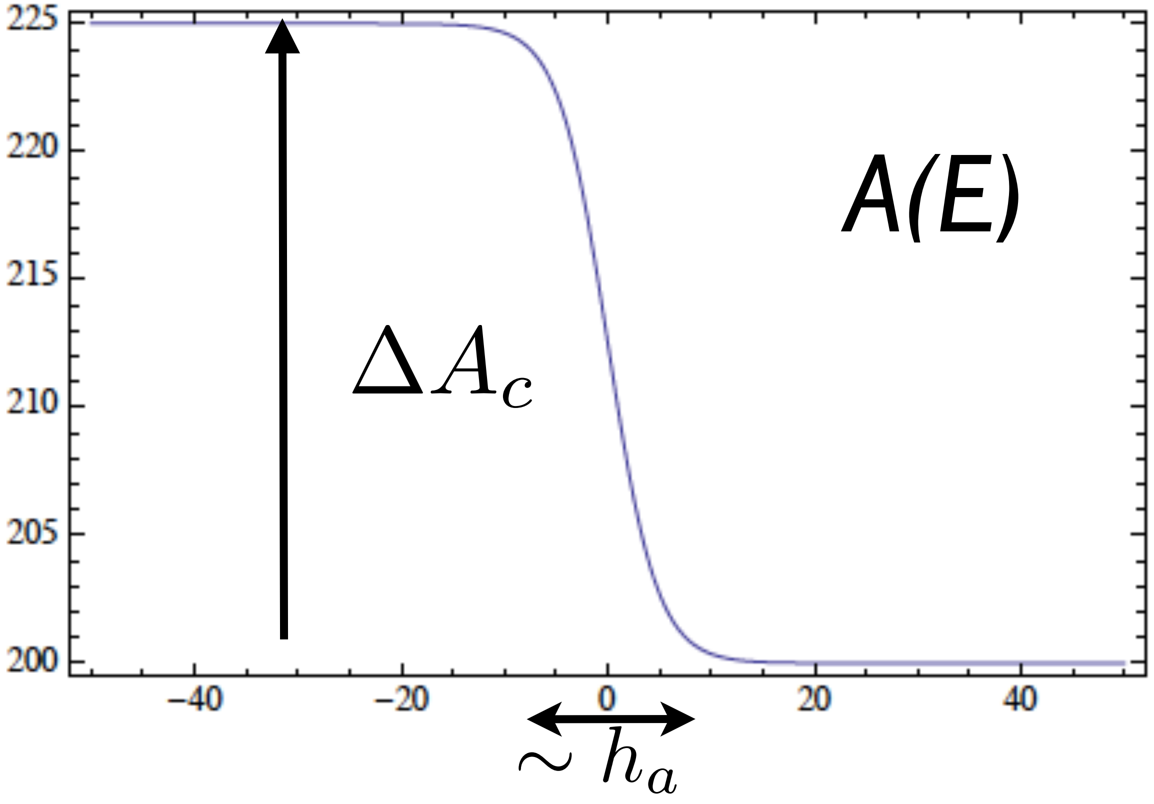
$$\frac{dE}{dt} = [1 - \alpha(E)]F_{solar}(t) + F_{bottom} + F_{south} + v_{ice}(E) - [A(\xcancel{E}) + BT(E)]$$

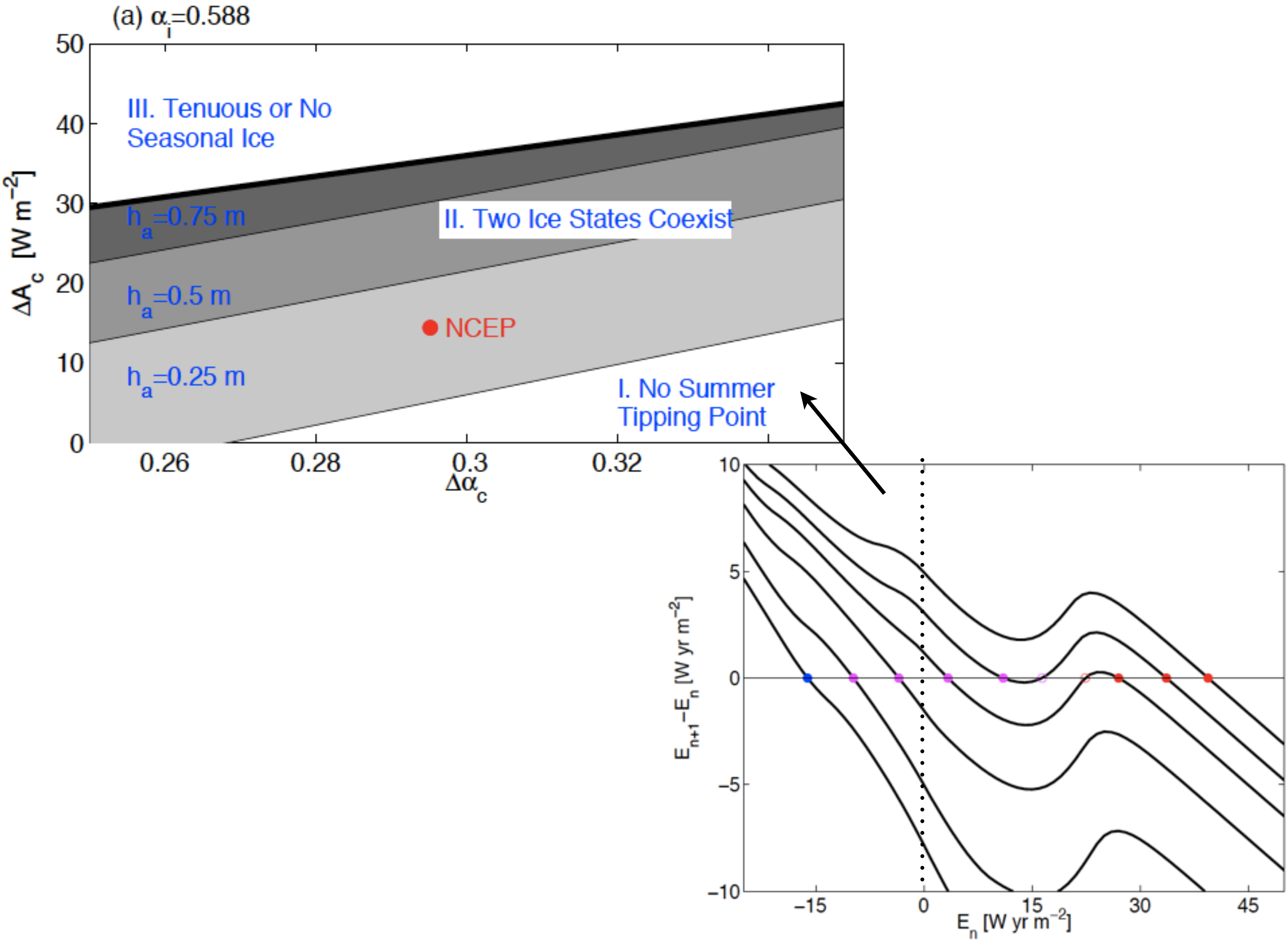
“cloud feedbacks”: no summer tipping point?

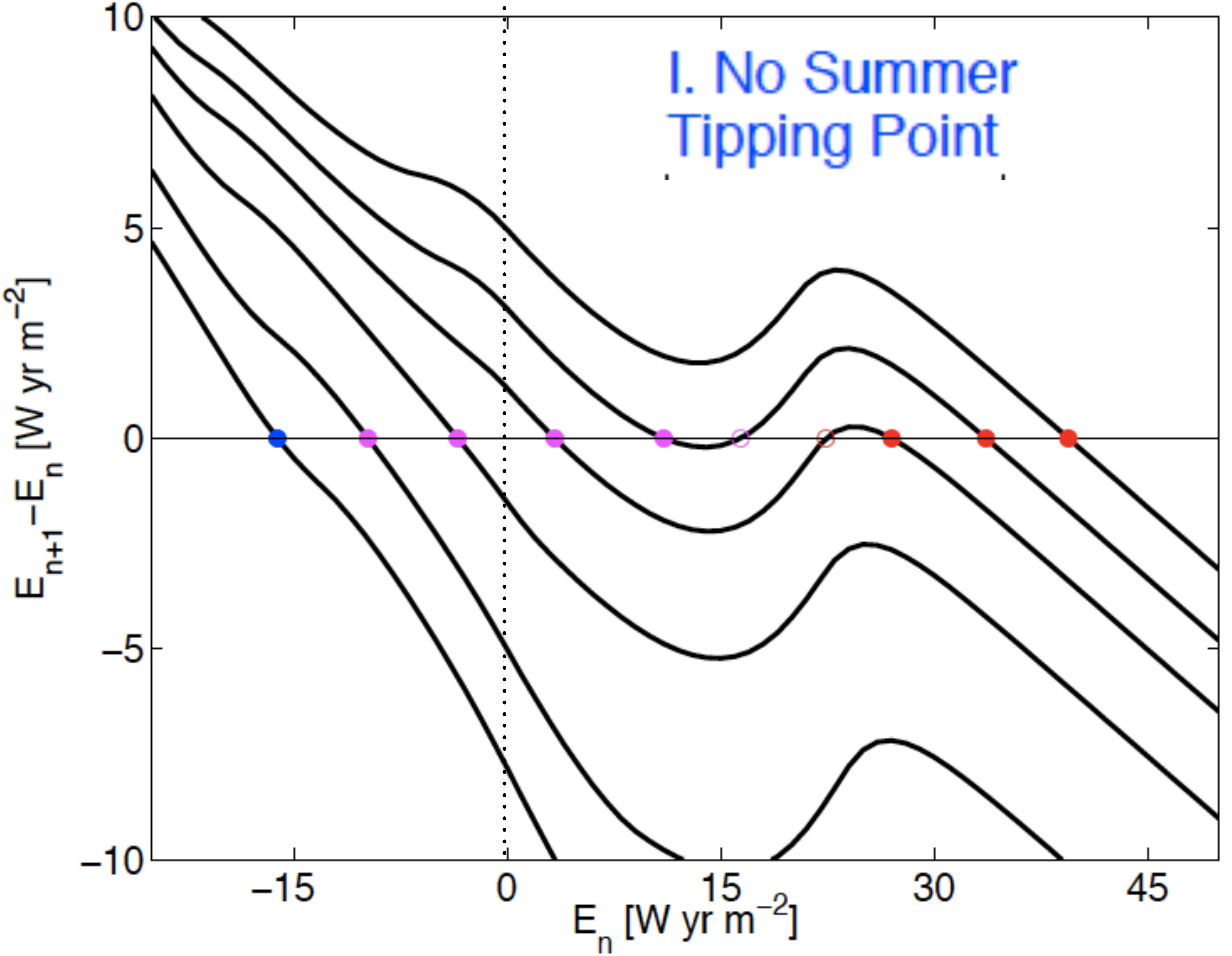


Figures from Abbot, Silber, Pierrehumbert 2010 preprint

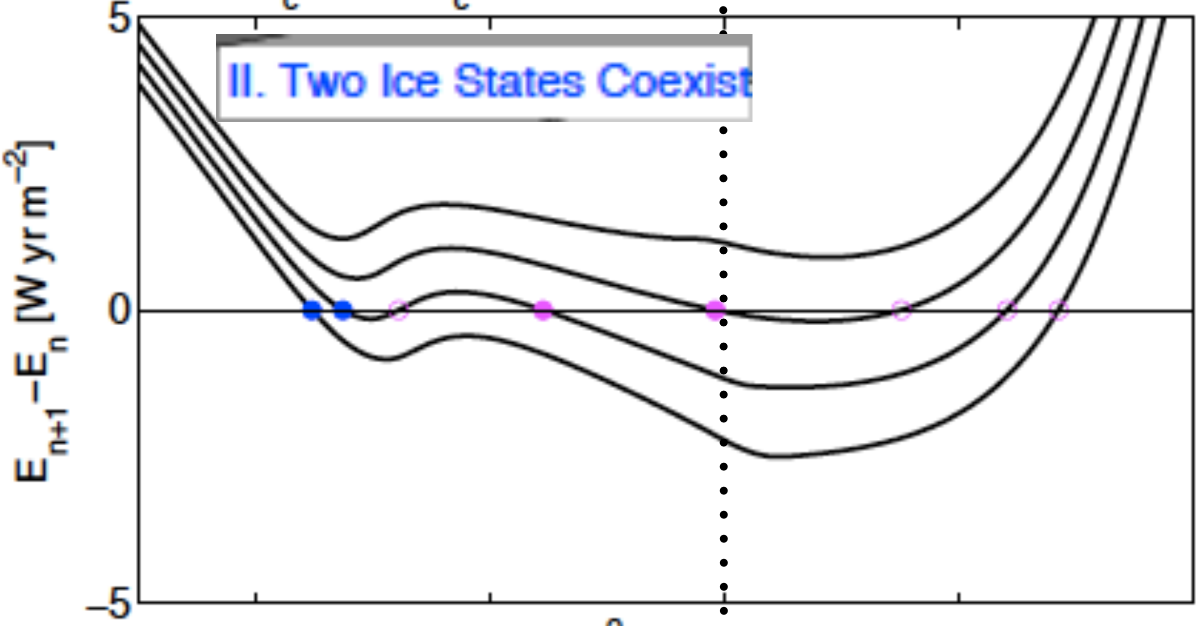




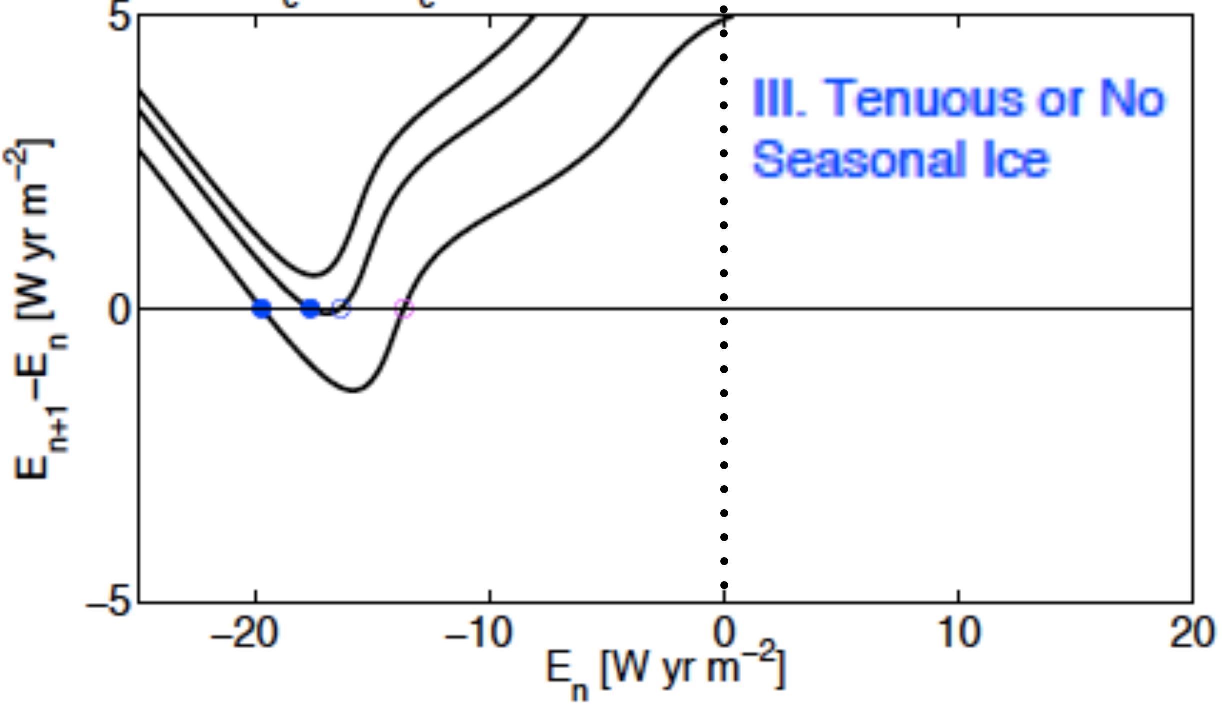

(a) α_i=0.588
III. Tenuous or No Seasonal Ice
II. Two Ice States Coexist
I. No Summer Tipping Point
h_a=0.75 m
h_a=0.5 m
h_a=0.25 m
NCEP
ΔA_c [W m^{-2}]
$\Delta\alpha_c$
0
10
20
30
40
50
0.26
0.28
0.3
0.32
$E_{n+1}-E_n$ [W yr m^{-2}]
E_n [W yr m^{-2}]
10
5
0
−5
−10
−15
0
15
30
45


I. No Summer
Tipping Point
10
5
0
−5
−10
−15
0
15
30
45
$E_{n+1}-E_n$ [W yr m^{-2}]
E_n [W yr m^{-2}]

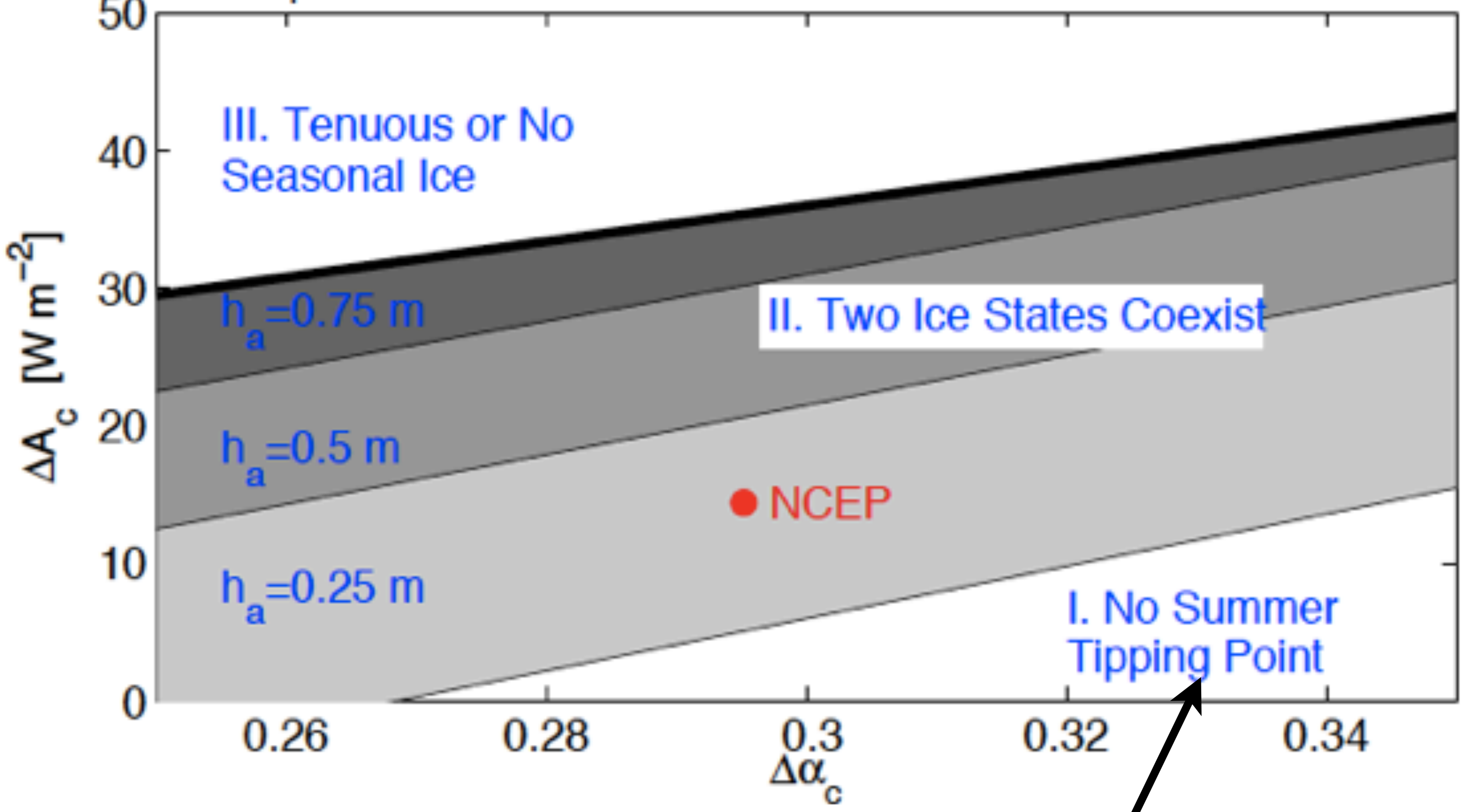
(a) $\Delta\alpha_c$=0.3, ΔA_c=25 [W m^{-2}]

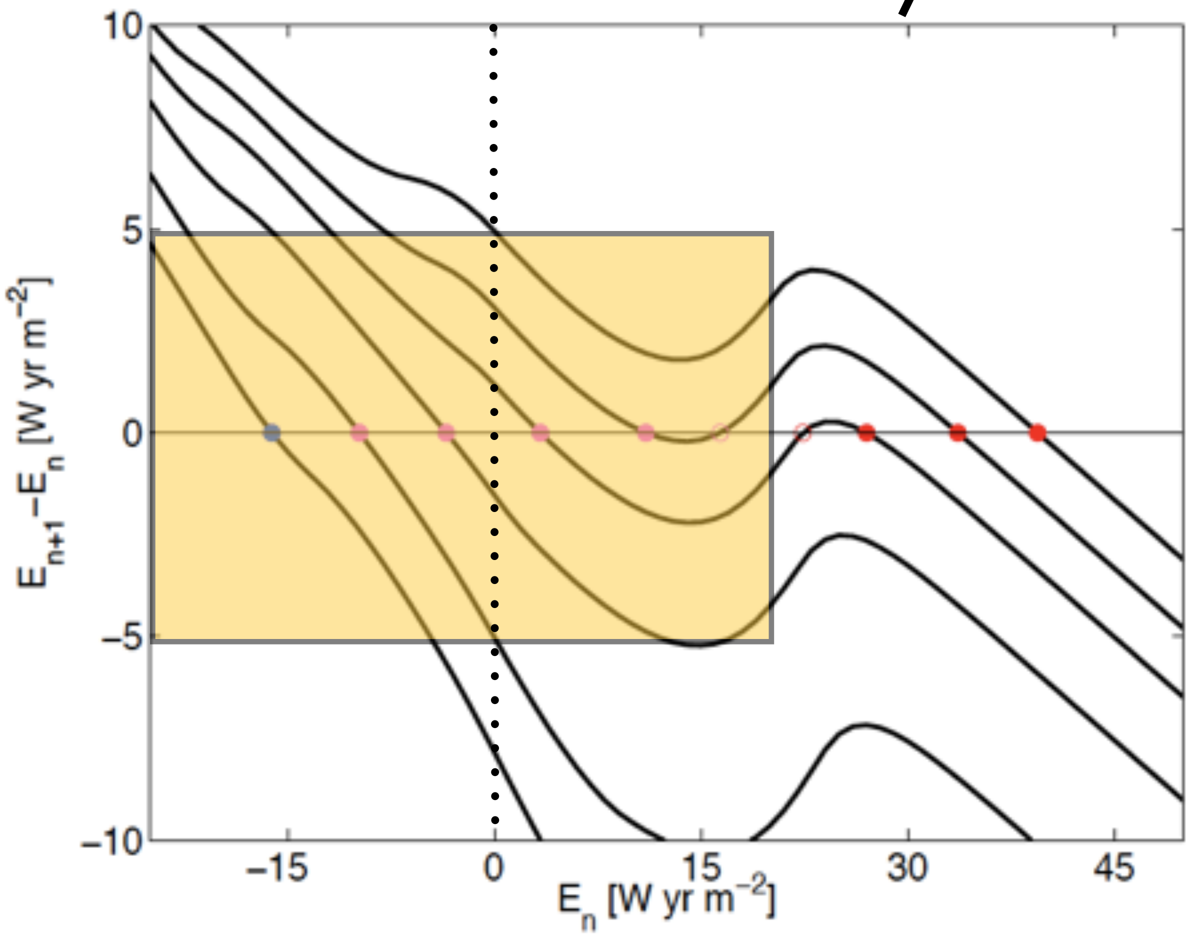


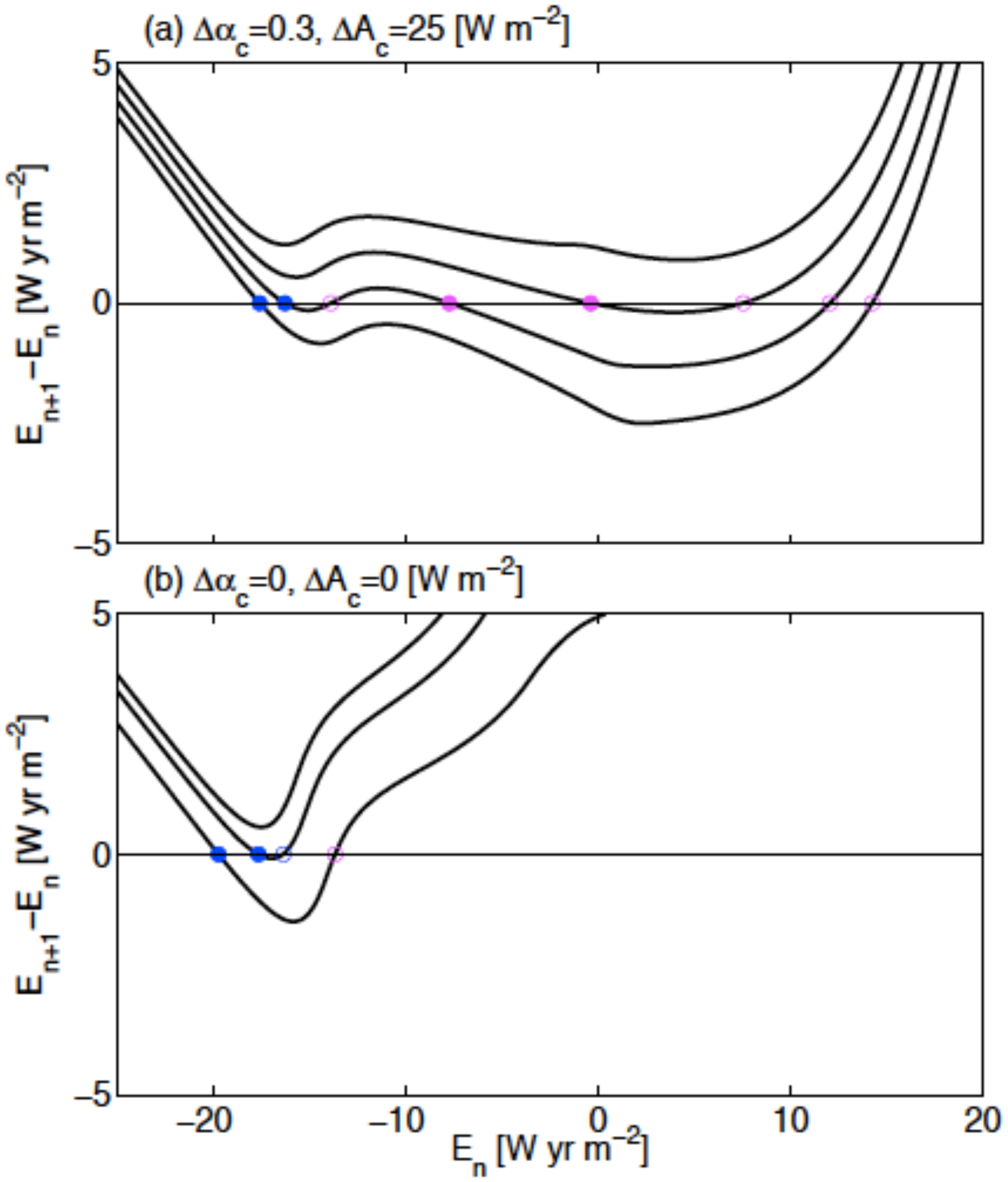
(b) $\Delta\alpha_c$=0, ΔA_c=0 [W m^{-2}]



(a) α_i=0.588







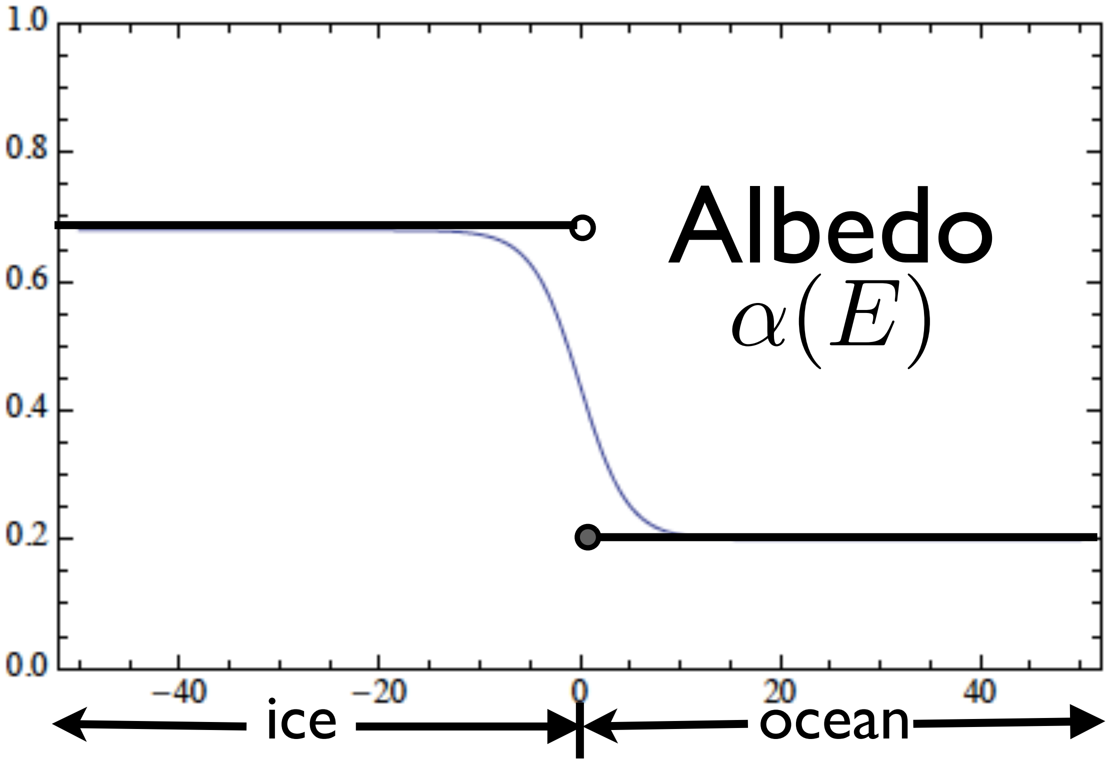
(a) $\Delta\alpha_c$=0.3, ΔA_c=25 [W m^{-2}]

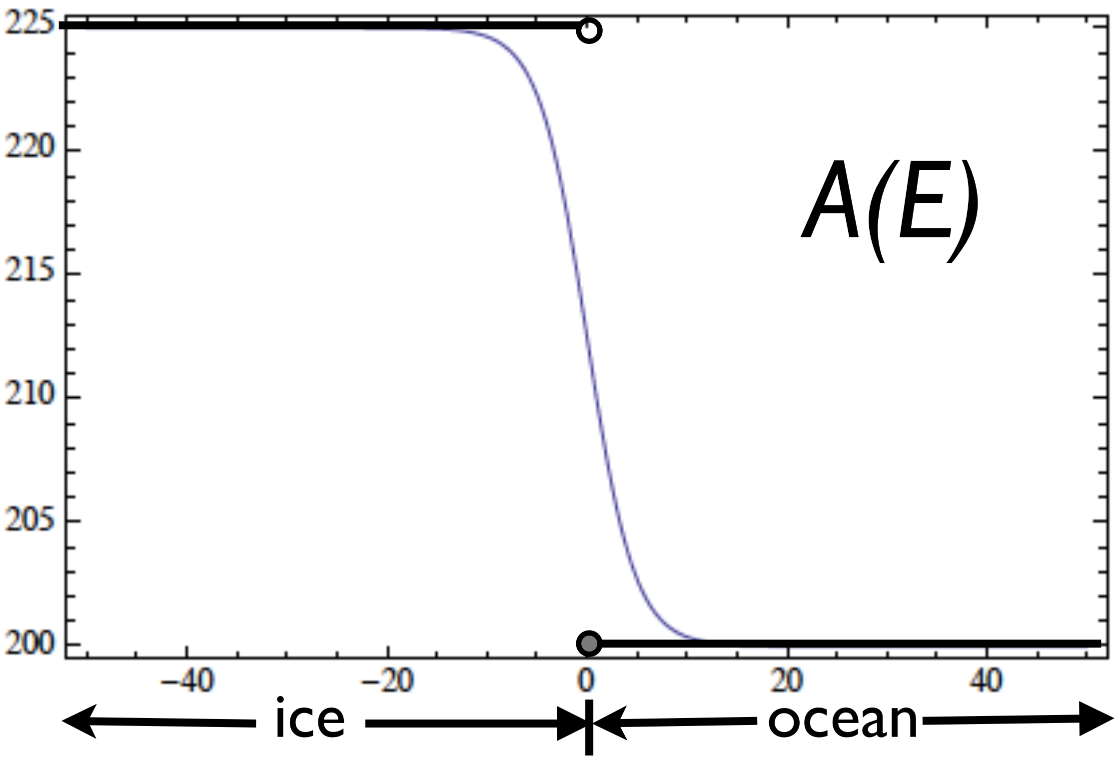
(b) $\Delta\alpha_c$=0, ΔA_c=0 [W m^{-2}]

Some analysis:

determining existence conditions for seasonally ice-free states

Approximation: piecewise constant α(*E*) and *A(E)*



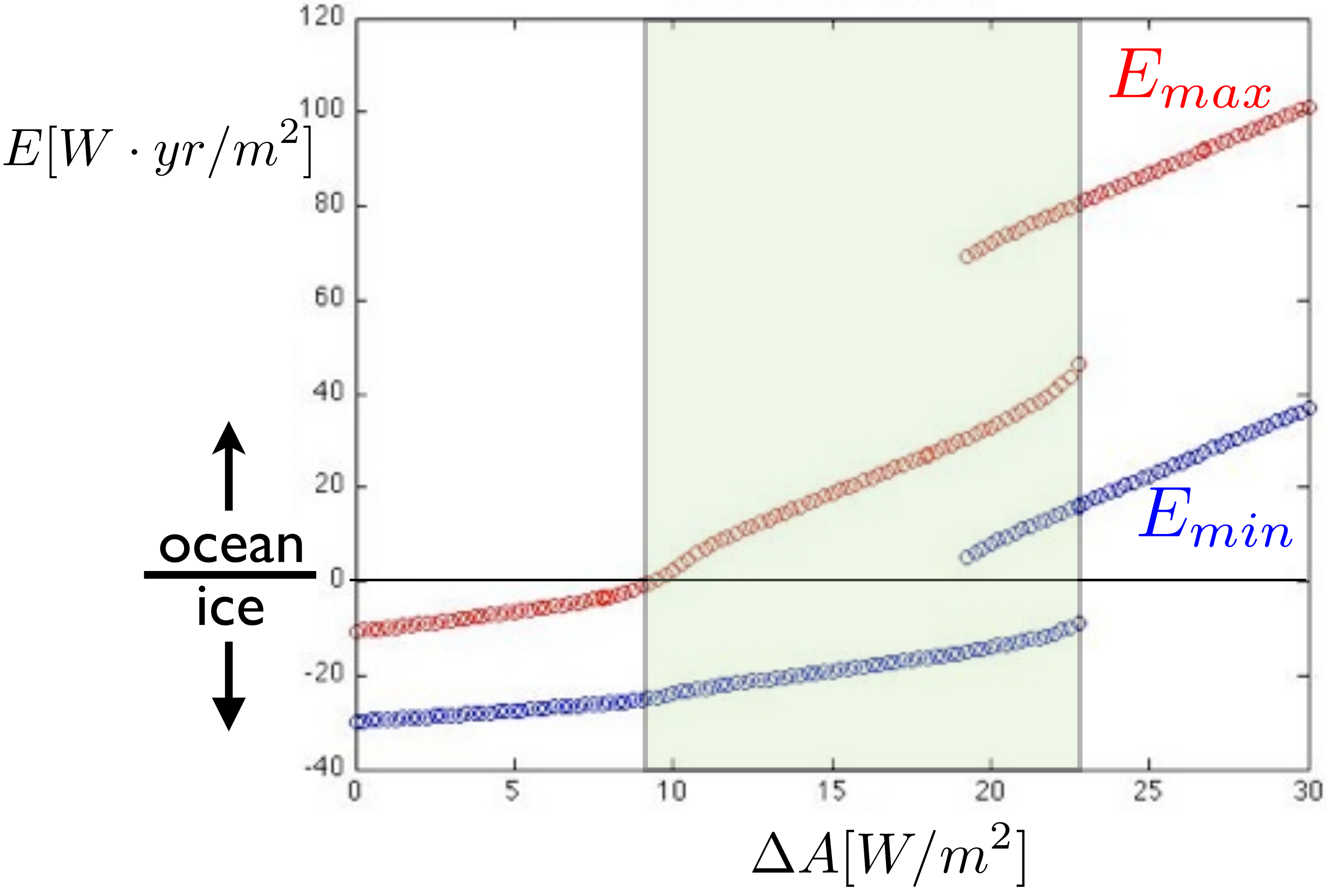


$$\frac{dE}{dt} = [1 - \alpha(E)]F_{solar}(t) + F_{bottom} + F_{south} + v_{ice}(E) - [A(E) + BT(E)]$$

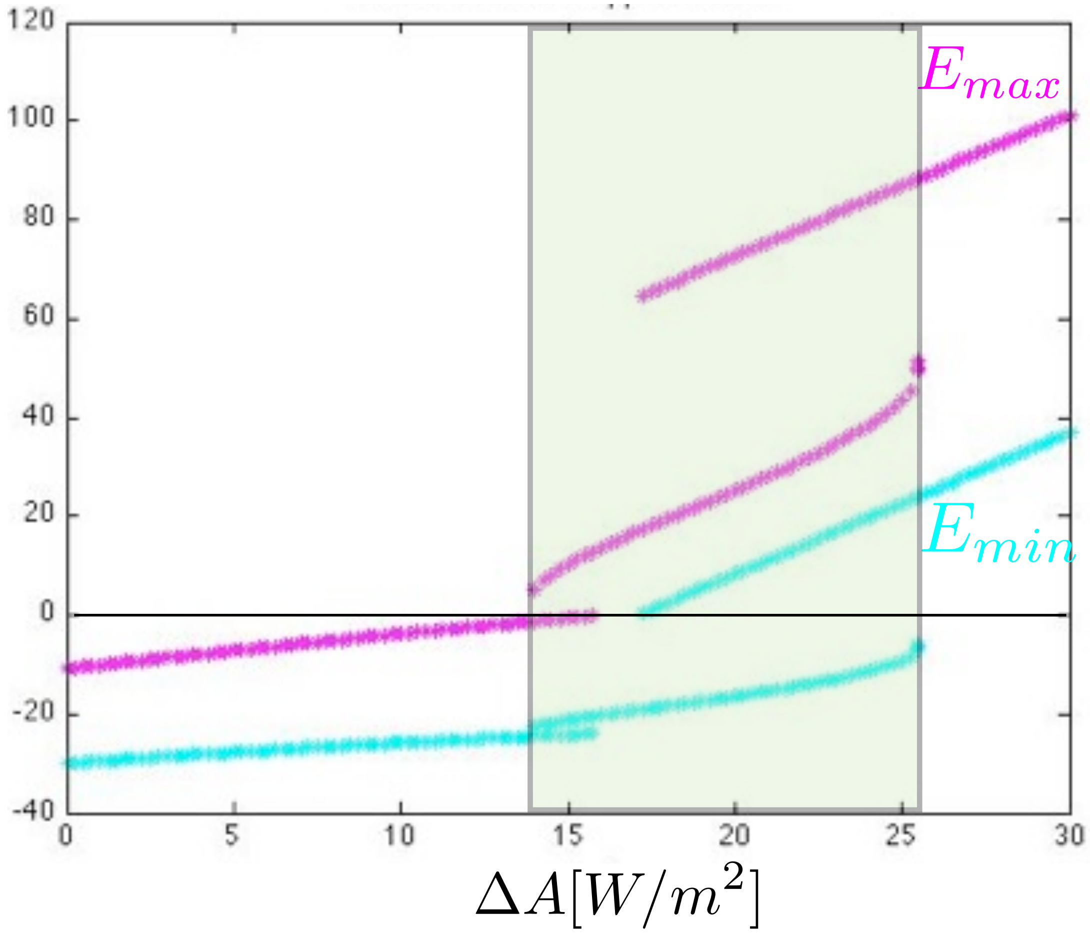
Some analysis:

determining existence conditions for seasonally ice-free states

Smoothed albedo attractors



Piece-wise constant albedo attractors

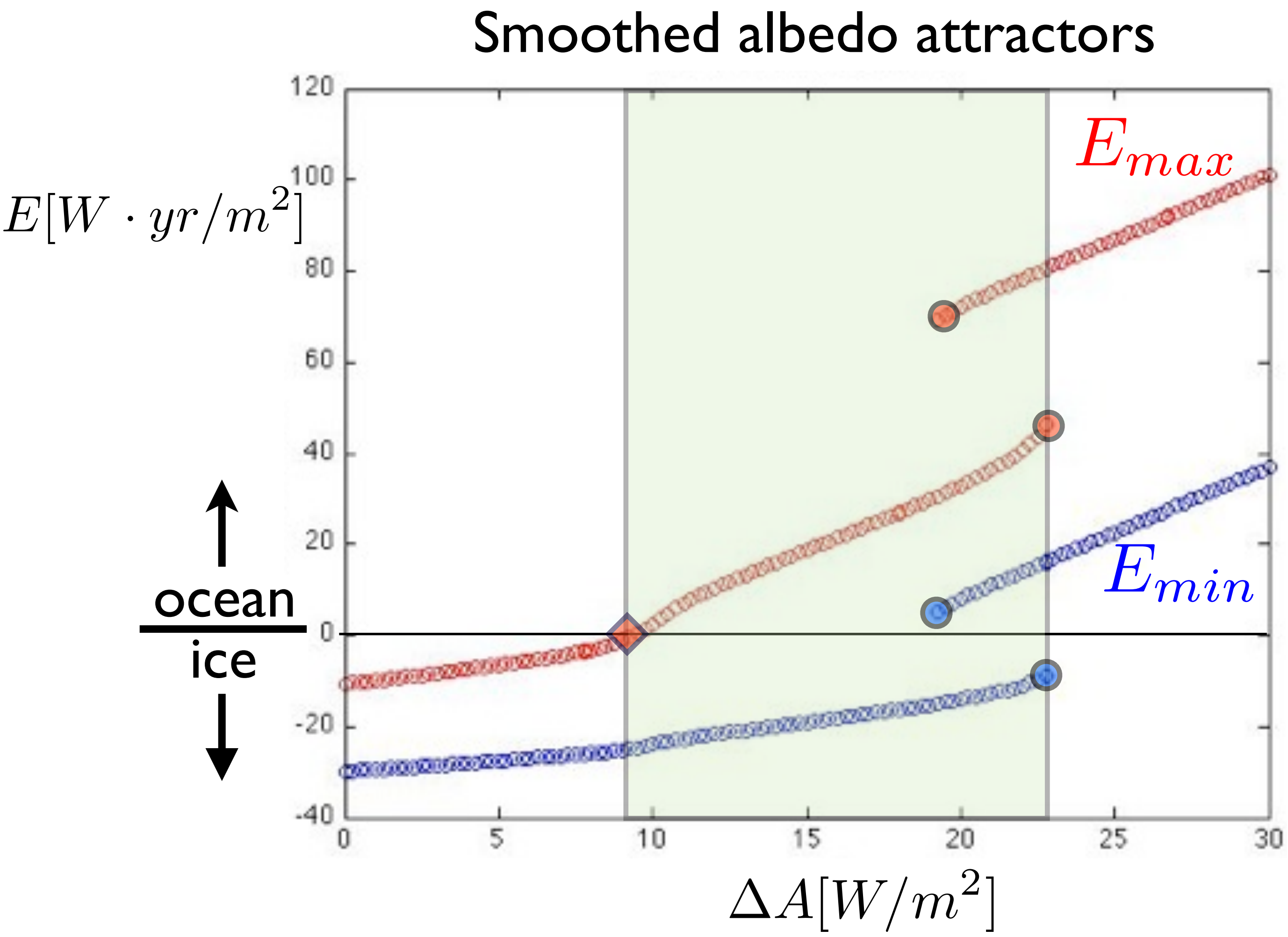


$$A = A_0 - \Delta A$$

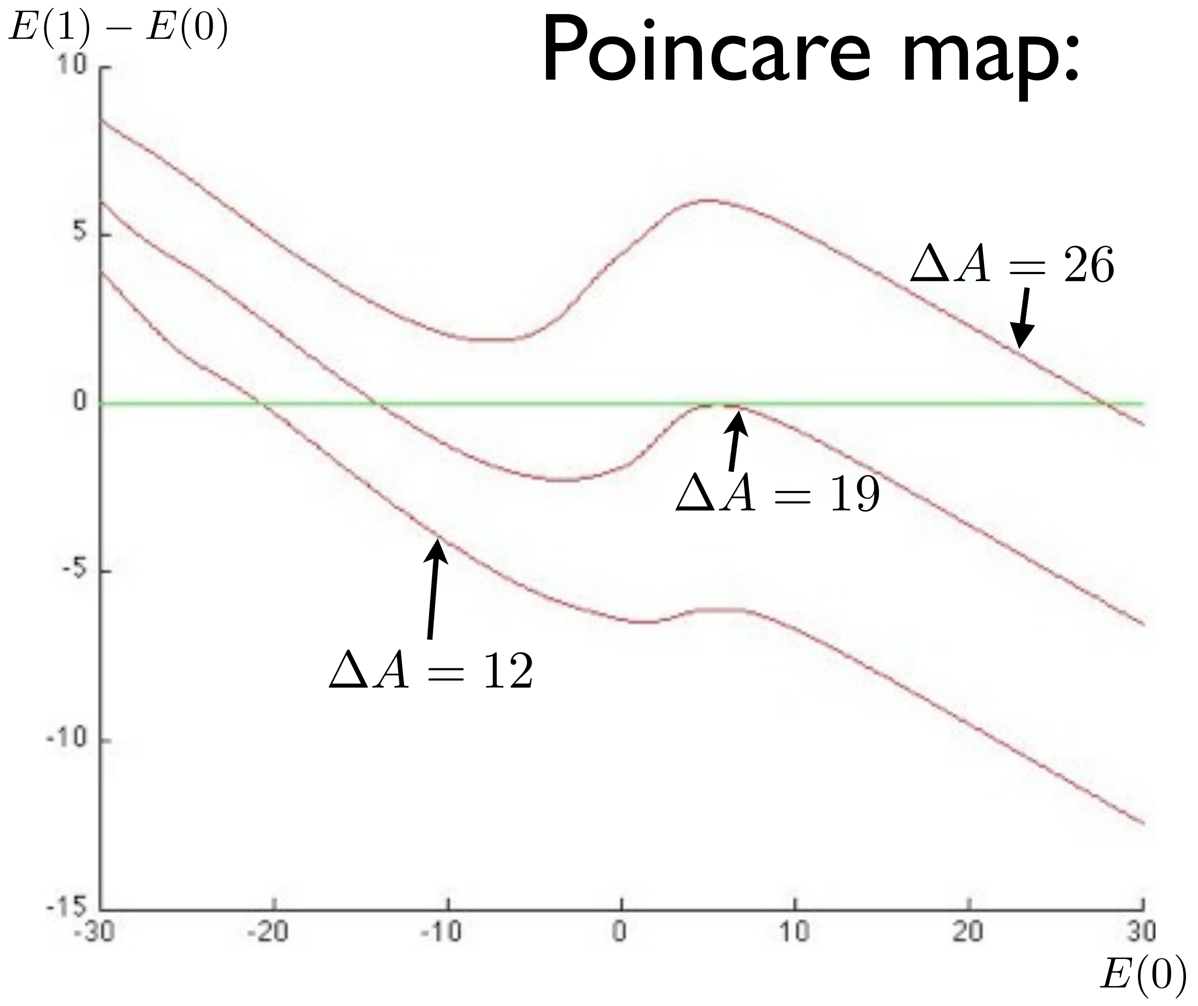
$$\frac{dE}{dt} = [1 - \alpha(E)]F_{solar}(t) + F_{bottom} + F_{south} + v_{ice}(E) - [A(\cancel{E}) + BT(E)]$$

Some analysis:

determining existence conditions for seasonally ice-free states



Smoothed albedo attractors



Poincare map:

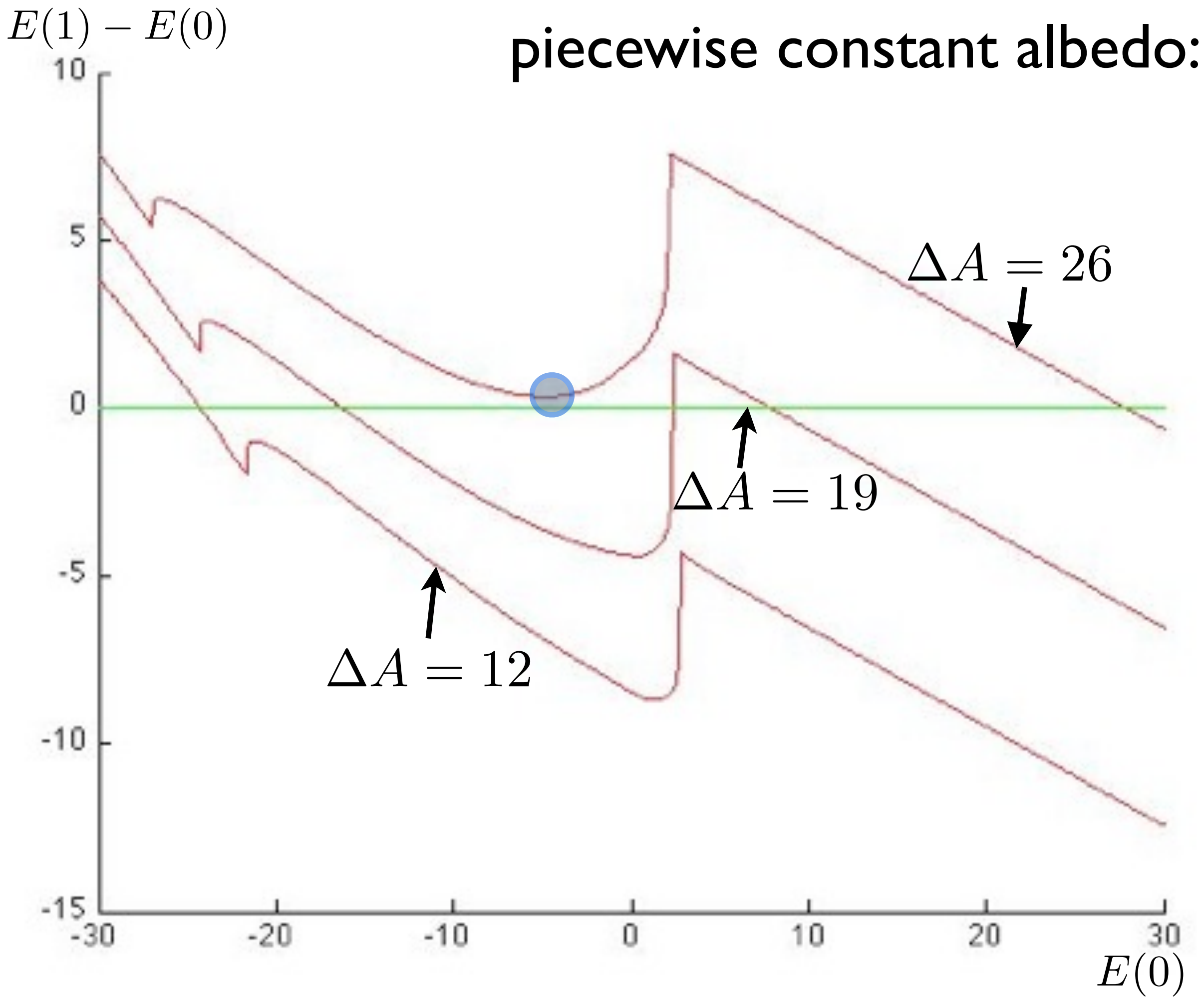
$$A = A_0 - \Delta A$$

$$\frac{dE}{dt} = [1 - \alpha(E)]F_{solar}(t) + F_{bottom} + F_{south} + v_{ice}(E) - [A(\cancel{E}) + BT(E)]$$

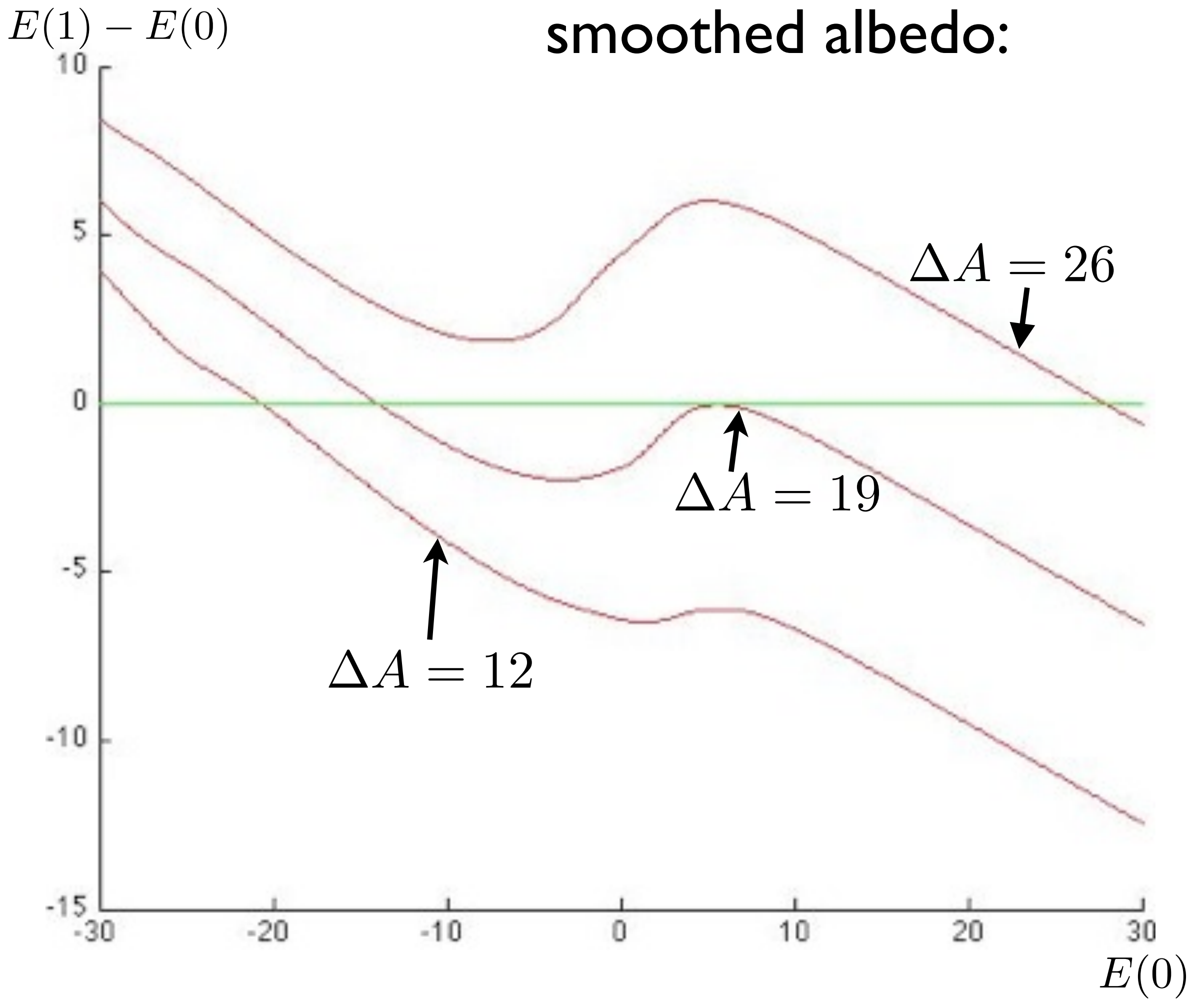
Some analysis:

determining existence conditions for seasonally ice-free states

piecewise constant albedo:



smoothed albedo:



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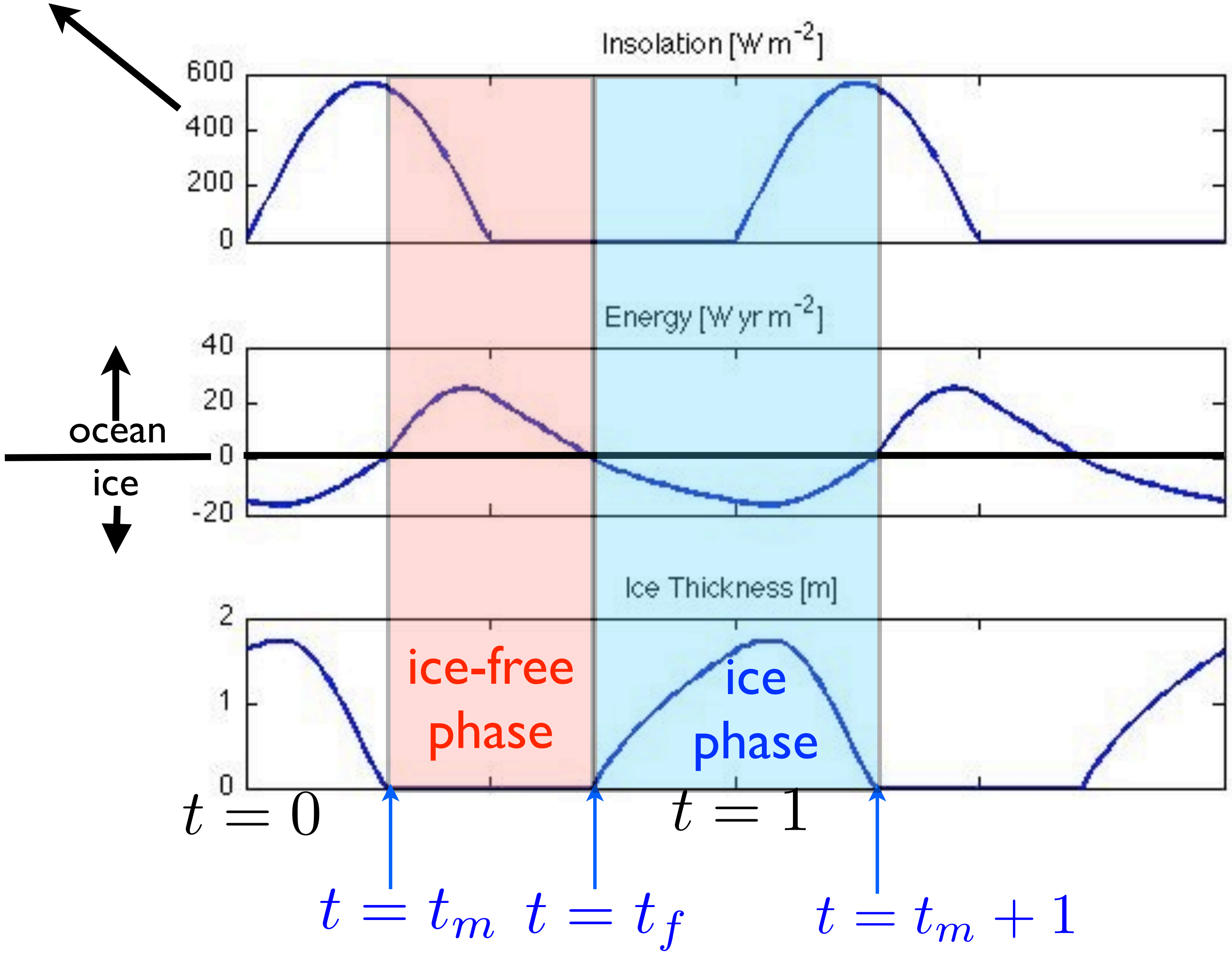
Existence conditions for seasonally ice-free states

$$\frac{dE}{dt} = [1-\alpha(E)]F_{solar}(t) + F_{bottom} + F_{south} + v_{ice}(E) - [A(E) + BT(E)]$$

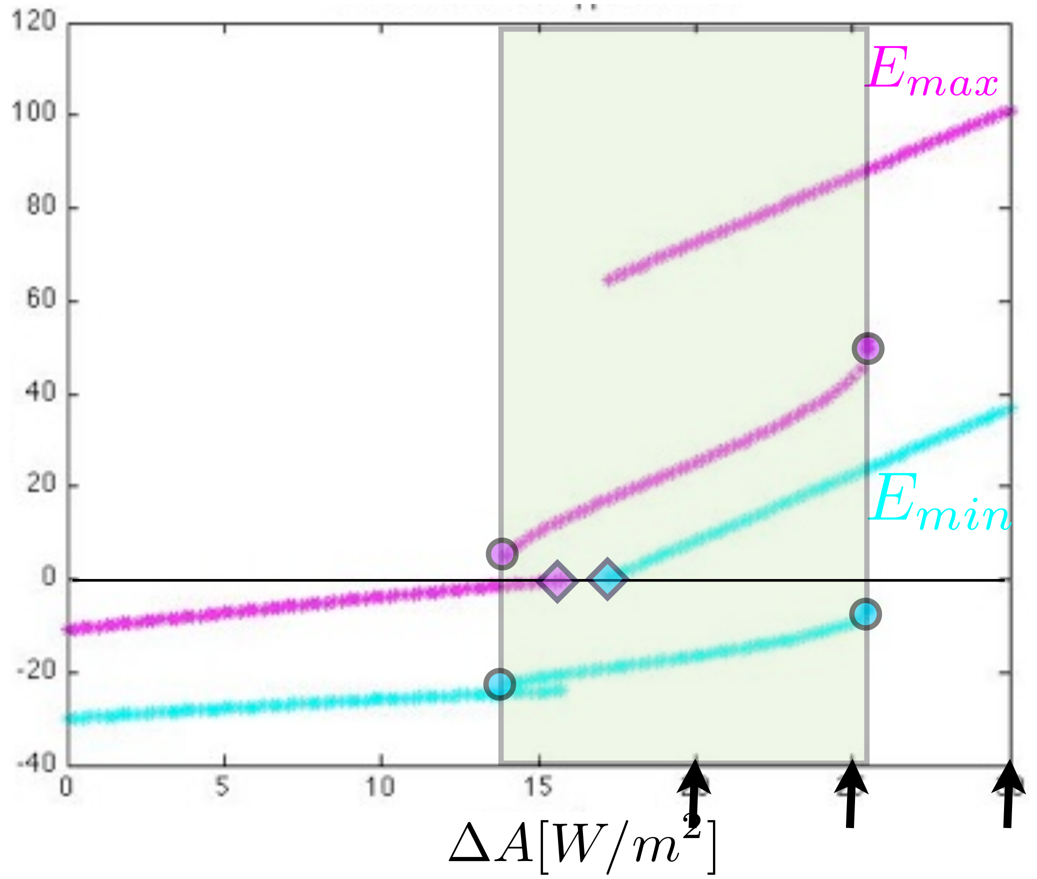


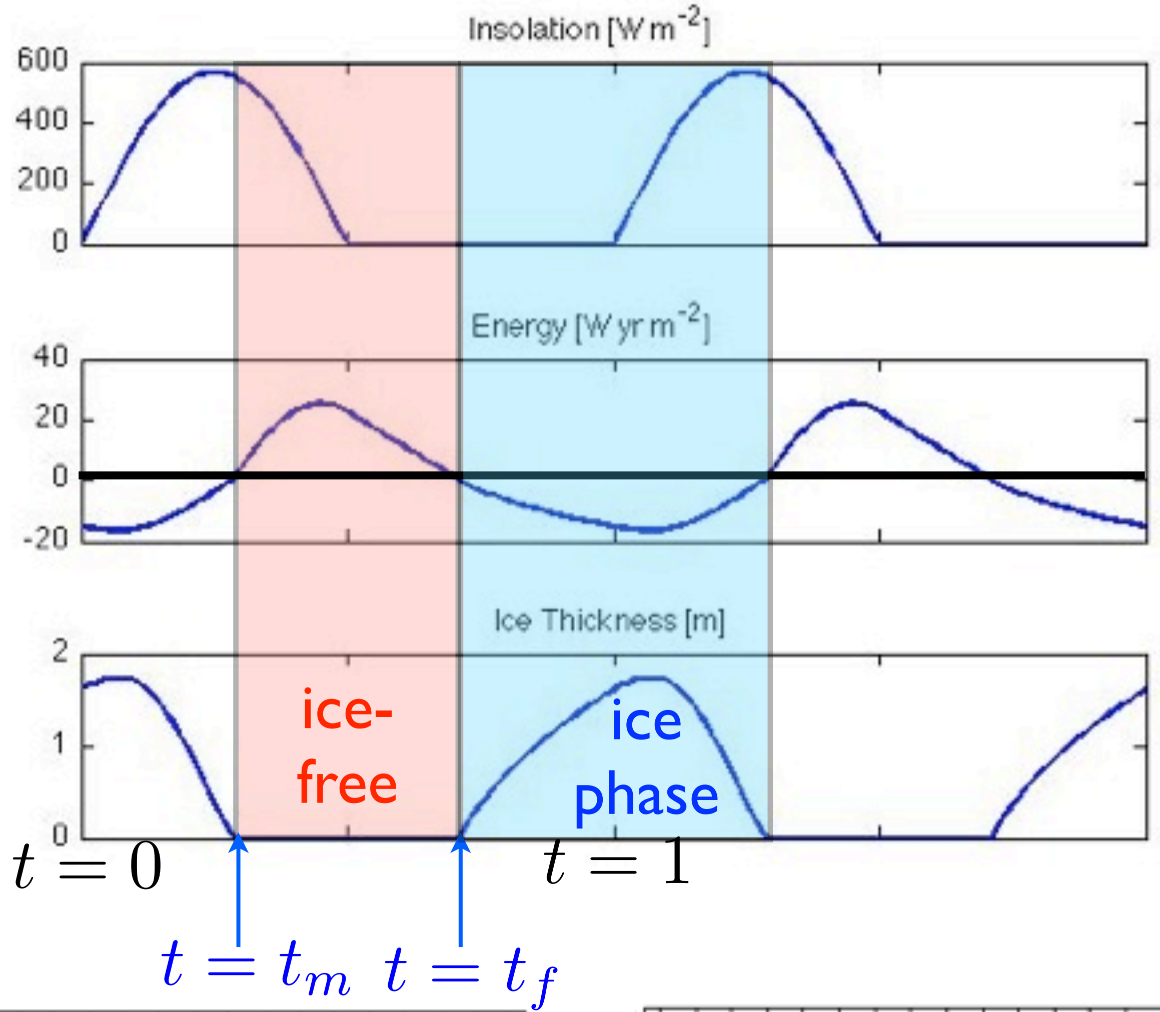
Periodic solutions: fixed points of appropriate Poincaré map $(P = P_i \circ P_o)$

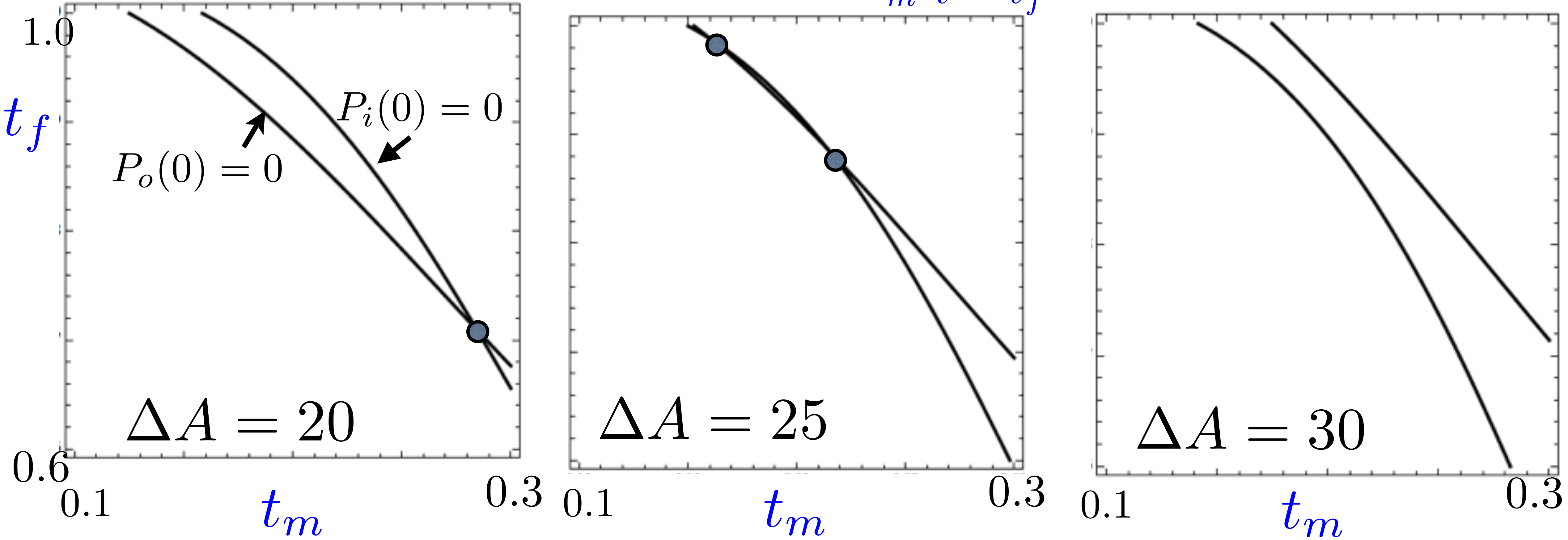
$$(t = t_m, E = 0) \underset{E \geq 0}{\overset{P_o}{\rightarrow}} (t = t_f, E = 0) \underset{E < 0}{\overset{P_i}{\rightarrow}} (t = t_m + 1, E = 0) \ldots ad\ infinitum$$

Existence conditions for seasonally ice-free states

Piece-wise constant albedo







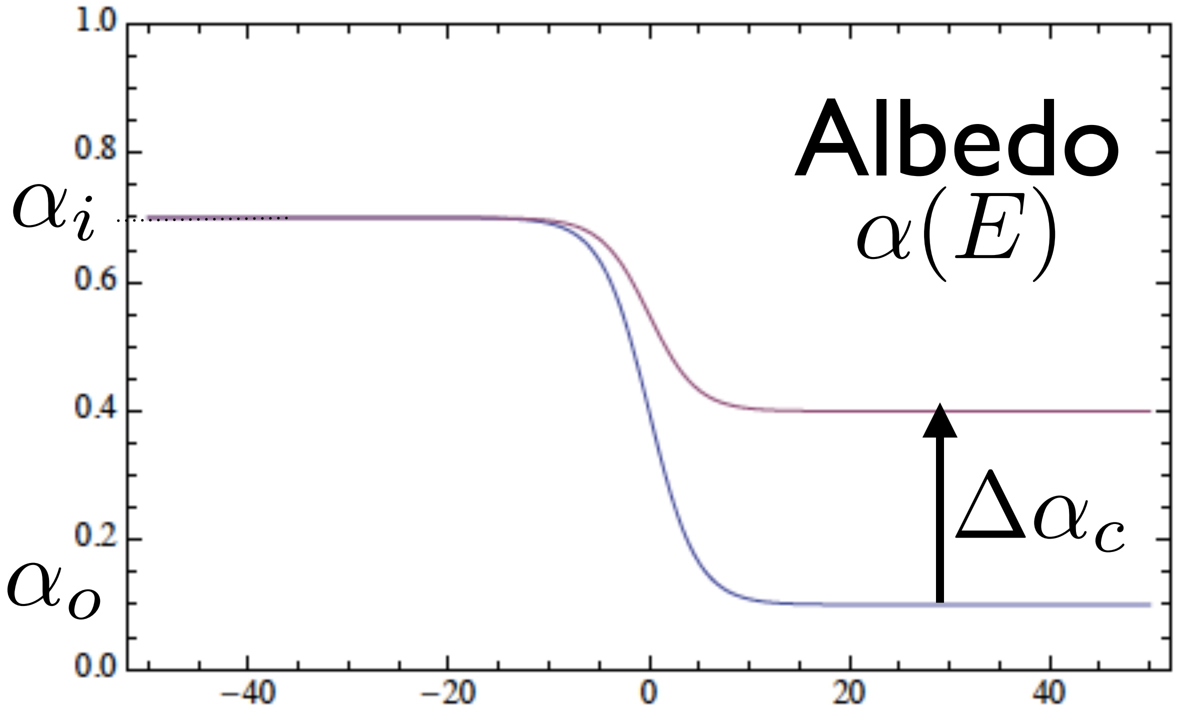
Existence conditions for seasonally ice-free states

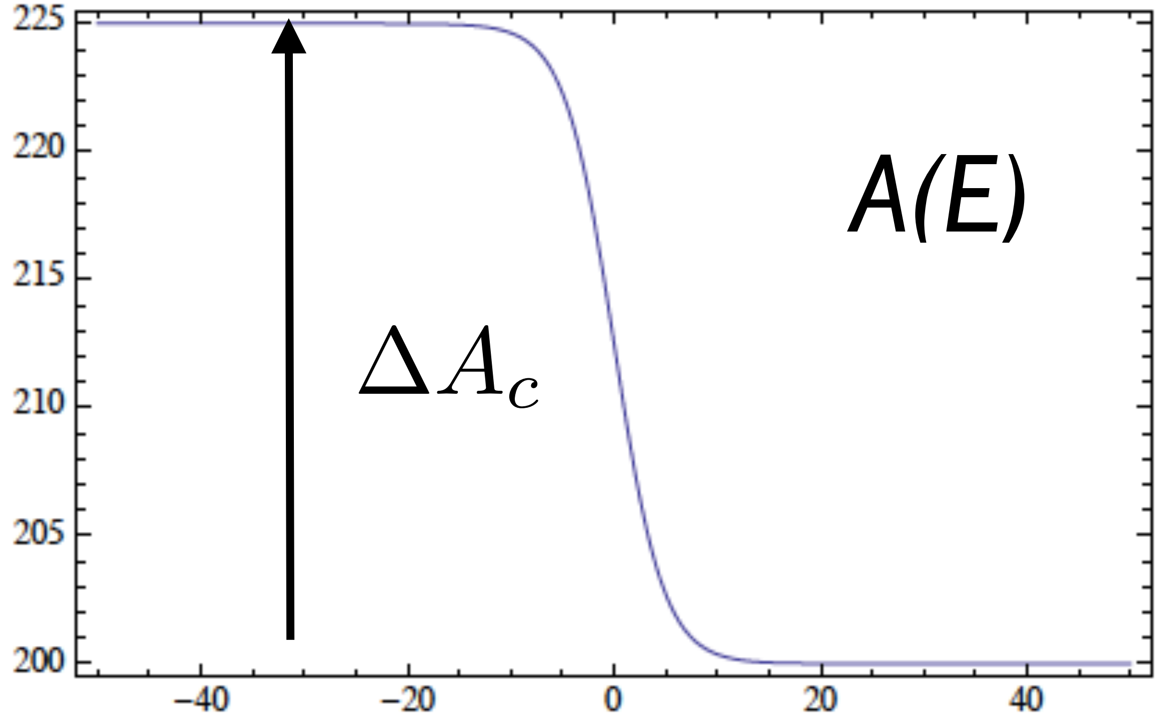
Periodic solutions: fixed points of appropriate Poincaré map $(P = P_i \circ P_o)$

$$(t = t_m, E = 0) \underset{E \geq 0}{\overset{P_o}{\rightarrow}} (t = t_f, E = 0) \underset{E < 0}{\overset{P_i}{\rightarrow}} (t = t_m + 1, E = 0) \ldots ad\ infinitum$$

$$P_o(E = 0; t_m, t_f, A - \underline{\Delta A_c}, \alpha_o + \underline{\Delta \alpha_c}) = 0$$

$$P_i(E = 0; t_m, t_f, A, \alpha_i) = 0$$





Existence conditions for seasonally ice-free states

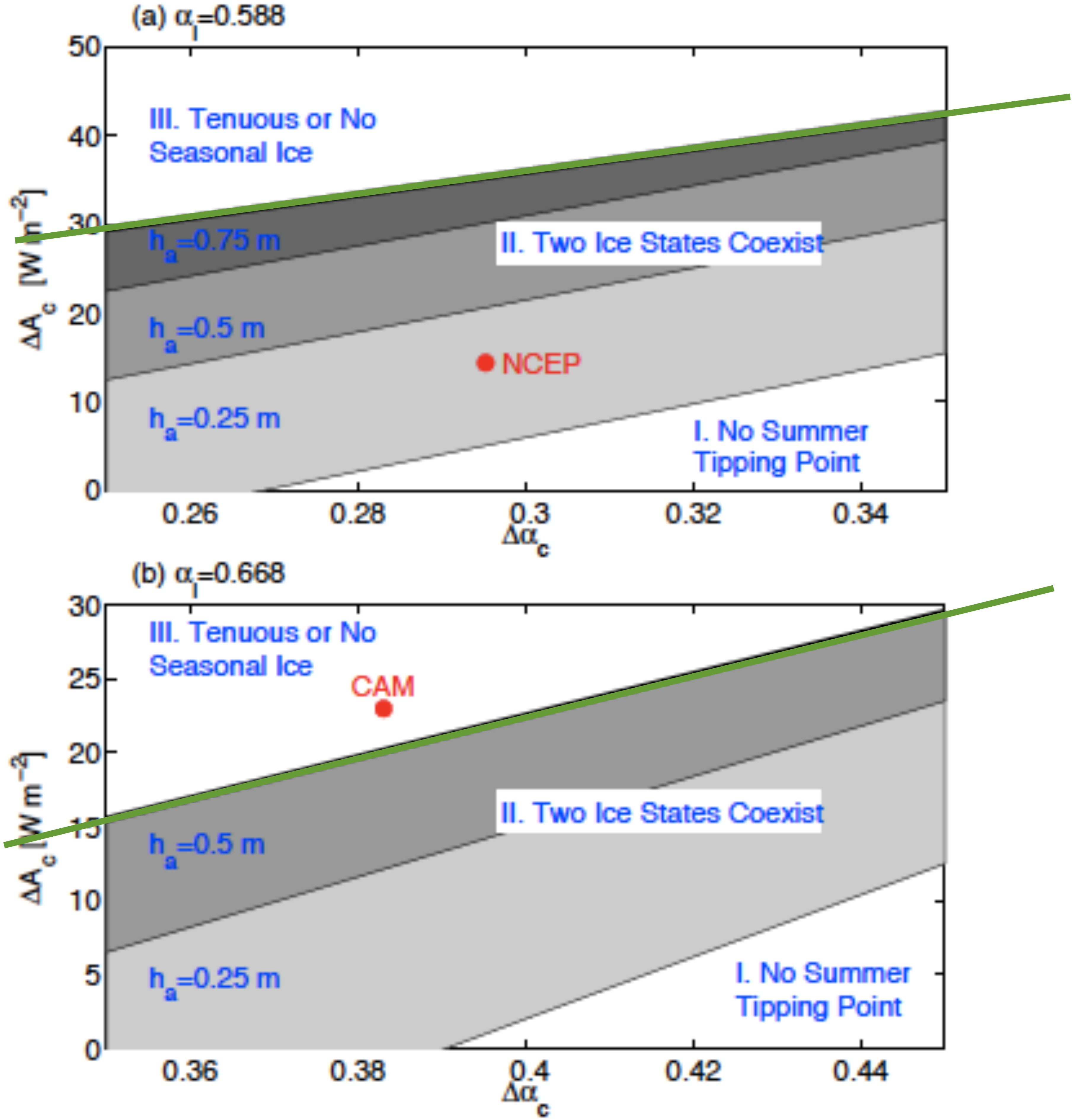
Periodic solutions: fixed points of appropriate Poincaré map $(P = P_i \circ P_o)$

$$(t = t_m, E = 0) \underset{E \geq 0}{\overset{P_o}{\rightarrow}} (t = t_f, E = 0) \underset{E < 0}{\overset{P_i}{\rightarrow}} (t = t_m + 1, E = 0) \ldots ad\ infinitum$$

$$P_o(E = 0; t_m, t_f, A - \underline{\Delta A_c}, \alpha_o + \underline{\Delta\alpha_c}) = 0$$

$$P_i(E = 0; t_m, t_f, A, \alpha_i) = 0$$

$$\begin{tikzcd}
\frac{dE}{dt} = [1 - \alpha(E)]F_{solar}(t) + F_{bottom} + F_{south} + v_{ice}(E) - [A(E) + BT(E)] \arrow[d, "P_o"] \\
\frac{dE}{dt} + \frac{BE}{C_s} = [1 - \alpha_o - \underline{\Delta\alpha_c}]F_{solar}(t) + [F_{bottom} + F_{south} - A + \underline{\Delta A_c}]
\end{tikzcd}$$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50
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30
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ΔA_c [W m^-2]
III. Tenuous or No Seasonal Ice
h_a=0.75 m
h_a=0.5 m
h_a=0.25 m
II. Two Ice States Coexist
NCEP
I. No Summer Tipping Point
0.26
0.28
0.3
0.32
0.34
Δα_c
(b) α_I=0.668
30
25
20
15
10
5
0
ΔA_c [W m^-2]
III. Tenuous or No Seasonal Ice
CAM
h_a=0.5 m
h_a=0.25 m
II. Two Ice States Coexist
I. No Summer Tipping Point
0.36
0.38
0.4
0.42
0.44
Δα_c