Multivariate Spatial Models

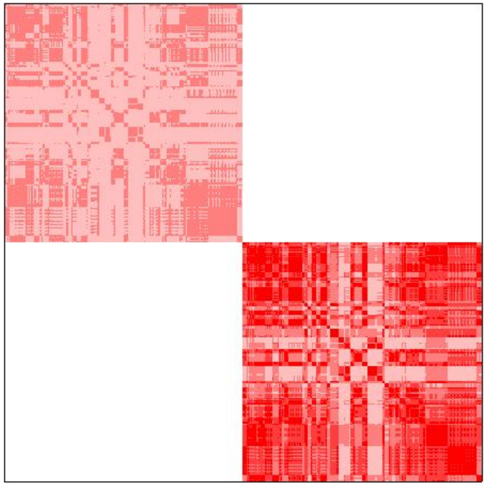
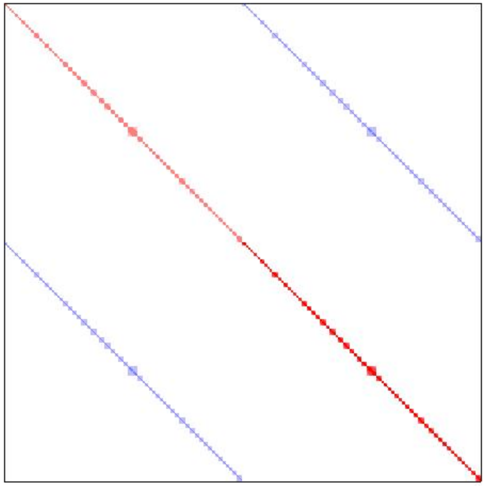
Stephan R. Sain

Geophysical Statistics Project
Institute for Mathematics Applied to Geosciences
National Center for Atmospheric Research
Boulder, CO

Supported by NSF ATM/DMS.

Outline

- Overview of multivariate spatial regression models.
- Case study: NC temperature and precipitation.
- Case study: pedotransfer functions and soil water profiles.

A Spatial Regression Model

- A spatial regression model:

$$\underset{(n \times 1)}{\mathbf{Y}} = \underset{(n \times q)(q \times 1)}{\mathbf{X}\boldsymbol{\beta}} + \underset{(n \times 1)}{\mathbf{h}} + \underset{(n \times 1)}{\boldsymbol{\epsilon}}$$

  where

  - E$[\mathbf{h}] = \mathbf{0}$, Var$[\mathbf{h}] = \boldsymbol{\Sigma}_{\mathbf{h}}$
  - E$[\boldsymbol{\epsilon}] = \mathbf{0}$, Var$[\boldsymbol{\epsilon}] = \sigma^2\mathbf{I}$.
  - $\mathbf{h}$ and $\boldsymbol{\epsilon}$ are independent.

- $\mathbf{Y} \sim \mathcal{N}(\mathbf{X}\boldsymbol{\beta}, \mathbf{V})$, $\mathbf{V} = \boldsymbol{\Sigma}_{\mathbf{h}} + \sigma^2\mathbf{I}$

- $\widehat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}^{-1}\mathbf{Y}$, $\widehat{\mathbf{h}} = \boldsymbol{\Sigma}_{\mathbf{h}}\mathbf{V}^{-1}(\mathbf{Y} - \mathbf{X}\widehat{\boldsymbol{\beta}})$

Multivariate Regression

- A multivariate, multiple regression model:

$$\underset{(n \times p)}{\mathbf{Y}} = \underset{(n \times q)(q \times p)}{\mathbf{X}\boldsymbol{\beta}} + \underset{(n \times p)}{\boldsymbol{\epsilon}}$$

  where

  – Each of the n rows of $\mathbf{Y}$ represents a p-vector observation.

  – Each of the p columns of $\boldsymbol{\beta}$ represent regression coefficients for each variable.

  – The rows of $\boldsymbol{\epsilon}$ represents a collection of iid error vectors with zero mean and common covariance matrix, $\mathbf{\Sigma}$.

Multivariate Regression

- MLEs are straightforward to obtain:

$$\underset{(q \times p)}{\widehat{\boldsymbol{\beta}}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}$$
$$\underset{(p \times p)}{\widehat{\boldsymbol{\Sigma}}} = \frac{1}{n}\mathbf{Y}'\mathbf{P}\mathbf{Y}$$

  where $\mathbf{P} = \mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$.

- Note that the columns of $\widehat{\boldsymbol{\beta}}$ can be obtained through p univariate regressions.

Vec and Kronecker

- The Kronecker product of an $m \times n$ matrix $\mathbf{A}$ and an $r \times q$ matrix $\mathbf{B}$ is an $mr \times nq$ matrix:

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \cdots & a_{2n}\mathbf{B} \\ \vdots & & \ddots & \vdots \\ a_{m1}\mathbf{B} & a_{m2}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{bmatrix}$$

- Some properties:

$$\begin{aligned} \mathbf{A} \otimes (\mathbf{B} + \mathbf{C}) &= \mathbf{A} \otimes \mathbf{B} + \mathbf{A} \otimes \mathbf{C} \\ \mathbf{A} \otimes (\mathbf{B} \otimes \mathbf{C}) &= (\mathbf{A} \otimes \mathbf{B}) \otimes \mathbf{C} \\ (\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) &= \mathbf{AC} \otimes \mathbf{BD} \\ (\mathbf{A} \otimes \mathbf{B})' &= \mathbf{A}' \otimes \mathbf{B}' \\ (\mathbf{A} \otimes \mathbf{B})^{-1} &= \mathbf{A}^{-1} \otimes \mathbf{B}^{-1} \\ |\mathbf{A} \otimes \mathbf{B}| &= |\mathbf{A}|^m |\mathbf{B}|^n \end{aligned}$$

Vec and Kronecker

- The vec-operator stacks the columns of a matrix:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad \text{vec}(\mathbf{A}) = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{12} \\ a_{22} \end{bmatrix}$$

- Some properties:

$$\begin{aligned} \text{vec}(\mathbf{AXB}) &= (\mathbf{B}' \otimes \mathbf{A})\,\text{vec}\,\mathbf{X} \\ \text{tr}(\mathbf{A}'\mathbf{B}) &= \text{vec}(\mathbf{A})'\,\text{vec}(\mathbf{B}) \\ \text{vec}(\mathbf{A}+\mathbf{B}) &= \text{vec}(\mathbf{A}) + \text{vec}(\mathbf{B}) \\ \text{vec}(\alpha\mathbf{A}) &= \alpha\,\text{vec}(\mathbf{A}) \end{aligned}$$

Multivariate Regression Revisited

- Rewrite the multivariate, multiple regression model:

$$\begin{array}{ccccc} \text{vec}(\mathbf{Y}) & = & (\mathbf{I}_p \otimes \mathbf{X})\,\text{vec}(\boldsymbol{\beta}) & + & \text{vec}(\boldsymbol{\epsilon}) \\ (np \times 1) & & (np \times qp)(qp \times 1) & & (np \times 1). \end{array}$$

- What is $\text{Var}[\text{vec}\,\boldsymbol{\epsilon}]$?

- What is the GLS estimator for $\text{vec}(\boldsymbol{\beta})$?

A Multivariate Spatial Model

- Extend the multivariate, multiple regression model:

$$\underset{(np \times 1)}{\text{vec}(\mathbf{Y})} = \underset{(np \times qp)(qp \times 1)}{(\mathbf{I}_p \otimes \mathbf{X})\,\text{vec}(\boldsymbol{\beta})} + \underset{(np \times 1)}{\text{vec}(\mathbf{h})} + \underset{(np \times 1),}{\text{vec}(\boldsymbol{\epsilon})}$$

where

$$\begin{aligned}
\text{Var}[\text{vec}(\mathbf{h})] &= \boldsymbol{\Sigma}_{\mathbf{h}} = \begin{bmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} & \cdots & \boldsymbol{\Sigma}_{1p} \\ \boldsymbol{\Sigma}'_{12} & \boldsymbol{\Sigma}_{22} & \cdots & \boldsymbol{\Sigma}_{2p} \\ \vdots & & \ddots & \vdots \\ \boldsymbol{\Sigma}'_{12} & \boldsymbol{\Sigma}'_{2p} & \cdots & \boldsymbol{\Sigma}_{pp} \end{bmatrix} \\
\text{Var}[\text{vec}(\boldsymbol{\epsilon})] &= \boldsymbol{\Sigma} \otimes \mathbf{I}_n
\end{aligned}$$

A Multivariate Spatial Model

- One simplification to the spatial covariance matrix is to use a Kronecker form:

$$\Sigma_{\mathbf{h}} = \boldsymbol{\rho} \otimes \mathbf{K}$$

  where

  - $\boldsymbol{\rho}$ is a $p \times p$ matrix of scale parameters
  - $\mathbf{K}$ is an $n \times n$ spatial covariance.

A Multivariate Spatial Model

- Extend the multivariate, multiple regression model:

$$\underset{(np\times 1)}{\text{vec}(\mathbf{Y})} = \underset{(np\times qp)(qp\times 1)}{(\mathbf{I}_p \otimes \mathbf{X})\,\text{vec}(\boldsymbol{\beta})} + \underset{(np\times 1)}{\text{vec}(\mathbf{h})} + \underset{(np\times 1)}{\text{vec}(\boldsymbol{\epsilon})}$$

OR

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{h} + \boldsymbol{\epsilon}$$

- Now everything follows...

Case Study: Pedotransfer Functions

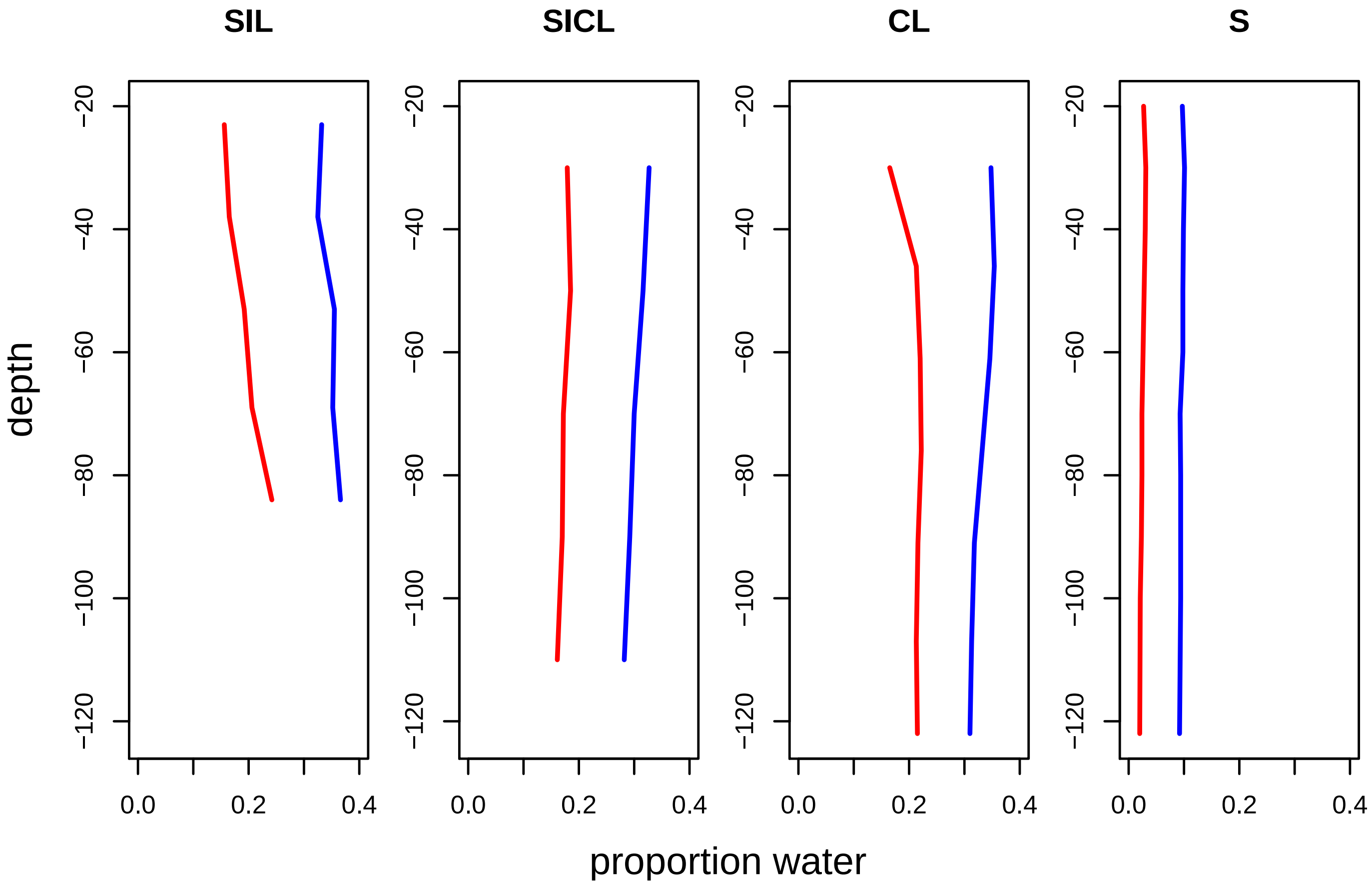
- Soil characteristics such as composition (clay, silt, sand) are commonly measured and easily obtainable.

- Unfortunately, crop models require water holding characteristics such as the wilting point or lower limit (LL) and the drained upper limit (DUL) which are not so easy to obtain.
  - Often the LL and DUL are a function of depth - soil water profile.

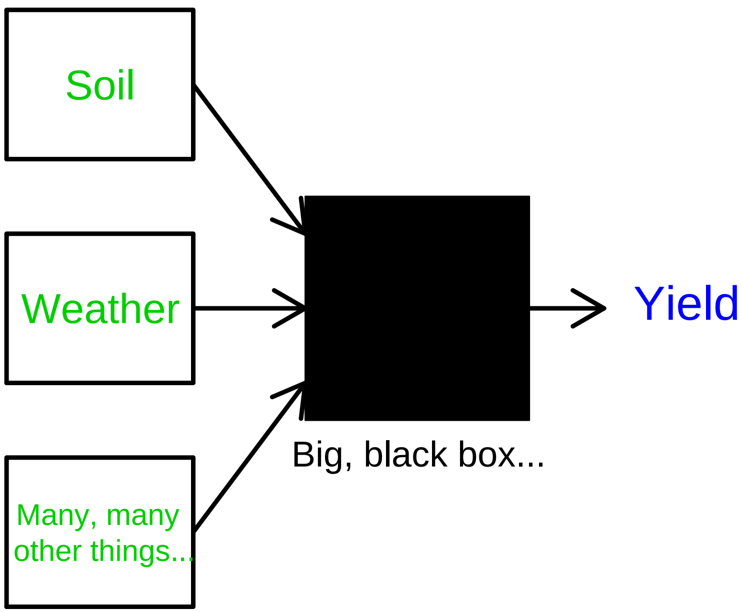
Case Study: Pedotransfer Functions

- *Pedotransfer functions* are commonly used to estimate LL and DUL.
  - Differential equations, regression, nearest neighbors, neural networks, etc.
  - Often specialized by soil type and/or region.
- Develop a new type of pedotransfer function that can capture the entire soil water profile (LL & DUL as a function of depth).
  - Characterize the variation!

Soil Water Profiles



The Big Picture



The CERES Crop Model

- Soil
  - Water holding characteristics
  - Bulk density
  - Etc.
- Weather (20 years)
  - Solar radiation
  - Temperature max/min
  - Precipitation

The Big Picture

- Given a complicated array of inputs, the CERES crop model will give the yields of, for example, maize.

- Deterministic output – variation in yields also of interest.

- Goals:
  - Establish a framework to study sources of variation in crop yields.
  - Assess impacts of climate change on crop yields.

Data

- $n = 272$ measurements on $N = 63$ soil samples
  - Gijsman et al. (2002)
  - Ratliff et al. (1983), Ritchie et al. (1987)
- Includes measurements of:
  - depth,
  - soil composition and texture
    - percentages of clay, sand, and silt
  - bulk density, organic matter, and
  - field measured values of LL and DUL.

Data

- The soil texture measurements form a composition

$$Z_{\text{clay}} + Z_{\text{silt}} + Z_{\text{sand}} = 1$$

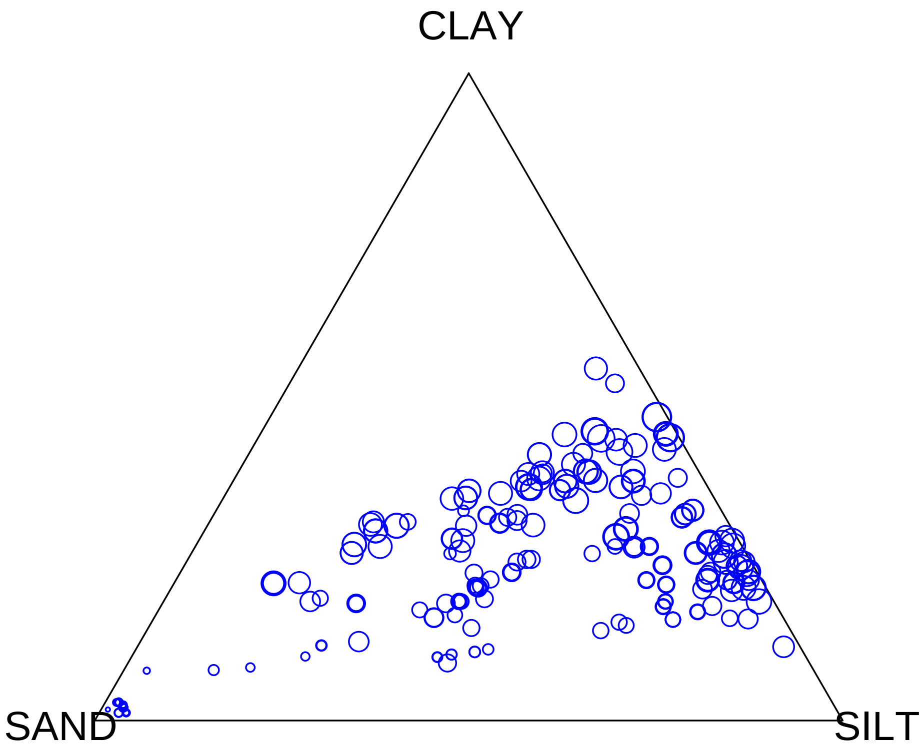
  and Z_{clay}, Z_{silt} ,Z_{sand} are the proportions of each soil component.

  - Not really three variables...

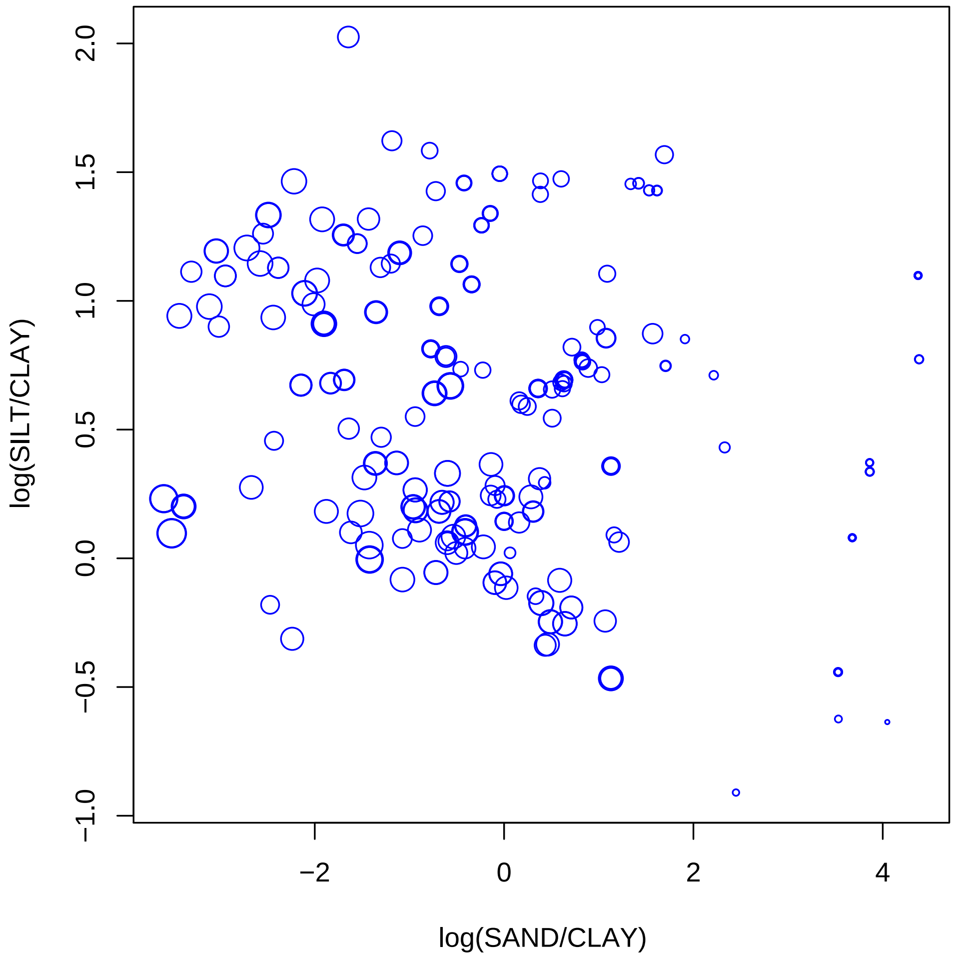
- To remove the dependence, the additive log-ratio transformation (Aitchinson, 1986) is applied, defining two new variables

$$X_1 = \log\left(\frac{Z_{\text{sand}}}{Z_{\text{clay}}}\right) \qquad X_2 = \log\left(\frac{Z_{\text{silt}}}{Z_{\text{clay}}}\right).$$

Data - Composition vs LL



Data - Composition vs LL



A Multi-objective Pedotransfer Function

- The model for the multi-objective pedotransfer function for a particular soil is

$$\mathbf{Y}_0 = \mathbf{T}_0\boldsymbol{\beta} + \mathbf{h}(\mathbf{X}_0) + \boldsymbol{\epsilon}(\mathbf{D}_0)$$

where

$$\mathbf{Y}_0 = \log \begin{bmatrix} \mathrm{LL}_1 \\ \vdots \\ \mathrm{LL}_d \\ \Delta_1 \\ \vdots \\ \Delta_d \end{bmatrix},$$

and d is the number of measurements (depths) and $\Delta_i = \mathrm{DUL}_i - \mathrm{LL}_i$.

A Multi-objective Pedotransfer Function

- The model for the multi-objective pedotransfer function for a particular soil is

$$\mathbf{Y}_0 = \mathbf{T}_0\boldsymbol{\beta} + \mathbf{h}(\mathbf{X}_0) + \boldsymbol{\epsilon}(\mathbf{D}_0)$$

where

$$\mathbf{T}_0 = \begin{bmatrix} \mathbf{1} & \mathbf{X}_0 & \mathbf{Z}_{\mathsf{LL},0} & & \mathbf{0} & \\ & \mathbf{0} & & \mathbf{1} & \mathbf{X}_0 & \mathbf{Z}_{\triangle,0} \end{bmatrix},$$

and

  - $\mathbf{X}_0$ is the transformed soil composition information
  - $\mathbf{Z}_{\mathsf{LL}}$ and $\mathbf{Z}_{\triangle}$ are additional covariates for LL and $\triangle$.
    * $\mathbf{Z}_{\mathsf{LL}}$ includes organic carbon
    * $\mathbf{Z}_{\triangle}$ includes linear and quadratic terms for depth

A Multi-objective Pedotransfer Function

- The model for the multi-objective pedotransfer function for a particular soil is

$$\mathbf{Y}_0 = \mathbf{T}_0\boldsymbol{\beta} + \mathbf{h}(\mathbf{X}_0) + \boldsymbol{\epsilon}(\mathbf{D}_0)$$

  where

  – $\mathbf{h}(\mathbf{X}_0)$ is a two-dimensional spatial process that controls the smoothness of the contribution of $\mathbf{X}$

  – $\boldsymbol{\epsilon}(\mathbf{D}_0)$ is an error process that

    * accounts for the dependence in LL and Δ for a particular depth and

    * accounts for dependence across depths (one-dimensional spatial process).

A Multi-objective Pedotransfer Function

- Letting

  $\mathbf{Y} = \log\,[\mathsf{LL}_{11} \ \cdots \ \mathsf{LL}_{1d_1} \ \mathsf{LL}_{21} \ \cdots \ \mathsf{LL}_{Nd_N} \ \Delta_{11} \ \cdots \ \Delta_{1d_1} \ \Delta_{21} \ \cdots \ \Delta_{Nd_N} \ ]'$,

  then $\mathbf{Y}$ is multivariate normal with

$$E[\mathbf{Y}] = \mathbf{T}\boldsymbol{\beta} \qquad \mathsf{Var}[\mathbf{Y}] = \boldsymbol{\Sigma}_{\mathbf{h}} + \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}$$

$$\begin{aligned} \boldsymbol{\Sigma}_{\mathbf{h}} &= \begin{bmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{bmatrix} \otimes \mathbf{K} \\ \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}} &= \mathbf{S} \otimes \mathbf{R}. \end{aligned}$$

  with

  - $K_{ij} = k(\mathbf{X}_i, \mathbf{X}_j)$
  - $\mathbf{S}$ is the covariance of (LL, Δ) at a fixed depth
  - $\mathbf{R}$ is the (spatial) covariance across depths

Covariance Structures

- The covariance function for h is the Matern family

$$C(d) = \sigma^2 \frac{2(\theta d/2)^{\nu} K_{\nu}(\theta d)}{\Gamma(\nu)}$$

where σ^2 is a scale parameter, θ represents the range, ν controls the smoothness.

  – $\sigma^2 = 1$ (the ρ controls the variances), $\nu = 1$, and θ is taken to be approximately the range of the data.

  – These choices represent a covariance structure that is consistent with the thin-plate spline estimator (large range, more smoothness).

  – Analogous to fixing the kernel and estimating the bandwidth with kernel estimators.

Covariance Structures

- The covariance function across depths is exponential

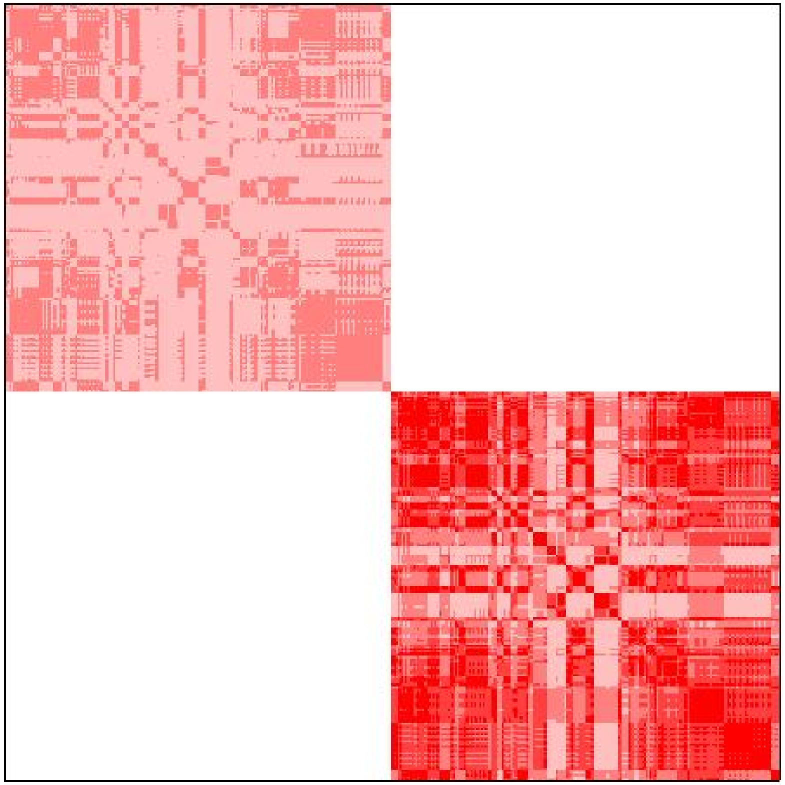
$$C(d) = \sigma^2 \exp\left(-d/\theta\right)$$

  where again σ^2 is a scale parameter and θ represents the range.

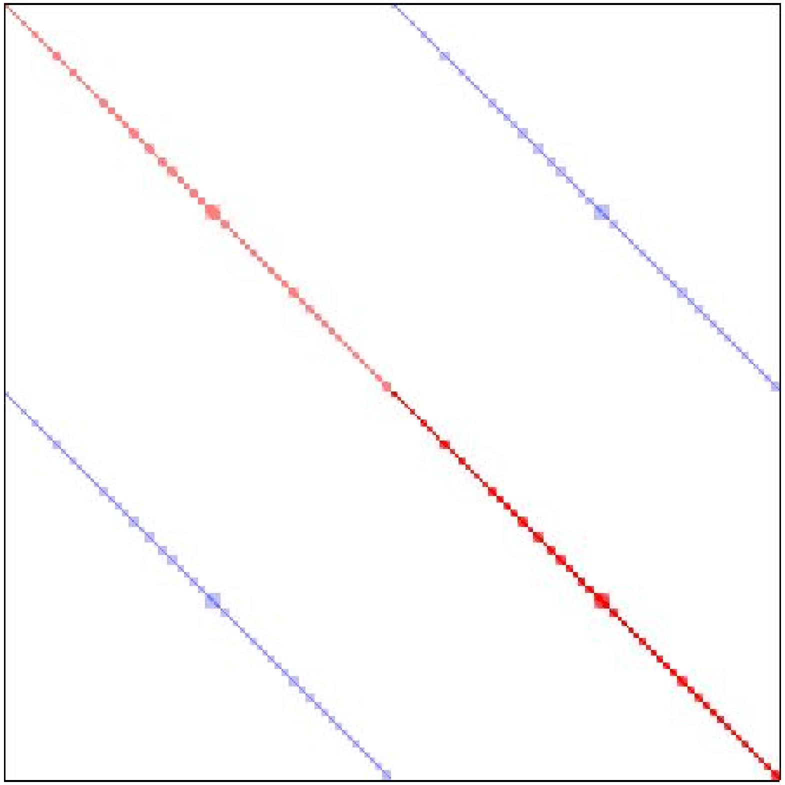
  – The parameters $\sigma^2 = 1$ (the matrix $\mathbf{S}$ controls the variances) and θ is estimated from the data.

  – The multiple realizations of the soils allow for improved ability to estimate both scale and range parameters.

Covariance Structures



$\Sigma_{\mathbf{h}}$



Σ_{ϵ}

Spatial Smoothing

- Write

$$
\begin{aligned}
\boldsymbol{\Sigma}_{\mathbf{h}} + \boldsymbol{\Sigma}_{\epsilon} &= \begin{bmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{bmatrix} \otimes \mathbf{K} + \begin{bmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{bmatrix} \otimes \mathbf{R} \\
&= s_{11} \left[\begin{bmatrix} \eta_1 & 0 \\ 0 & \eta_2 \end{bmatrix} \otimes \mathbf{K} + \begin{bmatrix} 1 & v_{12} \\ v_{12} & v_{22} \end{bmatrix} \otimes \mathbf{R} \right] \\
&= s_{11} \boldsymbol{\Omega}
\end{aligned}
$$

- The amount of smoothing is due to the relative contributions of the variance components, i.e. η_1 and η_2.

- Different degrees of smoothing are allowed for LL and Δ.

- Also, this construction allows for different degrees of variation in the error terms for LL and the Δ variables.

The Estimator

- The model suggests an estimator of the form

$$\widehat{\mathbf{Y}}_0 = \mathbf{T}_0\widehat{\boldsymbol{\beta}} + \mathbf{K}_0'\widehat{\boldsymbol{\delta}},$$

  where

$$\mathbf{K}_0' = \begin{bmatrix} \eta_1 & 0 \\ 0 & \eta_2 \end{bmatrix} \bigotimes \mathbf{K}.$$

- To fit the model, we must estimate:
  - η_1, η_2 and s_{11}
  - $\boldsymbol{\beta}$, $\boldsymbol{\delta}$
  - $\mathbf{R}$ and the other entries of $\mathbf{S}$

REML

- Take the QR decomposition of $\mathbf{T}$

$$\mathbf{T} = [\mathbf{Q}_1 \;\; \mathbf{Q}_2] \begin{bmatrix} \mathbf{R} \\ \mathbf{0} \end{bmatrix}.$$

- Then $\mathbf{Q}_2'\mathbf{Y}$ has zero mean and covariance matrix given by

$$\mathbf{Q}_2'(\boldsymbol{\Sigma}_{\mathbf{h}} + \boldsymbol{\Sigma}_{\epsilon})\mathbf{Q}_2.$$

- Maximize (numerically) the likelihood based on $\mathbf{Q}_2'\mathbf{Y}$ which is only a function of the covariance parameters.

- Estimates of $\boldsymbol{\beta}$ and $\boldsymbol{\delta}$ follow directly

$$\widehat{\boldsymbol{\beta}} = (\mathbf{T}'\widehat{\boldsymbol{\Omega}}^{-1}\mathbf{T})^{-1}\mathbf{T}'\widehat{\boldsymbol{\Omega}}^{-1}\mathbf{Y} \qquad \widehat{\boldsymbol{\delta}} = \widehat{\boldsymbol{\Omega}}^{-1}(\mathbf{Y} - \mathbf{T}\widehat{\boldsymbol{\beta}}).$$

An Iterative Approach

0. Initialize: compute $\mathbf{K}$ and set $\mathbf{S} = \mathbf{I}$ and $\mathbf{R} = \mathbf{I}$.

1. Estimate η_1 and η_2 (and s_{11}) via a simplified type of REML (grid search).

2. Then

$$\widehat{\boldsymbol{\beta}} = (\mathbf{T}'\widehat{\boldsymbol{\Omega}}^{-1}\mathbf{T})^{-1}\mathbf{T}^{-1}\widehat{\boldsymbol{\Omega}}^{-1}\mathbf{Y} \qquad \widehat{\boldsymbol{\delta}} = \widehat{\boldsymbol{\Omega}}^{-1}(\mathbf{Y} - \mathbf{T}\widehat{\boldsymbol{\beta}}).$$

3. Compute residuals and

   a. Update $\mathbf{S}$ ($\mathbf{R}$ fixed) – closed form solution.

   b. Update $\mathbf{R}$ ($\mathbf{S}$ fixed) – grid search for θ.

4. Repeat items 1-3 until convergence.

An Iterative Approach

- Let $\mathbf{Y} = \boldsymbol{\mu} + \mathbf{h} + \boldsymbol{\epsilon}$, where $\mathbf{h}$ and $\boldsymbol{\epsilon}$ are independent Gaussian random variables; the conditional distribution of $\mathbf{Y} - \boldsymbol{\mu} - \mathbf{h}$ given $\mathbf{h}$ is a zero mean Gaussian with covariance matrix $\boldsymbol{\epsilon}$.

- Thus, the log-likelihood associated with the residuals is given by

$$-\frac{n}{2}|\mathbf{S}| - |\mathbf{R}| - \text{vec}(\mathbf{U})'(\mathbf{S}^{-1} \otimes \mathbf{R}^{-1})\,\text{vec}(\mathbf{U})$$

- The quadratic form can be written as

$$\text{tr}(\mathbf{S}^{-1} \sum_i \sum_j r^{ij} \mathbf{u}_j \mathbf{u}_i')$$

  where r^{ij} is the ijth element of $\mathbf{R}^{-1}$ and $\mathbf{u}_i$ is the bivariate, unstacked residual for the ith observation.

An Iterative Approach

- An update for $\mathbf{S}$ can be written as

$$\widehat{\mathbf{S}} = \frac{1}{n} \sum_i \sum_j r^{ij} \mathbf{u}_j \mathbf{u}_i' = \frac{1}{n} \mathbf{U}' \mathbf{R}^{-1} \mathbf{U}$$

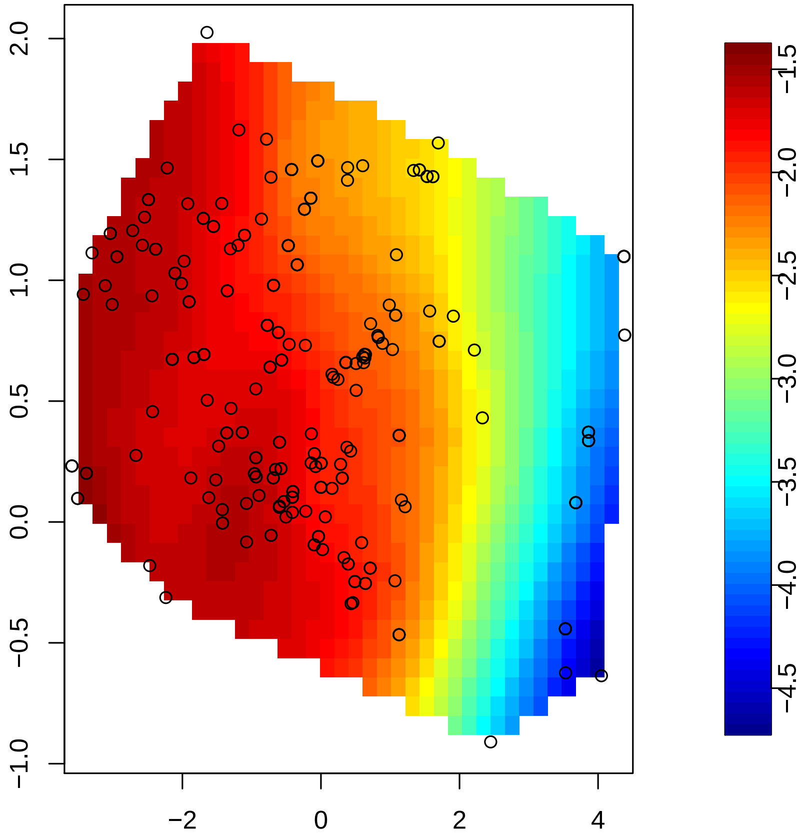
  where $\mathbf{U}$ is the $n \times 2$ matrix of unstacked residuals.

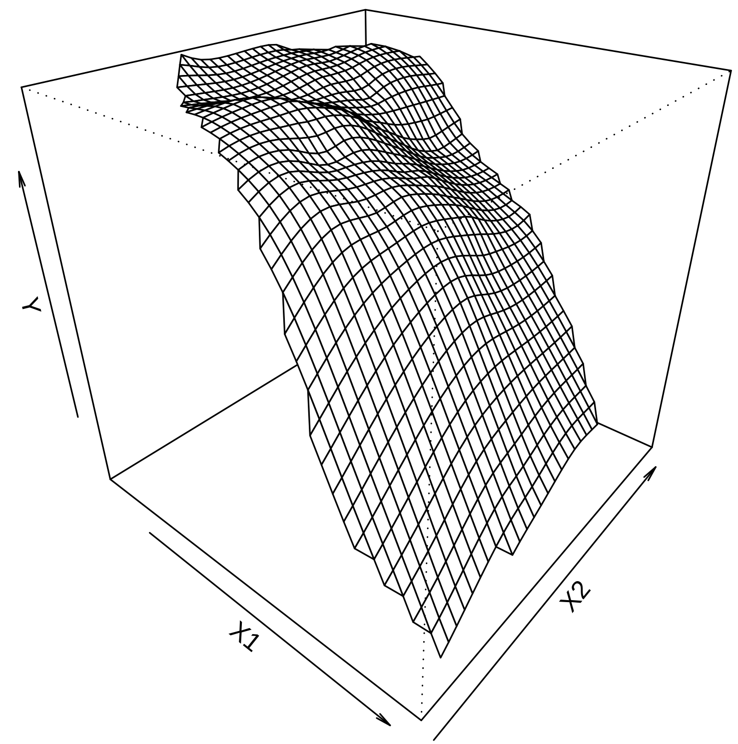
- Again, a simple grid search for θ is used to obtain a new value for $\mathbf{R}$.

Parameter Estimates

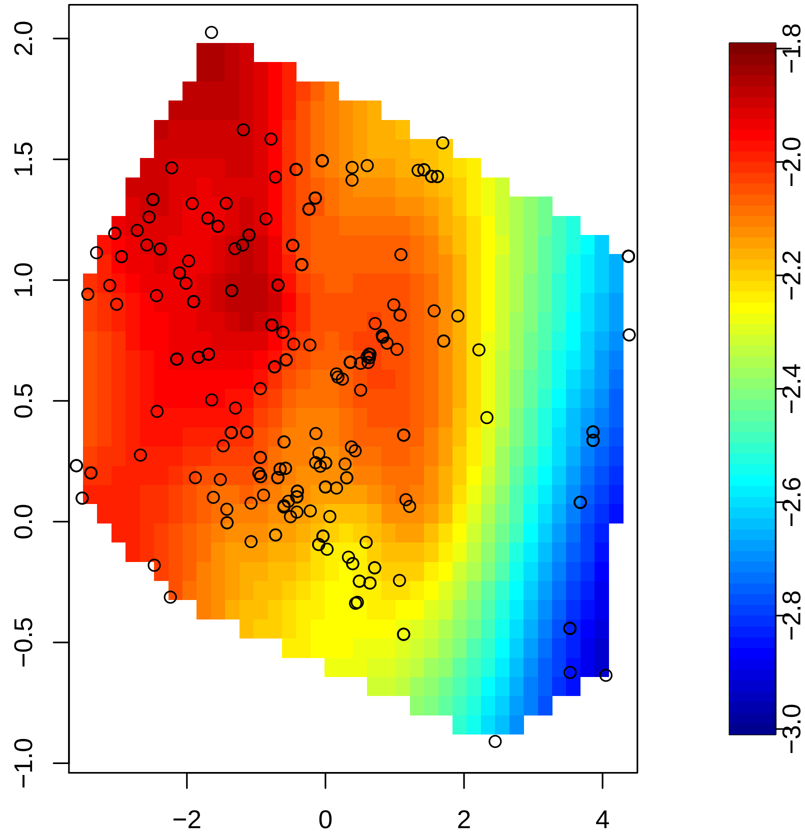
	η_1	η_2	S_{11}	S_{22}	S_{12}	θ
REML	5.84	1.66	0.0765	0.0483	-0.0222	134.6
Iterative	5.74	2.21	0.0697	0.0445	-0.0217	144.2

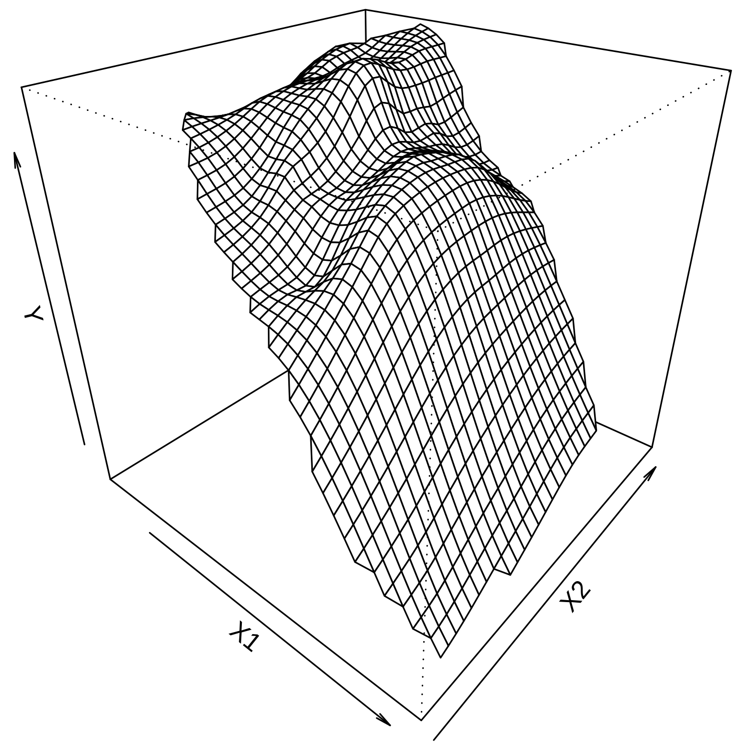
Soil Composition and LL



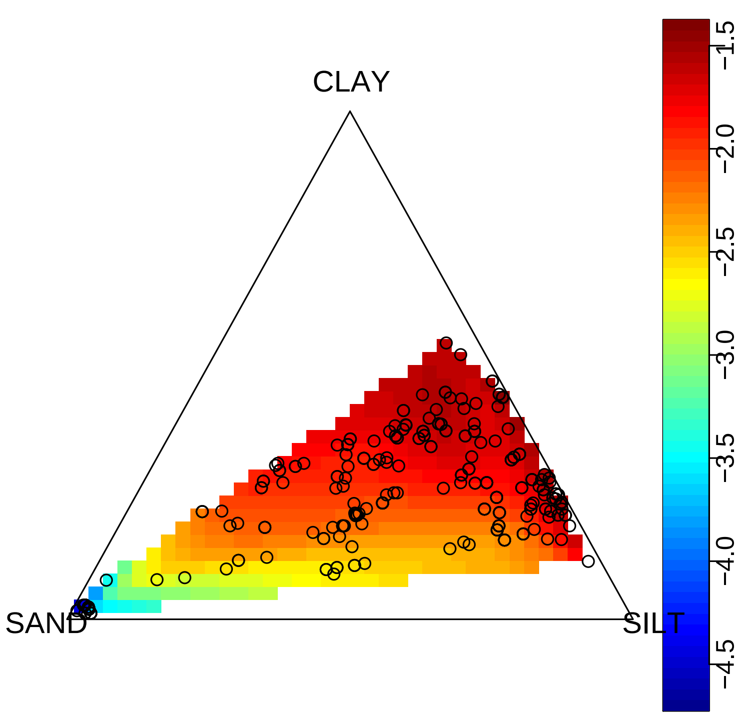


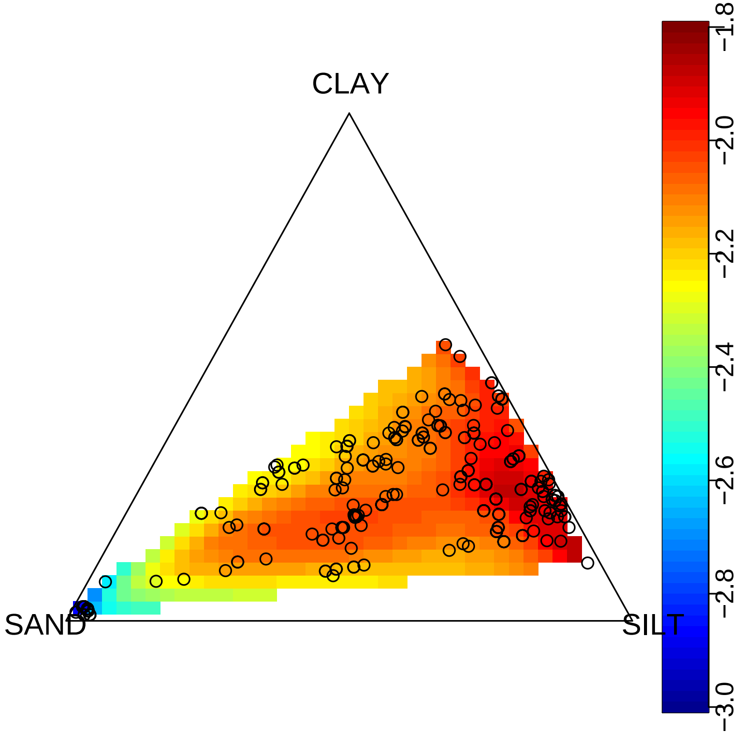
Soil Composition and Δ



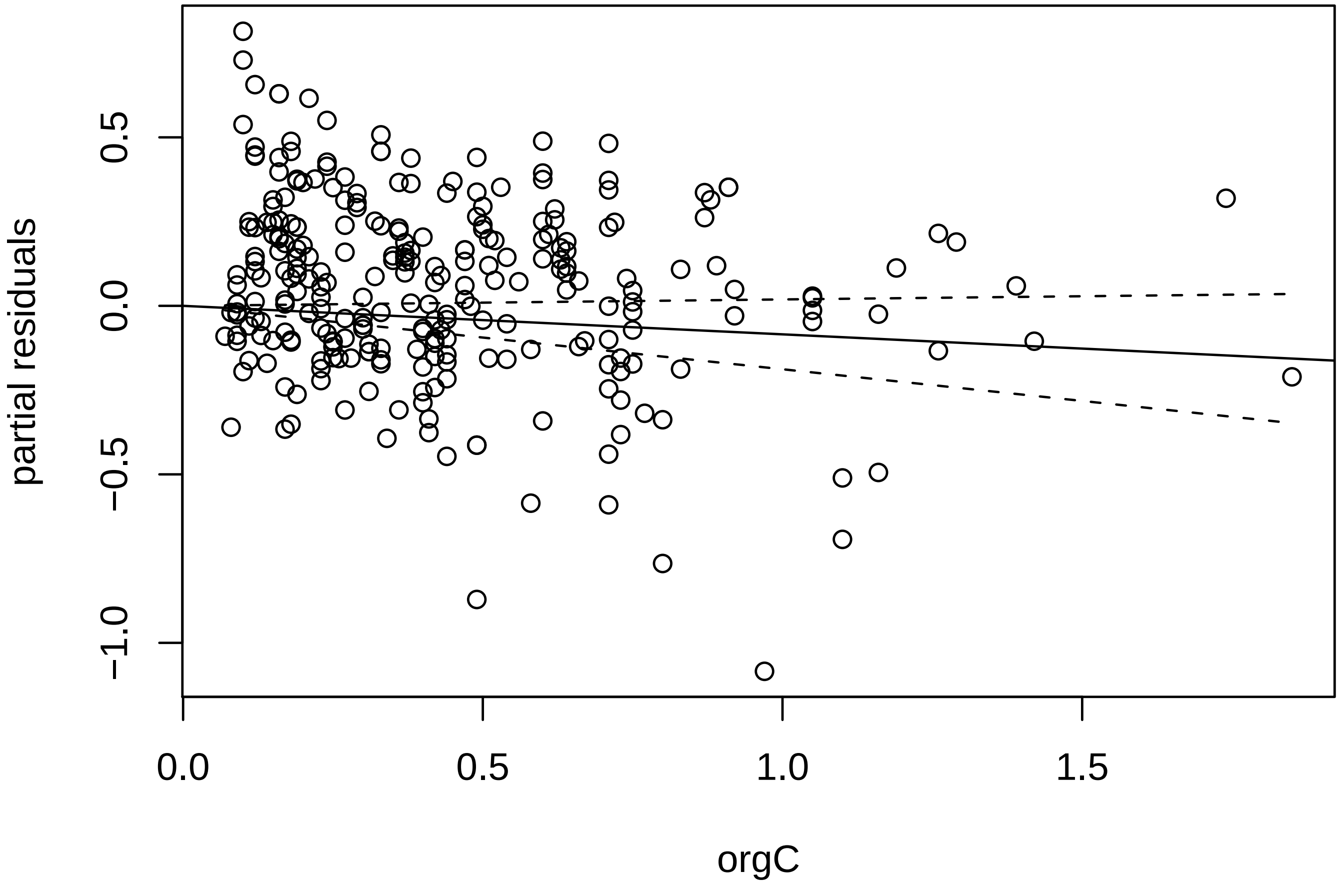


Soil Composition and LL/Δ

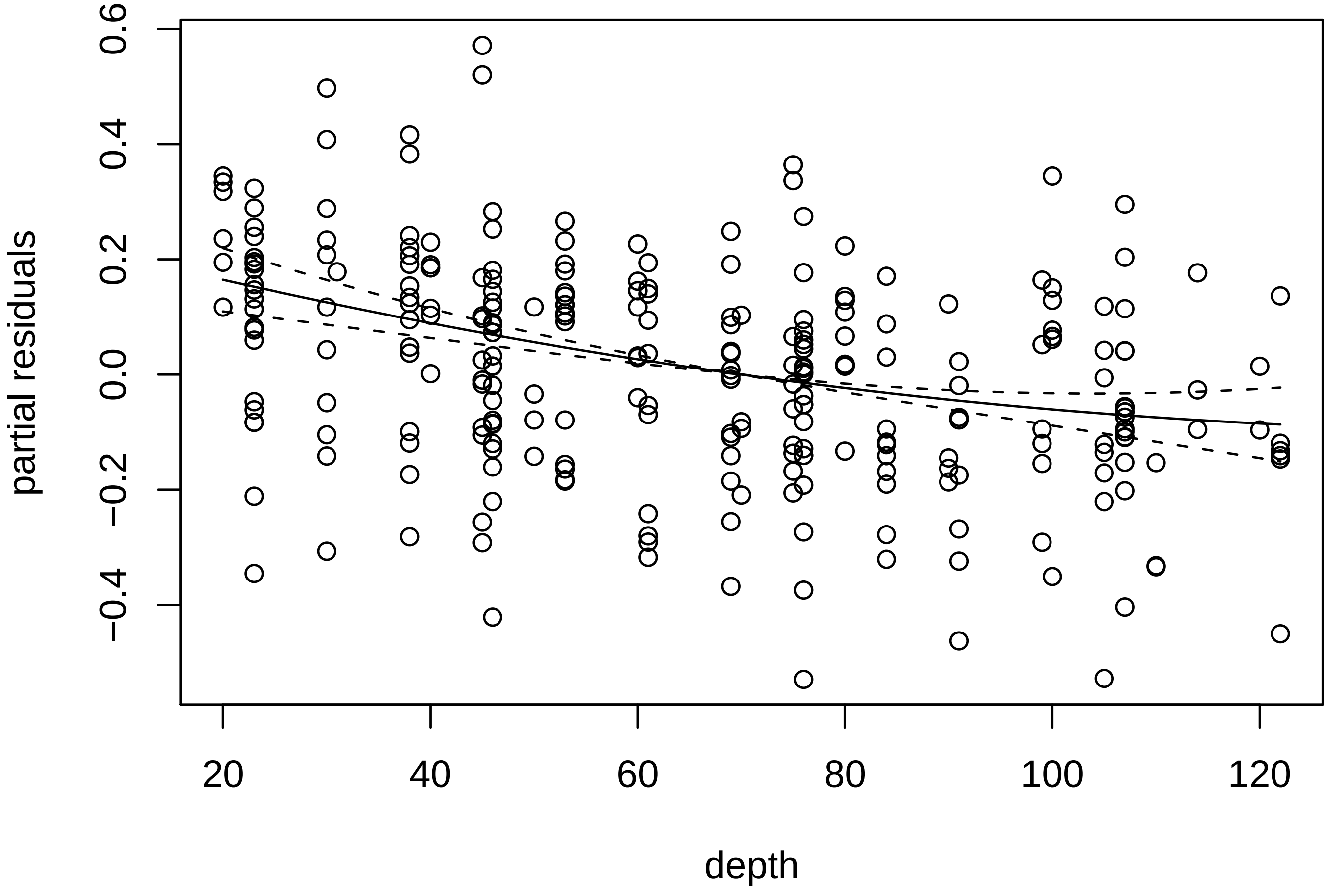




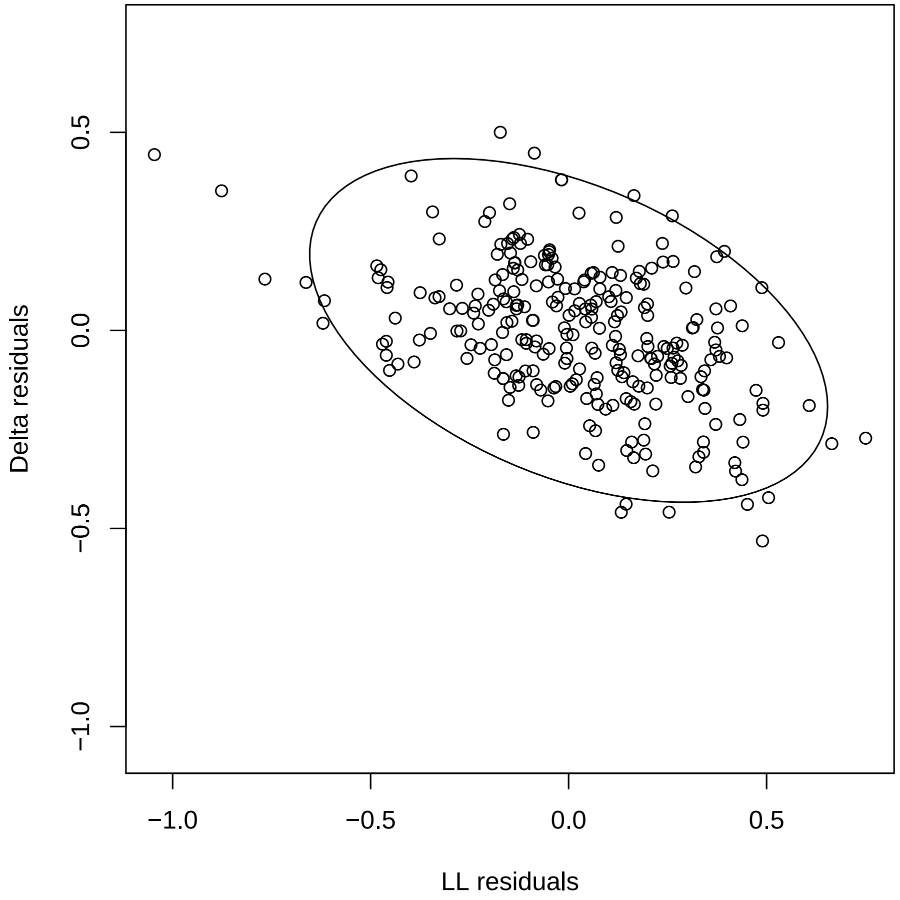
Organic Carbon and LL



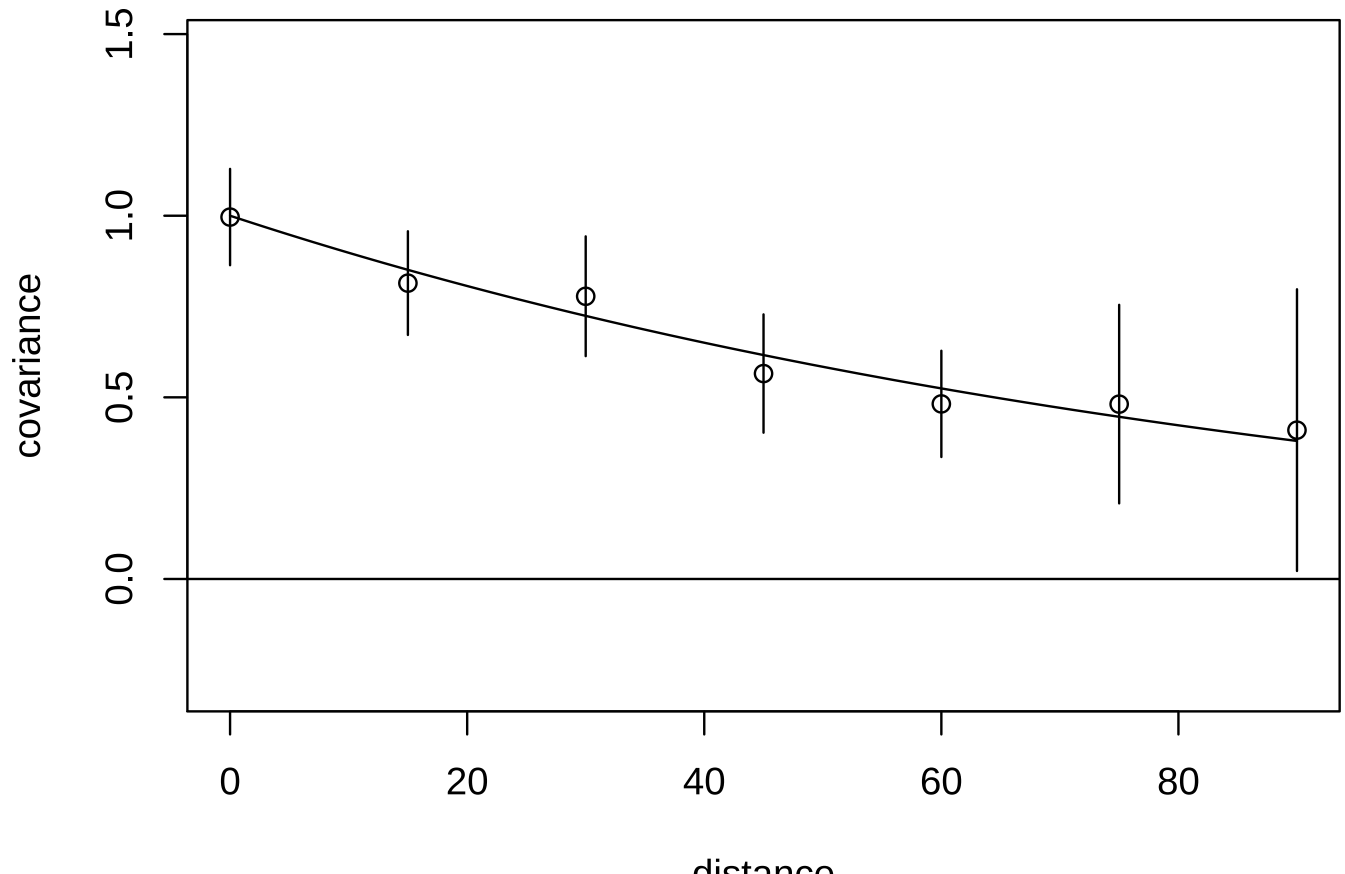
Depth and Δ



Residuals (Within Depth)



Spatial Covariance Across Depth



Prediction Error

- The real benefit of considering our estimator as a spatial process is in the interpretation with respect to the uncertainty of the estimator:
  - The thin-plate spline is a biased estimator with uncorrelated error – not easy to quantify the bias (interpolation error and smoothing error).
  - The spatial process estimator is unbiased, but with correlated error – more complicated error structure but conceptually straightforward to work with.

Prediction Error

- The estimator can be written as

$$\begin{aligned} \widehat{\mathbf{Y}}_0 &= \mathbf{T}_0\widehat{\boldsymbol{\beta}} + \mathbf{K}_0'\widehat{\boldsymbol{\delta}} \\ &= \mathbf{A}_0\mathbf{Y}, \end{aligned}$$

where

$$\begin{aligned} \mathbf{A}_0 &= \mathbf{T}_0(\mathbf{T}'\widehat{\boldsymbol{\Omega}}^{-1}\mathbf{T})^{-1}\mathbf{T}'\widehat{\boldsymbol{\Omega}}^{-1} \\ &+ \mathbf{K}_0\left(\widehat{\boldsymbol{\Omega}}^{-1} - \widehat{\boldsymbol{\Omega}}^{-1}\mathbf{T}(\mathbf{T}'\widehat{\boldsymbol{\Omega}}^{-1}\mathbf{T})^{-1}\mathbf{T}'\widehat{\boldsymbol{\Omega}}^{-1}\right). \end{aligned}$$

Prediction Error

- Hence,

$$\begin{aligned}\text{Var}(\mathbf{Y}_0 - \widehat{\mathbf{Y}}_0) &= \text{Var}(\mathbf{Y}_0 - \mathbf{A}_0\mathbf{Y}) \\ &= \text{Var}(\mathbf{Y}_0) + \mathbf{A}_0\text{Var}(\mathbf{Y})\mathbf{A}_0' - 2\mathbf{A}_0\text{Cov}(\mathbf{Y}, \mathbf{Y}_0).\end{aligned}$$

  - Var($\mathbf{Y}_0$) and Var($\mathbf{Y}$) are computed by plugging in parameters estimates for $\boldsymbol{\Sigma}_{\mathbf{h}}$ and $\boldsymbol{\Sigma}_{\epsilon}$.
  - The covariance between $\mathbf{Y}_0$ and $\mathbf{Y}$ comes from $\mathbf{h}$ and is based on the distance between the transformed composition data.

Generation of Soil Profiles

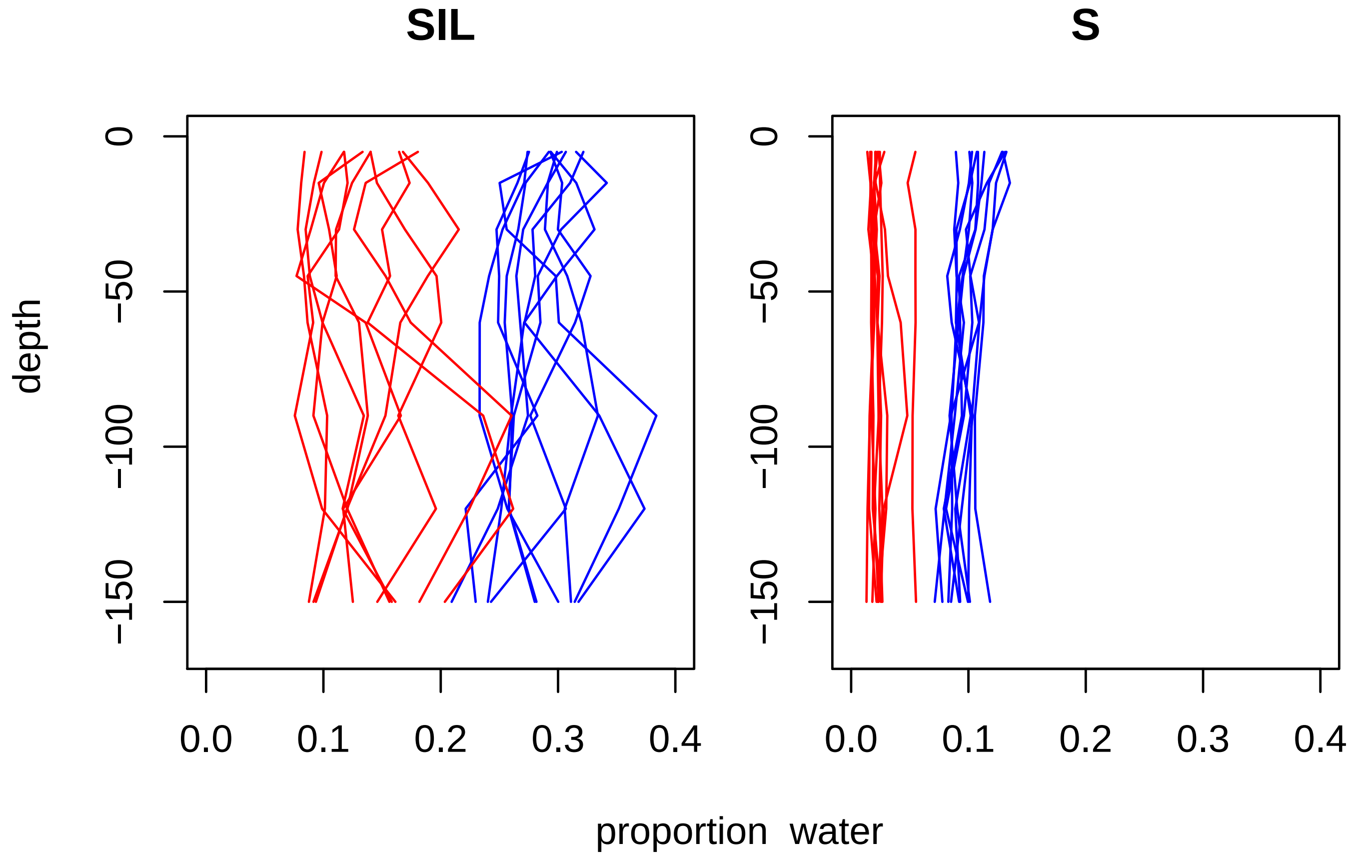
- Simulations of $\log LL$ and $\log \Delta$ were generated from a multivariate normal with mean $\mathbf{A}_0\mathbf{Y}$ and variance given by the prediction error.

- We use an average soil composition profile computed from the data and assumed to constant across all depths,

$$D = \{5, 15, 30, 45, 60, 90, 120, 150\}.$$

- Represent a draw from the posterior distribution of soil water profiles based on the estimated mean and covariance structure and the uncertainty gleaned from the data.

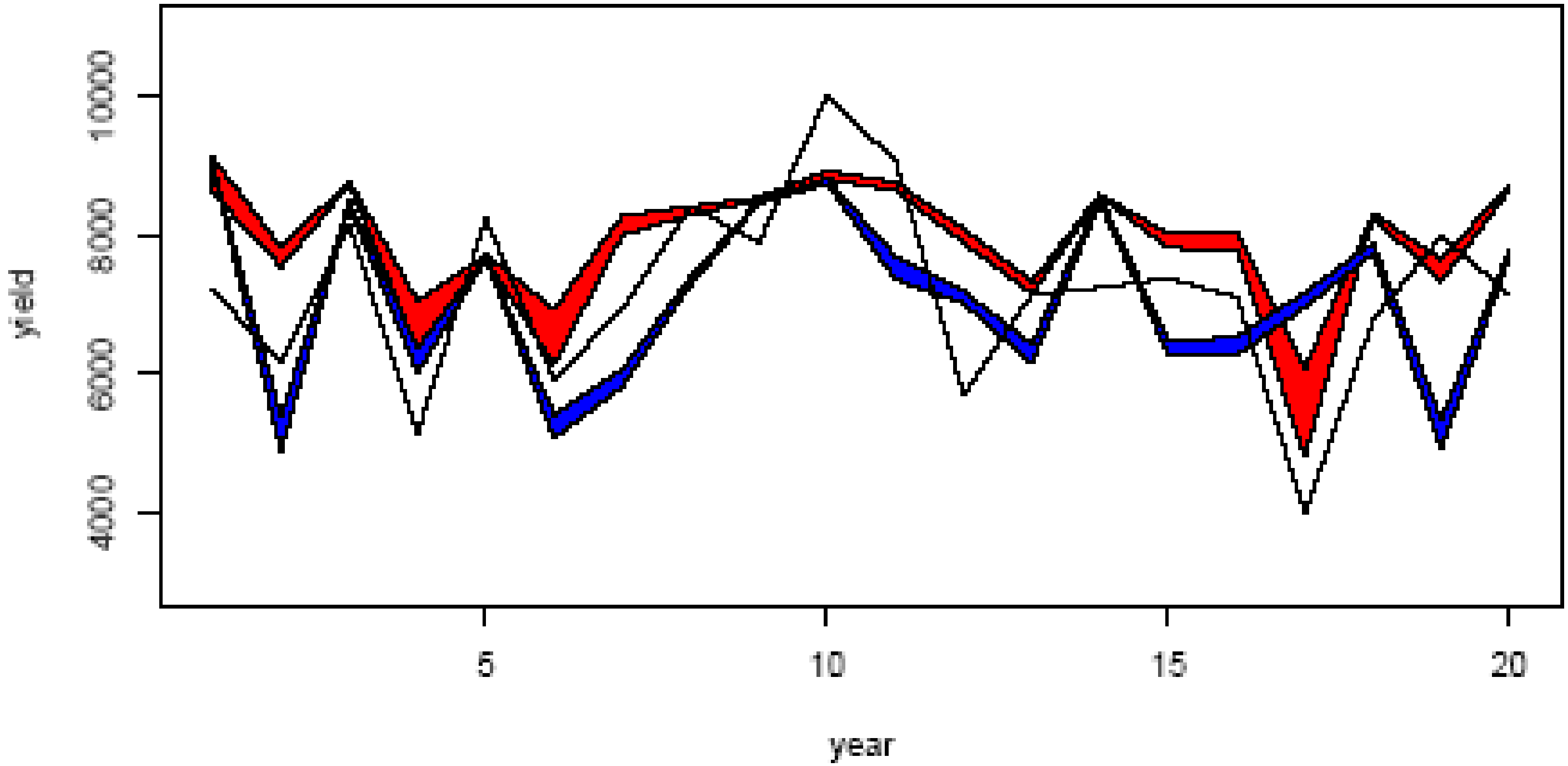
Generation of Soil Profiles



Application: Crop Models

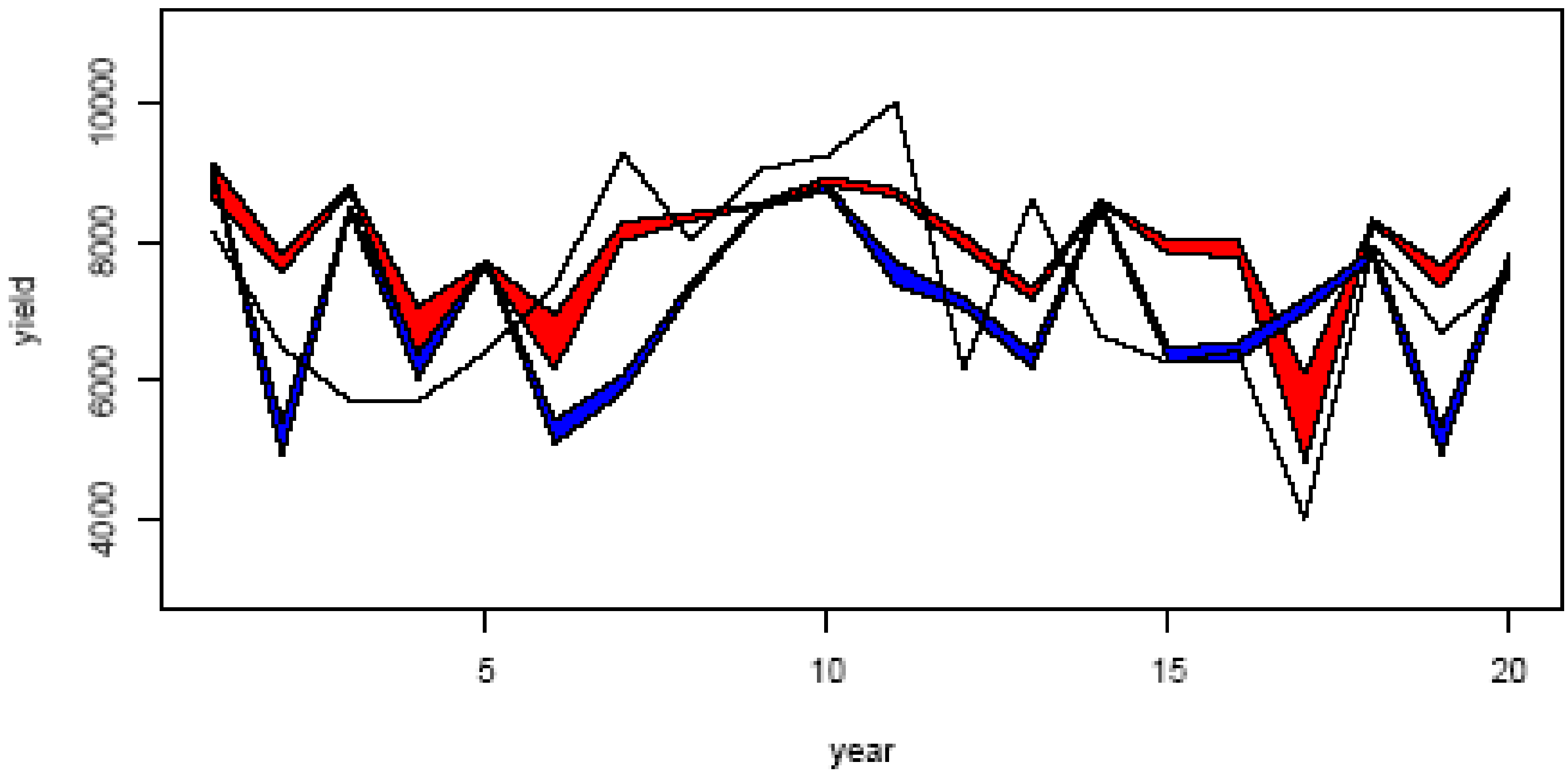
- Two soils (SIL, S)
  - Given the soil composition, organic carbon, depth, etc., 100 soil profiles (LL, DUL) were generated.
- Twenty years of weather (solar radiation, temperature min/max, and precipitation).
- Yield output generated from the CERES-Maize crop model.

Crop Yields



- SIL (red), S (blue), total annual precipitation (solid line)

Crop Yields



- SIL (red), S (blue), average annual temperature (solid line)

Thanks!

ssain@ucar.edu
www.image.ucar.edu/~ssain

- Sain, S.R., Mearns, L., Shrikant, J., and Nychka, D. (2006), "A multivariate spatial model for soil water profiles," *Journal of Agricultural, Biological, and Environmental Statistics*, **11**, 462-480.