# Statistical Models
for Climate Model Output

Motivation, means and ends

- Expensive ensembles of climate models (GCMs) are available. We should extract as much information as possible from them by reconciling and summarizing their different projections.

- We substitute formal statistical analysis for qualitative assessment based on heuristic criteria of model performance and inter-model agreement.

- The goal is to produce a probabilistic representation of the uncertainty in future temperature (precipitation) change, at regional scales, useful for impact research and decision making.

Which regional scales?
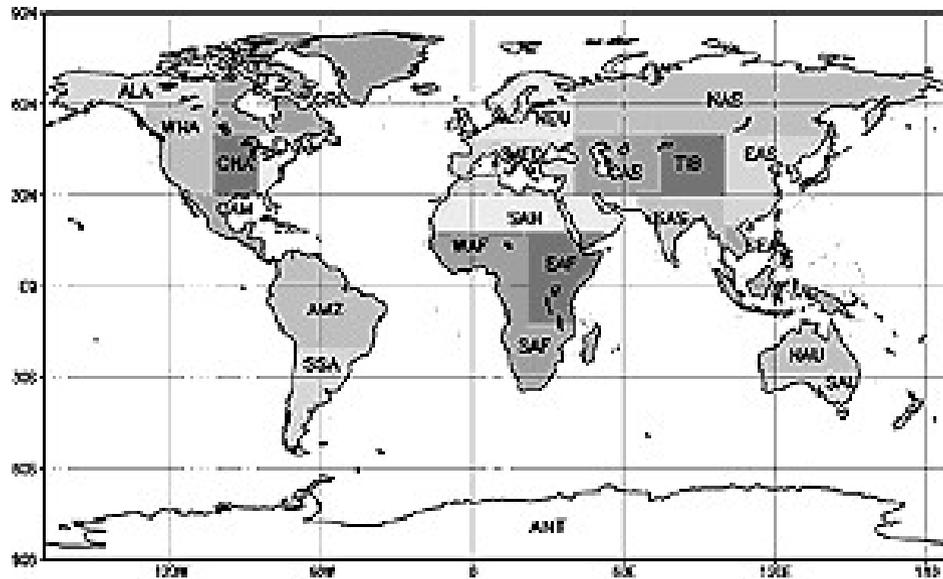
ALA	Alaska	NEU	Northern Europe	NAS	Northern Asia
WNA	Western North America	MED	Mediterranean	TIB	Tibet
CNA	Central North America	CAS	Central Asia	EAS	Eastern Asia
ENA	Eastern North America	SAH	Sahara	SAS	Southern Asia
GRL	Greenland	WAF	West Africa	SEA	South East Asia
CAM	Central America	EAF	East Africa	NAU	Northern Australia
AMZ	Amazonia	SAF	Southern Africa	SAU	Southern Australia
SSA	Southern South America	ANT	Antarctic		

A first attempt at synthesizing GCMs:
Reliability Ensemble Average (REA) method
(Giorgi and Mearns, J. of Climate, 2002)

- M climate models

- X_j - projections of current climate by GCM j

- Y_j - projections of future climate by GCM j

- X_0 - observed current true climate (uncertainty ϵ)

Define $\Delta T_j = Y_j - X_j$, and consider an estimator of the form

$$\widetilde{\Delta T} = \frac{\sum_{j=1}^{M} \lambda_j \Delta T_j}{\sum_{j=1}^{M} \lambda_j}$$

λ_j is − in Giorgi and Mearns' terms − the "reliability" of the j-th GCM

Two criteria bearing on λ_j are proposed: BIAS and CONVERGENCE

- Reward GCMs that perform well in reproducing current climate/discount GCMs that show a large bias

- Reward GCMs that form the consensus/downweight extreme projections

$$\lambda_j = (\lambda_{B,j}^m \lambda_{C,j}^n)^{1/mn}$$

where

$$\lambda_{B,j} = \min\left(1, \frac{\epsilon}{|X_j - X_0|}\right), \qquad \lambda_{C,j} = \min\left(1, \frac{\epsilon}{|Y_j - \tilde{Y}|}\right)$$

m, n control the relative importance of the two criteria (set to 1 in the paper)

Iterate to convergence. . .

See Nychka & Tebaldi (2003) for an interpretation of the REA weighted average.

Bayesian Univariate Model

$$
\begin{aligned}
X_0 &\sim N[\mu, \lambda_0^{-1}], \qquad (\lambda_0 \text{ known}) \\
X_j &\sim N[\mu, \lambda_j^{-1}], \\
Y_j | X_j &\sim N[\nu, (\theta \lambda_j)^{-1}],
\end{aligned}
$$

where $\mu, \nu, \theta, \lambda_j's$ have prior distributions

$$
\begin{aligned}
\mu, \nu &\sim U(-\infty, \infty), \\
\theta &\sim G[a, b], \\
\lambda_1, \ldots, \lambda_M &\sim G[a, b].
\end{aligned}
$$

The hyperparameters a, b are chosen to produce diffuse but proper priors. In practice, we set them $= 0.01$.

Define:

$$\tilde{\mu} \;=\; \frac{\lambda_0 X_0 + \sum \lambda_j X_j}{\lambda_0 + \sum \lambda_j + \theta \sum \lambda_j},$$

$$\tilde{\nu} \;=\; \frac{\sum \lambda_j Y_j}{\sum \lambda_j}.$$

Gibbs Sampler Updates

$$\mu \mid \text{rest} \quad \sim \quad N\left[\tilde{\mu}, \frac{1}{\lambda_0 + \sum \lambda_j}\right],$$

$$\nu \mid \text{rest} \quad \sim \quad N\left[\tilde{\nu}, \frac{1}{\theta \sum \lambda_j}\right],$$

$$\lambda_j \mid \text{rest} \quad \sim \quad Gam\left[a + 1, b + \tfrac{1}{2}(X_j - \mu)^2 + \tfrac{\theta}{2}(Y_j - \nu)^2\right],$$

$$\theta \mid \text{rest} \quad \sim \quad Gam\left[a + \tfrac{M}{2}, b + \tfrac{1}{2}\sum \lambda_j(Y_j - \nu)^2\right].$$

REA-like features of the posterior estimates:

μ^* is weighted average of observations and GCM output

ν^* is weighted average of GCM output

λ_j^*'s are weights and look like $\approx \frac{1}{\frac{1}{2}(X_j - \mu)^2 + \frac{\theta}{2}(Y_j - \nu)^2}$

Shortcomings of this model

Only two data points to estimate each $\boldsymbol{\lambda}_j$!

Very unstable estimates, and very "diverse" across GCMs.

No correlation between $\boldsymbol{X}_j$ and $\boldsymbol{Y}_j$

Bayesian Univariate Model (the return)

$$
\begin{aligned}
X_0 &\sim N[\mu, \lambda_0^{-1}], && (\lambda_0 \text{ known}) \\
X_j &\sim N[\mu, \lambda_j^{-1}], \\
Y_j | X_j &\sim N[\nu + \beta(X_j - \mu), (\theta \lambda_j)^{-1}],
\end{aligned}
$$

where $\mu, \nu, \beta, \theta, \lambda_j's$ have prior distributions

$$
\begin{aligned}
\mu, \nu, \beta &\sim U(-\infty, \infty), \\
\theta &\sim G[a, b], \\
\lambda_1, \ldots, \lambda_M &\sim G[a_\lambda, b_\lambda], \\
a_\lambda, b_\lambda &\sim G[a^*, b^*].
\end{aligned}
$$

The hyperparameters a, b, a^*, b^* are chosen to produce diffuse but proper priors. Here too we set all $= 0.01$.

Define:

$$\tilde{\mu} = \frac{\lambda_0 X_0 + \sum \lambda_j X_j - \theta\beta \sum \lambda_j (Y_j - \nu - \beta X_j)}{\lambda_0 + \sum \lambda_j + \theta\beta^2 \sum \lambda_j},$$

$$\tilde{\nu} = \frac{\sum \lambda_j \{Y_j - \beta(X_j - \mu)\}}{\sum \lambda_j},$$

$$\tilde{\beta} = \frac{\sum \lambda_j (Y_j - \nu)(X_j - \mu)}{\sum \lambda_j (X_j - \mu)^2}.$$

Gibbs Sampler Updates

$$\mu \mid \text{rest} \;\sim\; N\left[\tilde{\mu}, \frac{1}{\lambda_0 + \sum \lambda_j + \theta \beta^2 \sum \lambda_j}\right],$$

$$\nu \mid \text{rest} \;\sim\; N\left[\tilde{\nu}, \frac{1}{\theta \sum \lambda_j}\right],$$

$$\beta \mid \text{rest} \;\sim\; N\left[\tilde{\beta}, \frac{1}{\theta \sum \lambda_j (X_j - \mu)^2}\right],$$

$$\lambda_j \mid \text{rest} \;\sim\; Gam\left[a + 1, b + \tfrac{1}{2}(X_j - \mu)^2 + \tfrac{\theta}{2}\{Y_j - \nu - \beta(X_j - \mu)\}^2\right],$$

$$\theta \mid \text{rest} \;\sim\; Gam\left[a + \tfrac{M}{2}, b + \tfrac{1}{2}\sum \lambda_j\{Y_j - \nu - \beta(X_j - \mu)\}^2\right].$$

For the parameters a_λ, b_λ, use Metropolis

Improvements on first model

By imposing a common prior on the $\boldsymbol{\lambda}_j$'s we constrain them not to be too different from one another. We assume this family of GCMs is quite homogeneous.

The data will suggest if correlation exists between $\boldsymbol{X}_j$'s and $\boldsymbol{Y}_j$'s. Posterior PDF of $\boldsymbol{\beta}$ will tell . . .

Validation of the Univariate Model

We now have a predictive distribution for $\Delta T \equiv Y_0 - X_0$, how do we validate it?

Wait until the end of the century, observe actual temperature change and compare to predictive distribution.

Disadvantages? Advantages?

Cross-Validation in the Univariate Model

Rather, let's assume our GCMs are a sample from an infinite population

Based on the posterior distribution of all parameters we can compute a posterior predictive distribution for a new GCM's

$$\{Y^k - X^k\}_{k=M+1}.$$

(i) $\lambda_k \sim G(a_\lambda, b_\lambda)$

(ii) conditionally on λ_k, $Y^k - X^k \sim N[\nu - \mu, \frac{(\beta-1)^2 + \theta^{-1}}{\lambda_k}]$.

Mixing over the posterior of all parameters we obtain a full posterior predictive distribution for $Y^k - X^k$.

Do it by cross-validation

Cross-Validation Algorithm

For each $j = 1, \ldots, M$ perform analysis leaving GCM j out

At the nth iteration, use values of parameters
$a_\lambda^{(n)}, b_\lambda^{(n)}, \mu^{(n)}, \nu^{(n)}, \beta^{(n)}, \theta^{(n)}$ to generate

$$\lambda_j^{(n)} \sim Gam\left[a_\lambda^{(n)}, b_\lambda^{(n)}\right]$$

and compute

$$U_j^{(n)} = \Phi\left\{\frac{Y_j - X_j - (\nu^{(n)} - \mu^{(n)})}{\sqrt{(\lambda_j^{(n)})^{-1}(\beta^{(n)}-1)^2 + (\theta^{(n)})^{-1})}}\right\}$$

Take average $U_j = \sum_n U_j^{(n)}/N$

Plot U_j's across GCMs and regions, apply tests of fit, etc.

We have GCM output over the entire globe. Let's write down a multivariate model, synthesizing projections on a number of regions at once.

Bayesian multivariate model: Likelihood

Observed current temperature in region i

$$X_{i0} \sim N[\mu_0 + \zeta_i, \lambda_{0i}^{-1}], \ (\lambda_{0i} \text{ known})$$

Simulated current temperature for model j, in region i,

$$X_{ij} \sim N[\mu_0 + \zeta_i + \alpha_j, (\eta_{ij}\phi_i\lambda_j)^{-1}]$$

Simulated future temperature for model j, in region i,

$$Y_{ij} \mid X_{ij} \sim N[\nu_0 + \zeta_i' + \alpha_j' + \beta_i(X_{ij} - \mu_0 - \zeta_i - \alpha_j), (\eta_{ij}\theta_i\lambda_j)^{-1}]$$

Bayesian multivariate model: Prior distributions

$$
\begin{aligned}
\mu_0, \nu_0, \zeta_i, \zeta_i', \beta_i, \beta_0 &\sim U(-\infty, \infty), \\
\theta_i, \phi_i, \psi_0, \theta_0, c, \alpha_\lambda, \beta_\lambda &\sim Gam[a, b], \\
\lambda_j | \; a_\lambda, b_\lambda &\sim Gam[a_\lambda, b_\lambda], \\
\eta_{ij} &\sim Gam[c, c], \\
\alpha_j | \; \psi_0 &\sim N[0, \psi_0^{-1}], \\
\alpha_j' \; | \; \alpha_j, \beta_0, \theta_0, \psi_0 &\sim N[\beta_0 \alpha_j, (\theta_0 \psi_0)^{-1}]
\end{aligned}
$$

$$
a = b = 0.01.
$$

Features of this model

- Mean component:

  Model-specific biases → shrinkage

  Region-specific effects → no shrinkage

- Variance component:

  Model- and region-specific factors.

  Interaction effects − governed by parameter c.

  1. $c \to \infty$ implies $\boldsymbol{\eta}_{ij} \equiv \mathbf{1}$, i.e., no interaction

  2. $c \to \mathbf{0}$ implies no constraint on variances, i.e., univariate approach

  3. in our model a posterior distribution for c is estimated, i.e., the data tell us what the interaction level is.

Implementation

- MCMC algorithm for the posterior densities

- Cross-validation statistics U_{ij}

Both are calculated by generalizing the univariate model.

Some intermediate results (1)
# Univariate and multivariate densities
compared for 6 regions
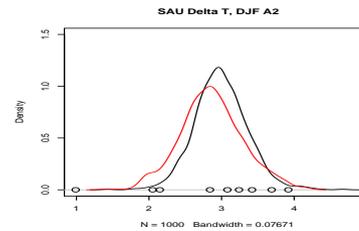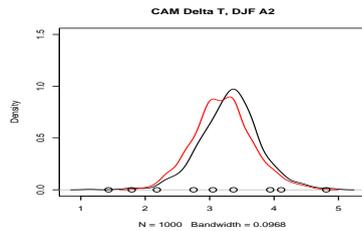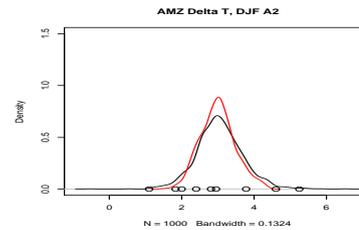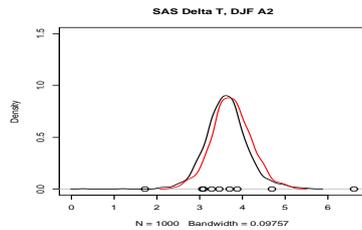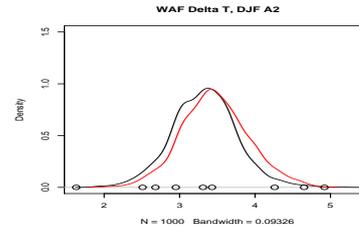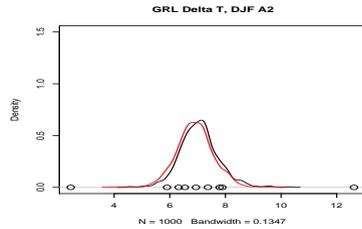
Some intermediate results (2)
Goodness of fit of the U statistics

For each region we have M values of U_{ij};

Use Kolmogorov-Smirnov test: are these M values from a
Uniform distribution?

Some intermediate results (3)
U statistics for the univariate
and multivariate model

UV: DJF, A2
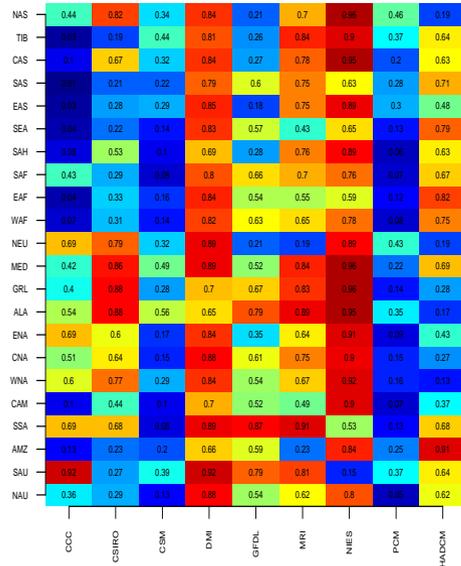
MV: DJF, A2
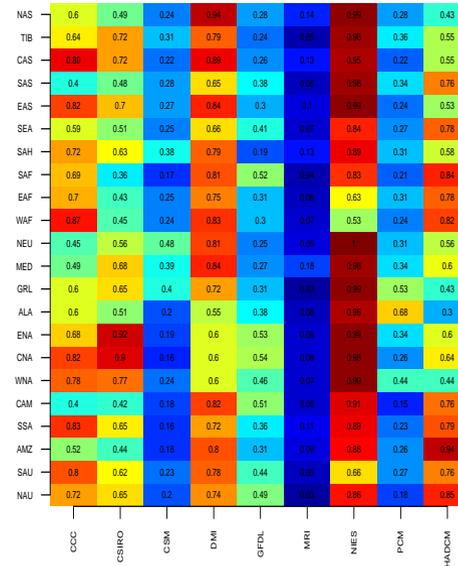
Do the rows look Uniform?

Some intermediate results (4)
Precision of the estimates

Are the multivariate model predictive distributions "tighter" than those derived through the univariate model?

Overall, yes (on average) but individual regional comparisons vary significantly:

	IQR	I15R	I5R
DJFA2	1.11 (13)	1.09(12)	1.12(15)
DJFB2	1.04 (13)	1.04(14)	1.05(12)
JJAA2	1.05 (13)	1.04(14)	1.00(14)
JJAB2	1.10 (15)	1.08(16)	1.08(14)

Some intermediate results (5)
Sensitivity analysis

First univariate model (Tebaldi et al. 2005, J. of Climate) produces PDFs highly sensitive to small perturbation in individal models' projections.

The two subsequent formulations do not suffer from the same problem.

UV original

N = 1000   Bandwidth = 0.02403

UVGM

N = 1000   Bandwidth = 0.1046

MV GM

N = 1000   Bandwidth = 0.1099
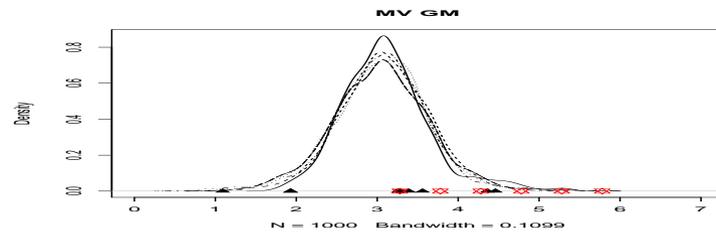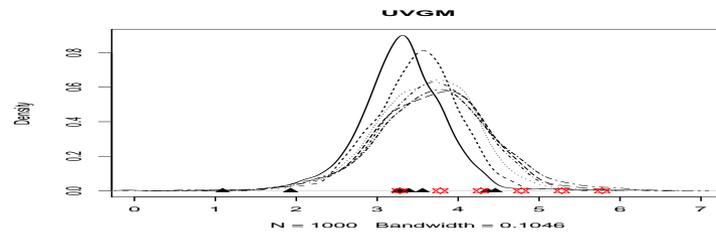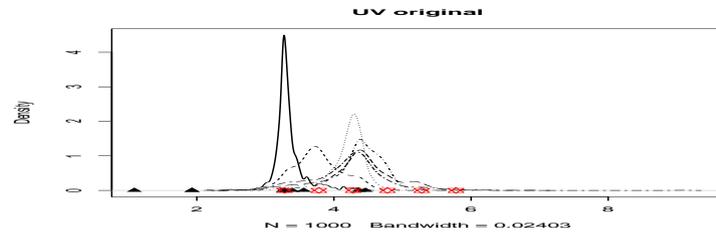
Final Results:
PDFs of Temperature Change

What this is all about, of course, is producing probabilistic projections of climate change:

$$\Delta T_i \;\equiv\; \nu_0 - \mu_0 + \zeta_i' - \zeta_i$$

We compute posterior PDFs for the 22 regions,

for all seasons,

under different SRES emission scenarios.

Temperature change in DJF, 2080–99 vs. 1980–99 under A2 and B2