Motivation

Introduce a sparseness structure in the covariance via tapering to gain computational advantages in large kriging problems constraint to maintaining asymptotic optimality.

Precipitation anomaly in April 1948
Motivation

Easy to think in one dimension: precipitation anomaly along 40° lat.
Motivation

Easy to think in one dimension: precipitation anomaly along 40° lat.

Ordinary kriging

Anomaly

Longitude

Boulder
Lafayette (15km)
Woodrow (156km)
Indianapolis (1632km)
Motivation

Easy to think in one dimension: precipitation anomaly along 40° lat.

Nearest neighbor kriging with 8 observations
Motivation

Easy to think in one dimension: precipitation anomaly along 40° lat.

Tapering
Objective

For an isotropic and stationary process with covariance $C_0(h)$, find a taper $C_\theta(h)$, such that kriging with the product $C_0(h)C_\theta(h)$ is asymptotically optimal.
Objective

For an isotropic and stationary process with covariance \( C_0(h) \),
find a taper \( C_\theta(h) \),
such that kriging with the product \( C_0(h)C_\theta(h) \)
is asymptotically optimal.

\[
\frac{\text{MSE}(x^*, C_0C_\theta)}{\text{MSE}(x^*, C_0)} \to 1
\]

\[
\frac{\varrho(x^*, C_0C_\theta)}{\text{MSE}(x^*, C_0)} \to \gamma
\]

\[
\varrho(x^*, C) = C(0) - c^* C^{-1} c^*
\]
Matérn Covariance

We need a broad, flexible class of covariances to describe spatial processes.

Matérn class covariance

$$C_{\alpha,\nu}(h) \propto (\alpha h)^\nu K_\nu(\alpha h) \quad h = \|h\|$$

and spectral density

$$f_{\alpha,\nu}(\rho) \propto \frac{1}{(\alpha^2 + \rho^2)^{\nu+d/2}} \quad \rho = \|\omega\|$$

Differentiability at the origin of the covariance is related to the tail behavior of the spectrum, i.e. the smoothness of the process.

The process is $m$ times mean squared differentiable iff $m < \nu$. 
Matérn Covariance

\[ \nu = 0.00 \]
\[ \nu = 0.5 \]
\[ \nu = 1 \]
\[ \nu = 1.5 \]
\[ \nu = 2 \]
\[ \nu = 5 \]
Taper Functions

We impose on the taper $C_\theta$ the conditions

- $C_\theta$ is a positive definite function in $\mathbb{R}^d$
- $C_\theta(h) = 0$ for $h > \theta$
Taper Functions

We impose on the taper $C_\theta$ the conditions

- $C_\theta$ is a positive definite function in $\mathbb{R}^d$
- $C_\theta(h) = 0$ for $h > \theta$

For example:

- triangular: $C_\theta(h) = \left(1 - \frac{|h|}{\theta}\right)_+ \quad (x)_+ = \max(0, x)$

- spherical: $C_\theta(h) = \left(1 - \frac{|h|}{\theta}\right)^2 \left(1 + \frac{|h|}{\theta}\right)$

- Wendland-type: $C_\theta(h) = \left(1 - \frac{|h|}{\theta}\right)^{\ell+k} \cdot \text{polynom in } \frac{|h|}{\theta} \text{ of deg } k$
Examples

![Graph showing covariance and lag with Exponential and Spheric lines](image-url)
Examples

Covariance vs. Lag

- Exponential
- Spheric
- Exp * Spheric

Covariance vs. Lag

- Matern (1.5)
- Wendland
- Matern * Wendland
Misspecified Covariances

In a series of (Annals) papers, Stein gives asymptotic results for misspecified covariances.

Suppose the true covariance $C_0$ and spectrum $f_0$. If we krig with the misspecified covariance $C_1$ characterized by $f_1$ then under the Tail Condition

$$\frac{f_1(\omega)}{f_0(\omega)} = \gamma \quad 0 < \gamma < \infty \quad \text{as} \quad \|\omega\| \to \infty \quad \text{and} \ldots$$

we have asymptotic optimality

$$\frac{\text{MSE}(x^*, C_1)}{\text{MSE}(x^*, C_0)} \to 1 \quad \frac{\varrho(x^*, C_1)}{\text{MSE}(x^*, C_0)} \to \gamma$$
Tapered Covariances

Tapering is a form of misspecification if

\[ \frac{\mathcal{F}(C_{a,\nu}(h)C_{\theta}(h))}{\mathcal{F}(C_{a,\nu}(h))} \to \gamma \text{ as } \|\omega\| \to \infty \]

Which taper satisfies this condition?

The taper has to be

- as differentiable at the origin as the original covariance
- more differentiable throughout the domain than at the origin
**Taper Theorem**

**Infill Condition:** Let \( x^* \in \mathbb{D} \) and \( x_1, x_2, \ldots \) be a dense sequence in \( \mathbb{D} \).

**Taper Condition:** Let \( f_\theta \) be the spectral density of the taper covariance, \( C_\theta \), and for some \( \epsilon > 0 \) and \( M < \infty \)

\[
f_\theta(\rho) < \frac{M}{(1 + \rho^2)^{\nu + d/2 + \epsilon}}
\]

**Taper Theorem:** Assume that \( C_{\alpha, \nu} \) is a Matérn covariance with smoothness parameter \( \nu \) and the Infill and Taper Conditions hold. Then

\[
\lim_{n \to \infty} \frac{\text{MSE}(x^*, C_{\alpha, \nu}C_\theta)}{\text{MSE}(x^*, C_{\alpha, \nu})} = 1 \quad \lim_{n \to \infty} \frac{\varrho(x^*, C_{\alpha, \nu}C_\theta)}{\text{MSE}(x^*, C_{\alpha, \nu})} = 1 (= \gamma)
\]
The principal irregular term (PIT) relates the tail behavior of the spectrum and the behavior at the origin of the covariance.

Formally, the PIT of $C$ is the first term as a function of $h$ in this series expansion about zero that is not raised to an even power.

**Conjecture:** Assume a polynomial isotropic covariance function $C_\theta$ in $\mathbb{R}^d$ that is integrable with PIT $B h^\mu$ and . . . . Then the PIT and the tail behaviour are related by

$$\lim_{\rho \to \infty} \rho^{\mu+d} f_\theta(\rho) = \left| B \cdot T(\mu, d) / \theta^{d-1} \right|$$
Simulation Study

When does infill asymptotics kick in?

Simulation setup:
- \( n \) equispaced or random (100 samples) observations in \([0, 1]^2\)
- \( C_{\alpha, \nu} \): Matérn covariance, eff. range 0.4
- \( C_\theta \): spherical and Wendland-type, range \( \theta = 0.4 \)
- prediction on \((0.5, 0.5)\).
Simulation Study

When does infill asymptotics kick in?

<table>
<thead>
<tr>
<th>ν</th>
<th>Spherical</th>
<th>Wendland type</th>
<th>Wendland type</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>MSE</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>1</td>
<td>MSE</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>1.5</td>
<td>MSE</td>
<td>1.0</td>
<td>1.0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>n (sqrt-scale)</th>
<th>64</th>
<th>144</th>
<th>256</th>
<th>400</th>
<th>576</th>
<th>784</th>
</tr>
</thead>
<tbody>
<tr>
<td>ν = 0.5</td>
<td>MSE</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>ν = 1</td>
<td>MSE</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>ν = 1.5</td>
<td>MSE</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
</tr>
</tbody>
</table>
Simulation Study

What is an efficient taper?

Simulation setup:
- \( n = 400 \) equispaced or random observations in \([0, 1]^2\)
- \( C_{\alpha,\nu} \): Matérn covariance, eff. range 0.4
- \( C_{\theta} \): spherical, Wendland-type, range \( \theta \)
- NN-kriging with neighborhood \( \theta \)
- prediction on \((0.5, 0.5)\).
Simulation Study

What is an efficient taper?

Spherical

$\nu = 0.5$

Wendland type

$\nu = 0.5$

Wendland type

$\nu = 1$

Wendland type

$\nu = 1.5$
Conclusion

Tapering is an (asymptotically and computationally) efficient technique to create sparse covariance matrices.

Taper range can be justified by computing resources. However, 20–30 locations within the taper range is often sufficient.
References

For example:
