Some Applications of HOMs

Amik St-Cyr
IMAGe/NCAR
Overview:

- Preconditioning: Optimized Schwarz for climate modeling
- Adaptive mesh refinements (on the sphere)
- Efficient time-stepping for AMR and SEM
- Discontinuous Galerkin for non-hydrostatic modeling
Suppose we need to solve:

\[ \mathcal{L}u = f \text{ in } \Omega, \quad \mathcal{B}u = g \text{ on } \partial\Omega \]

Partition the original domain into 2 domains:

\[ \mathcal{L}u_1^{n+1} = f \text{ in } \Omega_1, \quad \mathcal{L}u_2^{n+1} = f \text{ in } \Omega_2, \]
\[ \mathcal{B}(u_1^{n+1}) = g \text{ on } \partial\Omega_1, \quad \mathcal{B}(u_2^{n+1}) = g \text{ on } \partial\Omega_2, \]
\[ u_1^{n+1} = u_2^n \text{ on } \Gamma_{12}, \quad u_2^{n+1} = u_1^n \text{ on } \Gamma_{21}. \]
The Robin method

Lions (1990)

Used to accelerate convergence of Schwarz

Free positive parameter: how to find its correct value?

Convergence rate not demonstrated theoretically

\[ \mathcal{L} u_{j}^{k+1} = u_{j}^{k+1} - \Delta u_{j}^{k+1} = f_{j} \]

\[ p u_{j}^{k+1} + \frac{\partial u_{j}^{k+1}}{\partial n_{j l}} = p u_{l}^{k} + \frac{\partial u_{l}^{k}}{\partial n_{j l}} \text{ on } \partial \Omega_{j} \cap \partial \Omega_{l} \text{ for } l \in \mathcal{N}(\Omega_{j}) \]

\[ u_{j}^{k+1} = u_{0} \text{ on } \partial \Omega_{j} \cap \partial \Omega \]
Optimized approach

Inspired by the Robin problem:

\[
(\eta - \Delta)u_1^{n+1} = 0 \quad \text{in } \Omega_1, \quad (\eta - \Delta)u_2^{n+1} = 0 \quad \text{in } \Omega_2,
\]

\[
(\partial_x + S_1)u_1^{n+1} = (\partial_x + S_1)u_2^n \quad \text{on } \Gamma_{12}, \quad (\partial_x + S_2)u_2^{n+1} = (\partial_x + S_2)u_1^n \quad \text{on } \Gamma_{21}.
\]

We are looking for the best possible forms of in Fourier space

Proceeding as before leads to the solutions: \((\sigma_r(k) = \mathcal{F}(S_r))\)

\[
\hat{u}_1^n(x, k) = \frac{\sigma_1(k) - \sqrt{k^2 + \eta}}{\sigma_1(k) + \sqrt{k^2 + \eta}} e^{-\sqrt{k^2 + \eta} (x-L)} \hat{u}_2^{n-1}(L, k), \quad \hat{u}_2^n(x, k) = \frac{\sigma_2(k) + \sqrt{k^2 + \eta}}{\sigma_2(k) - \sqrt{k^2 + \eta}} e^{-\sqrt{k^2 + \eta} x} \hat{u}_1^{n-1}(0, k)
\]

New convergence rate:

\[
\rho_{opt} = \rho_{opt}(k, \eta, L) = \frac{\sigma_1(k) - \sqrt{k^2 + \eta}}{\sigma_1(k) + \sqrt{k^2 + \eta}} \frac{\sigma_2(k) + \sqrt{k^2 + \eta}}{\sigma_2(k) - \sqrt{k^2 + \eta}} e^{-2 \sqrt{k^2 + \eta} L}
\]
Optimized Schwarz: algebraic results

- SGT 2006: show how to modify existing Schwarz algorithm to yield optimized versions

- The augmented or “enhanced” system is rediscovered

- Spectral elements are natural candidates:
  - Overlapping grids are cumbersome to construct
  - Block preconditioning costly: FDM when possible
  - Optimal preconditioner is known (SD Kim 2006)
  - Q1-GLL based problem costly to invert does not scale: use MG or other solver to invert
Creating the augmented system from a weak form

Consider:

\[ w_j^{k+1} - \Delta w_j^{k+1} = G_j(x, y) \]
\[ pw_j^{k+1} + \frac{\partial w_j^{k+1}}{\partial n_{jl}} = pw_l^k + \frac{\partial w_l^k}{\partial n_{jl}} \text{ on } \partial \Omega_j \cap \partial \Omega_l \text{ for } l \in \mathcal{N}(\Omega_j) \]
\[ w_j^{k+1} = w_0 \text{ on } \partial \Omega_j \cap \partial \Omega \]

To be solved for all \( k \) on any \( \Omega_j \): it converges (Lions 1990).

Weak form:

\[ \int_{\Omega_j} \phi_j w_j^{k+1} + \int_{\Omega_j} \nabla \phi_j \cdot \nabla w_j^{k+1} - \int_{\partial \Omega_j} \phi_j \left( \frac{\partial w_j}{\partial n} \right)^{k+1} = \int_{\Omega_j} \phi_j G_j. \]

Where test functions are in:

\[ H^1(Q_h) = \{ v \in L^2(\Omega) | v|_Q \in H^1(Q) \forall Q \in Q_h \} \]

Decomposition:

\[ Q_h = \bigcup_j \Omega_j \]
Creating the augmented system from a weak form

Define:
\[
a_j(w_j^{k+1}, \phi_j) \equiv \int_{\Omega_j} \phi_j w_j^{k+1} + \int_{\Omega_j} \nabla \phi_j \cdot \nabla w_j^{k+1}
\]
\[
f_j(\phi_j) \equiv \int_{\Omega_j} \phi_j G_j
\]
\[
T_j(w_j^{k+1}, \phi_j) \equiv \sum_{l \in N(\Omega_j)} \int_{\Gamma_{jl}} \phi_j \left( \frac{\partial w_j}{\partial n_{jl}} \right)^{k+1}
\]

Leads to:
\[
a_j(w_j^{k+1}, \phi_j) - T_j(w_j^{k+1}, \phi_j) = f_j(\phi_j)
\]
Creating the augmented system from a weak form

Define:

\[ a_j(w_j^{k+1}, \phi_j) \equiv \int_{\Omega_j} \phi_j w_j^{k+1} + \int_{\Omega_j} \nabla \phi_j \cdot \nabla w_j^{k+1} \]

\[ f_j(\phi_j) \equiv \int_{\Omega_j} \phi_j G_j \]

\[ T_j(w_j^{k+1}, \phi_j) \equiv \sum_{l \in N(\Omega_j)} \int_{\Gamma_{jl}} \phi_j \left( \frac{\partial w_j}{\partial n_{jl}} \right)^{k+1} \]

Leads to:

\[ a_j(w_j^{k+1}, \phi_j) - T_j(w_j^{k+1}, \phi_j) = f_j(\phi_j) \]

Remains to introduce the artificial transmission condition...
Creating the augmented system from a weak form

- The normal derivative can be written in terms of the original bilinear operator (Toselli, Widlund 2005)

- Avoids the difficult duality pairing for functions on the edges of the subdomains

\[
T_j(w_j^{k+1}, \phi_j) = \int_{\Omega_j} \nabla \phi_j \cdot \nabla w_j^{k+1} + \int_{\Omega_j} \phi_j \Delta w_j^{k+1} \\
= \int_{\Omega_j} \nabla \phi_j \cdot \nabla w_j^{k+1} + \int_{\Omega_j} \phi_j w_j^{k+1} - f_j(\phi_j) \\
= a_j(w_j^{k+1}, \phi_j) - f_j(\phi_j)
\]

Where we pick \( \phi_j \in H^1(\partial \Omega_j) \)
Creating the augmented system from a weak form

Boundary condition is:

\[ T_j(w_j^{k+1}, \phi_j) = \sum_{l \in N(\Omega_j)} T_j(w_j^{k+1}, \phi_j|_{\Gamma_{ji}}) \]

\[ = \sum_{l \in N(\Omega_j)} \left\{ \int_{\Gamma_{ji}} p\phi_j(w_i^k - w_i^{k+1}) - T_i(w_i^k, \phi_j|_{\Gamma_{ji}}) \right\} \]

\[ = - \int_{\Omega_j} p\phi_j w_j^{k+1} + \sum_{l \in N(\Omega_j)} \left\{ \int_{\Gamma_{ji}} p\phi_j w_i^k - T_i(w_i^k, \phi_j|_{\Gamma_{ji}}) \right\} \]

where a sum on neighbors appears.
Creating the augmented system from a weak form

Boundary condition is:

\[ T_j(w_j^{k+1}, \phi_j) = \sum_{l \in \mathcal{N}(\Omega_j)} T_j(w_j^{k+1}, \phi_j|_{\Gamma_{jl}}) \]

\[ = \sum_{l \in \mathcal{N}(\Omega_j)} \left\{ \int_{\Gamma_{jl}} p\phi_j(w_i^k - w_i^{k+1}) - T_l(w_i^k, \phi_j|_{\Gamma_{jl}}) \right\} \]

\[ = - \int_{\Omega_j} p\phi_jw_j^{k+1} + \sum_{l \in \mathcal{N}(\Omega_j)} \left\{ \int_{\Gamma_{jl}} p\phi_jw_i^k - T_l(w_i^k, \phi_j|_{\Gamma_{jl}}) \right\} \]

where a sum on neighbors appears.

Leads to the form required by the algorithm

\[ a_j(w_j^{k+1}, \phi_j) + \int_{\Omega_j} p\phi_jw_j^{k+1} = f_j(\phi_j) + \sum_{l \in \mathcal{N}(\Omega_j)} f_l(\phi_l|_{\Gamma_{jl}}) \]

\[ - \sum_{l \in \mathcal{N}(\Omega_j)} a_l(w_i^k, \phi_l|_{\Gamma_{jl}}) + \sum_{l \in \mathcal{N}(\Omega_j)} \int_{\Gamma_{jl}} p\phi_lw_i^k \]
Creating the augmented system from a weak form

\[ a_j(w_j^{k+1}, \phi_j) + \int_{\Omega_j} p\phi_j w_j^{k+1} = f_j(\phi_j) + \sum_{l \in N(\Omega_j)} f_l(\phi_l|\Gamma_{jl}) \]

\[ - \sum_{l \in N(\Omega_j)} a_l(w_l^k, \phi_l|\Gamma_{jl}) + \sum_{l \in N(\Omega_j)} \int_{\Gamma_{jl}} p\phi_l w_l^k \]

After (any) discretization:

\[ \tilde{A}_j u_j^{n+1} = f_j + \sum_{k=1}^{J} \tilde{B}_{jk} u_k^n, \quad j = 1, \ldots, J \]
Creating the augmented system from a weak form

\[
a_j(w_j^{k+1}, \phi_j) + \int_{\Omega_j} p\phi_j w_j^{k+1} = f_j(\phi_j) + \sum_{l \in N(\Omega_j)} f_l(\phi_l|\Gamma_{jl}) - \sum_{l \in N(\Omega_j)} a_l(w_l^k, \phi_l|\Gamma_{jl}) + \sum_{l \in N(\Omega_j)} \int_{\Gamma_{jl}} p\phi_l w_l^k
\]

After (any) discretization:

\[
\tilde{A}_j u_j^{n+1} = f_j + \sum_{k=1}^J \tilde{B}_{jk} u_k^n, \quad j = 1, \ldots, J
\]
Creating the augmented system from a weak form

\[
a_j(w_j^{k+1}, \phi_j) + \int_{\Omega_j} p\phi_j w_j^{k+1} = f_j(\phi_j) + \sum_{l \in N(\Omega_j)} f_l(\phi_l|\Gamma_{jl}) - \sum_{l \in N(\Omega_j)} a_l(w_l^k, \phi_l|\Gamma_{jl}) + \sum_{l \in N(\Omega_j)} \int_{\Gamma_{jl}} p\phi_l w_l^k
\]

After (any) discretization:

\[
\tilde{A}_j\mathbf{u}_j^{n+1} = \mathbf{f}_j + \sum_{k=1}^{J} \tilde{B}_{jk} \mathbf{u}_k^n, \quad j = 1, \ldots, J
\]
Creating the augmented system from a weak form

\[ a_j(w_j^{k+1}, \phi_j) + \int_{\Omega_j} p\phi_j w_j^{k+1} = f_j(\phi_j) + \sum_{l \in \mathcal{N}(\Omega_j)} f_l(\phi_l|\Gamma_{jl}) - \sum_{l \in \mathcal{N}(\Omega_j)} a_l(w_l^k, \phi_l|\Gamma_{jl}) + \sum_{l \in \mathcal{N}(\Omega_j)} \int_{\Gamma_{jl}} p\phi_l w_l^k \]

After (any) discretization:

\[ \tilde{A}_j u_j^{n+1} = f_j + \sum_{k=1}^{J} \tilde{B}_{jk} u_k^n, \quad j = 1, \ldots, J \]
Creating the augmented system from a weak form

\[ a_j(w_j^{k+1}, \phi_j) + \int_{\Omega_j} p\phi_j w_j^{k+1} = f_j(\phi_j) + \sum_{l \in N(\Omega_j)} f_l(\phi_l|\Gamma_{jl}) \]

\[ - \sum_{l \in N(\Omega_j)} a_l(w_l^k, \phi_l|\Gamma_{jl}) + \sum_{l \in N(\Omega_j)} \int_{\Gamma_{jl}} p\phi_l w_l^k \]

After (any) discretization:

\[ \tilde{A}_j u_j^{n+1} = f_j + \sum_{k=1}^{J} \tilde{B}_{jk} u_k^n, \quad j = 1, \ldots, J \]

Possible to create augmented system!
Creating the augmented system from a weak form

\[ a_j(w_j^{k+1}, \phi_j) + \int_{\Omega_j} p\phi_j w_j^{k+1} = f_j(\phi_j) + \sum_{l \in N(\Omega_j)} f_l(\phi_l|\Gamma_{jl}) \]

\[- \sum_{l \in N(\Omega_j)} a_l(w_l^k, \phi_l|\Gamma_{jl}) + \sum_{l \in N(\Omega_j)} \int_{\Gamma_{jl}} p\phi_l w_l^k \]

After (any) discretization:

\[ \tilde{A}_j u_j^{n+1} = f_j + \sum_{k=1}^{J} \tilde{B}_{jk} u_k^n, \quad j = 1, .., J \]

Possible to create augmented system!
Not mentioned: difficulties at corners
Creating the augmented system from a weak form

\[ a_j(w_j^{k+1}, \phi_j) + \int_{\Omega_j} p\phi_j w_j^{k+1} = f_j(\phi_j) + \sum_{l \in N(\Omega_j)} f_l(\phi_l|\Gamma_{jl}) \]

\[ - \sum_{l \in N(\Omega_j)} a_l(w_l^k, \phi_l|\Gamma_{jl}) + \sum_{l \in N(\Omega_j)} \int_{\Gamma_{jl}} p\phi_l w_l^k \]

After (any) discretization:

\[ \tilde{A}_j u_j^{n+1} = \mathbf{f}_j + \sum_{k=1}^{J} \tilde{B}_{jk} u_k^n, \quad j = 1, \ldots, J \]

Possible to create augmented system!

Not mentioned: difficulties at corners

Not mentioned: “under” integration for SEM
SEM simple problem

\[ L u = (\eta - \Delta) u = f, \quad \text{in } \Omega, \]

Gander 2006
SEM simple problem
SEM simple problem

![Graph showing GMRES iterations vs. total number of elements](image)
Primitive equations

Momentum: \[ \frac{d\mathbf{v}}{dt} + f \mathbf{k} \times \mathbf{v} + \nabla \Phi + RT \nabla \ln p = 0 \]

Thermodynamic: \[ \frac{dT}{dt} - \frac{\kappa T \omega}{p} = 0 \]

Continuity: \[ \frac{\partial}{\partial t} \left( \frac{\partial p}{\partial \eta} \right) + \nabla \cdot \left( \mathbf{v} \frac{\partial p}{\partial \eta} \right) + \frac{\partial}{\partial \eta} \left( \dot{\eta} \frac{\partial p}{\partial \eta} \right) = 0 \]

HOMME: high order multiscale modeling environment
Primitive equations: SI

Hydrostatic assumption: \( \frac{\partial \Phi}{\partial \eta} = -\frac{RT}{p} \frac{\partial p}{\partial \eta} \).

Linearization (barotropic state): \( T^r = 300K, \ p^r_s = 1000hPa \)
Semi-Implicit:
\[
\frac{dX}{dt} = \mathcal{M}(X)
\]

Add zero: \( \frac{dX}{dt} = \mathcal{M}(X) + \mathcal{L}X - \mathcal{L}X = \mathcal{N}(X) - \mathcal{L}X \)

\[
\frac{X^{n+1} - X^{n-1}}{2\Delta t} = \mathcal{N}(X^n) - \frac{1}{2} \mathcal{L}(X^{n+1}+X^{n-1}) = \mathcal{M}(X^n) + \mathcal{L}X^n - \frac{1}{2} \mathcal{L}(X^{n+1}+X^{n-1})
\]

\[
\frac{X^{n+1} - X^{n-1}}{2\Delta t} = \mathcal{M}(X^n) - \frac{1}{2} \Delta t \mathcal{L}X
\]

“Time diffusion”
**PE: vertical structure matrix**

Results of hydrostatic assumption and vertical coordinate choice: \[ p(\eta, p_s) = A(\eta)p_0 + B(\eta)p_s \]

\[ A = RH^r T + RT^r P, \]

\[ G^r - \Delta t^2 A \nabla^2 G^r = B - \Delta t A \nabla \cdot \mathbf{v} \]

Solve for each \( k \):

\[
\left( \nabla^2 - \frac{1}{\Delta t^2 \lambda_k} \right) \Gamma_k^r = C_k
\]

Time dependence

Series of 2D Helmholtz

Barotropic eigenmodes of atmosphere

(Thomas and Loft 2005)
PE: vertical structure matrix

Results of hydrostatic assumption and vertical coordinate choice: \[ p(\eta, p_s) = A(\eta)p_0 + B(\eta)p_s \]

\[
A = R\mathbf{H}^r \mathbf{T} + RT^r P, \quad \text{Diagonalize}
\]

\[ G^r - \Delta t^2 A \nabla^2 G^r = B - \Delta t A \nabla \cdot \mathbf{v} \]

Solve for each k:

\[
\left( \nabla^2 - \frac{1}{\Delta t^2 \lambda_k} \right) \Gamma_k^r = C_k
\]

Backsub:

\[
D = \Delta t^{-1} A^{-1} (B - G^r)
\]
\[
\ln p_s = \mathcal{P} - \Delta t P \cdot D
\]
\[
T = T - \Delta t \mathbf{T} D
\]
\[
\mathbf{v} = \mathcal{V} - \Delta t \nabla G^r
\]

Series of 2D Helmholtz Barotropic eigenmodes of atmosphere (Thomas and Loft 2005)
Cubed sphere

- Equiangular projection
- Sadourny (72), Rancic (96), Ronchi (96)
- Most models moving towards this approach
- SFC: Dennis 2003

Metric tensor

\[
g_{ij} = \frac{1}{r^4 \cos^2 x_1 \cos^2 x_2} \left[ \begin{array}{cc} 1 + \tan^2 x_1 & -\tan x_1 \tan x_2 \\ -\tan x_1 \tan x_2 & 1 + \tan^2 x_2 \end{array} \right].
\]

Rewrite div and vorticity

\[
g \nabla \cdot \mathbf{v} = \frac{\partial}{\partial x^j} (g u^j), \quad g \zeta = \epsilon_{ij} \frac{\partial u_j}{\partial x^i}.
\]
Cubed sphere

- Equiangular projection
- Sadourny (72), Rancic (96), Ronchi (96)
- Most models moving towards this approach
- SFC: Dennis 2003

Metric tensor

\[ g_{ij} = \frac{1}{r^4 \cos^2 x_1 \cos^2 x_2} \left[ \begin{array}{cc} 1 + \tan^2 x_1 & -\tan x_1 \tan x_2 \\ -\tan x_1 \tan x_2 & 1 + \tan^2 x_2 \end{array} \right] \].

Rewrite div and vorticity

\[ g \nabla \cdot \mathbf{v} = \frac{\partial}{\partial x^j} (g u^j), \quad g \zeta = \epsilon_{ij} \frac{\partial u_j}{\partial x^i}. \]
Held-Suarez numerical experiment: with moisture

Galewski, Sobel and Held 2004
Convergence per mode

- Optimized algorithm: no maxing out
- Communication cost identical
- Twice the cost of CG per iteration
- Diagonal $O(N)$ while OS is $O(N^{\frac{3}{2}})$
- Best strategy: use OS on first few barotropic modes and diagonal elsewhere
- No coarse solver needed: because of time dependance
Diagonal preconditioning

![Graph showing iterations over simulated days for different modes](image-url)
Optimized Schwarz
New approach

![Graph showing comparison between Diagonal and Opt Schwarz methods.](image)
New approach
New approach

![Bar Chart]

- **OAS**
  - Diagonal
  - Opt Schwarz

- **Diagonal**

- **Opt Schwarz**

- **Values**:
  - 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20
Large scale runs

Max 5TF

W. Spotz and M. Taylor: Sandia National Labs
Si vs Exp: Red Storm

W. Spotz : Sandia National Labs
Si vs Exp: Blue gene

ne=32, 40km
Si vs Exp: Blue gene

\[
\text{ne}=32, 40\text{km}
\]
Adaptive Mesh Refinements
Conforming SEM

Linear problem associated with elliptic problem discretized with SEM

\[ Au = f \]

\[ A_L = block\{A_1, A_2, A_3, A_4\} \]

\[ v^T A u = v^T Q^T A_L Q u = v^T Q^T M_L Q f = v^T f \]

\[ v^T Q^T A_L u_L = v^T Q^T M_L f_L \]

\[ Q Q^T A_L u_L = Q Q^T M_L f_L \]
Conforming SEM

Linear problem associated with elliptic problem discretized with SEM

\[ Au = f \]

\[ A_L = \text{block}\{A_1, A_2, A_3, A_4\} \]

\[ v^T A u = v^T Q^T A_L Q u = v^T Q^T M_L Q f = v^T f \]

Direct stiffness summation: represents boolean operations
Nonconforming SEM

Fischer, Kruse and Loth 2002

Boolean matrix $Q$ is redefined as $Q = J_L \tilde{Q}$

DSS is conceptually the same:

$$v^T A u = v^T (\tilde{Q}^T J_L^T) A_L (J_L \tilde{Q}) u = v^T Q^T A_L Q u.$$
SEM vs FVM

a) SEM
b) FV

Standard test suite is employed
Cosine bell advection

Alpha = 0

<table>
<thead>
<tr>
<th>Resolution</th>
<th>$l_1$</th>
<th>$l_2$</th>
<th>$l_\infty$</th>
<th>$h(m)$ max/min</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.5</td>
<td>0.0341</td>
<td>0.0301</td>
<td>0.0317</td>
<td>949.1/0</td>
</tr>
<tr>
<td>1.25</td>
<td>0.0097</td>
<td>0.0103</td>
<td>0.0150</td>
<td>984.2/0</td>
</tr>
<tr>
<td>0.625</td>
<td>0.0016</td>
<td>0.0021</td>
<td>0.0044</td>
<td>995.0/0</td>
</tr>
<tr>
<td>0.3125</td>
<td>0.0003</td>
<td>0.0005</td>
<td>0.0014</td>
<td>998.4/0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Resolution</th>
<th>$l_1$</th>
<th>$l_2$</th>
<th>$l_\infty$</th>
<th>$h(m)$ max/min</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.5</td>
<td>0.0503</td>
<td>0.0269</td>
<td>0.0195</td>
<td>991.6/-15.1</td>
</tr>
<tr>
<td>1.25</td>
<td>0.0085</td>
<td>0.0056</td>
<td>0.0057</td>
<td>997.5/-4.2</td>
</tr>
<tr>
<td>0.625</td>
<td>0.0019</td>
<td>0.0014</td>
<td>0.0019</td>
<td>999.1/-1.1</td>
</tr>
<tr>
<td>0.3125</td>
<td>0.0008</td>
<td>0.0006</td>
<td>0.0015</td>
<td>999.7/-0.9</td>
</tr>
</tbody>
</table>

 documento de imagen
SWTC1 45 degrees

(a) SEM

(b) FV
SWTC2

a) SEM

Normalized $\|\Phi_h\|$ vs. Days

- 2.5° x 2.5° uniform
- static ref. (180°E, 45°N)
- static ref. (135°E, 30°N)

b) FV

Normalized $\|\Phi_h\|$ vs. Days

- 2.5° x 2.5° uniform
- static ref. (180°E, 45°N)
- static ref. (135°E, 30°N)

Tolstykh (2.5°), JCP 2002
SWTC5

a) SEM

0 72 144 216 288 360

Normalized $l_2(h)$

0.001
0.002
0.003
0.004
0.005
0.006

Hours

5° x 5° uniform
2.5° x 2.5° uniform
1.25° x 1.25° uniform
adaptive grid, 3 ref. levels (0.625°)

b) FV

5° x 5° uniform
2.5° x 2.5° uniform
0.625° x 0.625° uniform
adaptive grid, 3 ref. levels (0.625°)

Geopotential height [m]

-90 -45 0 45 90

-180 -90 0 90 180

Day

0

5

10

15

Latitude

Longitude

5000 5200 5400 5600 5800 6000

5000 5200 5400 5600 5800 6000

35
SWTC5
a) SEM

-90
-45
0
45
90
Latitude
0 90 180 270 360

b) FV

-90
-45
0
45
90
Latitude
0 90 180 270 360

c) NCAR reference

-90
-45
0
45
90
Latitude
0 90 180 270 360

Longitude
8500 9000 9500 10000 10500

Geopotential height [m]

0
0.002
0.004
0.006
0.008
Normalized $l_2(h)$

0 48 96 144 192 240 288 336

Hours

SEM uniform 3.2° x 3.2°

SEM 3.2°, static refinement

SEM uniform 2.5° x 2.5°

SEM 2.5°, static refinement

FV uniform 2.5° x 2.5°

FV 2.5°, static refinement

SWTC6
Time-stepping
Goal: SEM based AMR for primitive equations

Oliger and Sundstrom show ill posedness for any kind of boundary conditions

Cannot use local time stepping: Berger Oliger (84)

Semi-implicit semi-Lagrangian approach? (Robert 81)
OIFS+AMR

- Goal: SEM based AMR for primitive equations
- Oliger and Sundstrom show ill posedness for any kind of boundary conditions
- Cannot use local time stepping: Berger Oliger (84)
- Semi-implicit semi-Lagrangian approach? (Robert 81)
- SISL is rather inefficient on modern computers
- Attempts were made by Berhens: shmem only (96)
Material derivative (hide advective term)

Spatial position is now a function of time in the Lagrangian frame

**Characteristic equation:**

\[
\frac{dX}{dt} = u(X, t)
\]

\[
X^n = X^{n+1} - \frac{\Delta t}{2}(u(X^n, t^n) + u(X^{n+1}, t^{n+1})), \quad \text{with} \quad X(t^{n+1}) = x
\]
SISL

\[ t^n + \Delta t \]

\[ \frac{1}{a} \]

\[ x_j \quad x'_i \]

\[ t^n \]
SISL

$t^n + \Delta t$

\[
\frac{1}{a}
\]

$x_j \rightarrow x'_i$

$t^n$
SISL

\[ t^n + \Delta t \]

\[ \frac{1}{a} \]

\( x_j \) \( x'_j \)
Operator integrating factor splitting

- Maday, Patera, Ronquist (90): OIFS.
- $K$ elements of order $N$, $KN^d$ grid points
- Interpolation $KN^{2d}$
- Scalar advection requires $dKN^{d+1}$
- OIFS more efficient if sub-step $< N^{d-1}$ "times"
- Purely Eulerian: regular communication patterns
- Nonlinear OIFS: St-Cyr and Thomas (05)
- Euler (MC2): Girard, Thomas and St-Cyr (07)
Operator integrating factor splitting

- Maday, Patera, Ronquist (90): OIFS.
- \( K \) elements of order \( N \), \( KN^d \) grid points
- Interpolation \( KN^{2d} \)
- Scalar advection requires \( dKN^{d+1} \)
- OIFS more efficient if sub-step \( < N^{d-1} \) "times"
- Purely Eulerian: regular communication patterns
- Nonlinear OIFS: St-Cyr and Thomas (05)
- Euler (MC2): Girard, Thomas and St-Cyr (07)
Operator integrating factor splitting

\[ t^n + \Delta t \]

\[ \frac{1}{a} \]

\[ x_j \quad x \quad t^n \]
Operator integrating factor splitting

\[ t^n + \Delta t \]

\[ \frac{1}{a} \]

\[ x_j \quad x \]

\[ t^n \]
Operator integrating factor splitting

\[ t^n + \Delta t \]

\[ \frac{1}{a} \]

\[ x_j \quad x \]

\[ t^n \]
Operator integrating factor splitting

\[ t^n + \Delta t \]

\[ \frac{1}{a} \]

\[ x_j \]

\[ x \]
Operator integrating factor splitting

ODE resulting from SEM discretization (MOL)

\[
\frac{du(t)}{dt} = S(u(t)) + F(u(t)), \quad t \in [0, T]
\]

with initial condition \( u(0) = u_0 \)

**Problem:** find integrating factor, \( Q_{S}^{t^*}(t) \) such that \( Q_{S}^{t^*}(t^*) = I \),

\[
\frac{d}{dt} Q_{S}^{t^*}(t) \cdot u = Q_{S}^{t^*}(t) \cdot F(u).
\]

To find the action of \( Q_{S}^{t^*}(t) \) solve:

\[
\frac{dv^{(t^*,t)}(s)}{ds} = S(v^{(t^*,t)}), \quad 0 \leq s \leq t - t^*
\]

with initial condition \( v^{(t^*,t)}(0) = u(t) \)
Nonlinear OIFS

St-Cyr and Thomas (2005) sub-step

\[
\frac{\partial \tilde{v}}{\partial s} + \tilde{\zeta} k \times \tilde{v} + \frac{1}{2} \nabla (\tilde{v} \cdot \tilde{v}) = 0
\]

\[
\frac{\partial \tilde{\Phi}}{\partial s} + \nabla \cdot (\tilde{\Phi} \tilde{v}) = 0
\]

with initial conditions \( \tilde{v}(x, t^{n-q}) = v(x, t^{n-q}) \),
\( \tilde{\Phi}(x, t^{n-q}) = \Phi(x, t^{n-q}) \).
Nonlinear OIFS

Integration factor applied to the SWE's

\[
\frac{d}{dt} Q_S^*(t) \begin{bmatrix} v \\ \Phi \end{bmatrix} = -Q_S^*(t) \begin{bmatrix} f k \times v + \nabla \Phi \\ \Phi_0 \nabla \cdot v \end{bmatrix}
\]

Backward Differentiation Formula (BDF-2):

\[
\frac{3v^n - 4\tilde{v}^{n-1} + \tilde{v}^{n-2}}{2\Delta t} = -M f v^n - \nabla \Phi_n
\]

\[
\frac{3\Phi^n - 4\tilde{\Phi}^{n-1} + \tilde{\Phi}^{n-2}}{2\Delta t} = -\Phi_0 \nabla \cdot v^n
\]

Non-symmetric due to implicit Coriolis: CGS.
Implicit/Explicit
Implicit/Explicit
Implicit/Explicit
Implicit/Explicit
Implicit/Explicit

$\Delta s$
Implicit/Explicit
Implicit/Explicit

\[ \Delta s \]
Implicit/Explicit

\[
\frac{3 \mathbf{v}^n - 4 \dot{\mathbf{v}}^n + \mathbf{v}^{n-2}}{2 \Delta t} = -Mf \mathbf{v}^n - \nabla \Phi^n
\]

\[
\frac{3 \mathbf{\Phi}^n - 4 \dot{\mathbf{\Phi}}^n + \mathbf{\Phi}^{n-2}}{2 \Delta t} = -\Phi_0 \nabla \cdot \mathbf{v}^n
\]

\[
\frac{\partial \dot{\mathbf{v}}}{\partial s} + \zeta \mathbf{k} \times \dot{\mathbf{v}} + \frac{1}{2} \nabla (\dot{\mathbf{v}} \cdot \dot{\mathbf{v}}) = 0
\]

\[
\frac{\partial \dot{\mathbf{\Phi}}}{\partial s} + \nabla (\dot{\mathbf{\Phi}} \cdot \dot{\mathbf{v}}) = 0
\]
$dt = 120s$
$dt=360s$
$dt = 720\text{s}$
\textbf{dt=120s}
Total number of elements

![Graph showing total number of elements vs. adaptation cycle with 120, 360, and 720 seconds]

- 120 secs
- 360 secs
- 720 secs

Adaptation cycle

Total number of elements
Comparison with reference solution from NCAR pseudo spectral
Comparison with reference solution from NCAR pseudo spectral

$\epsilon = O(\Delta t^k + \frac{\Delta x^p}{\Delta t})$

Falcone and Ferretti 98:
Discontinuous Galerkin

Conservation law:

\[ u_t + \nabla \cdot \mathcal{F}(u) = S(u) \]

Weak form:

\[
\frac{d}{dt} \int_{\Omega_k} \varphi_h u_h \, d\Omega = \int_{\Omega_k} \varphi_h S(u_h) \, d\Omega + \int_{\Omega_k} \mathcal{F}(u_h) \cdot \nabla \varphi_h \, d\Omega - \int_{\partial \Omega_k} \varphi_h \mathcal{F} \cdot \hat{n} \, ds
\]

Numerical flux:

\[
\widehat{\mathcal{F}}(u_h^+, u_h^-) = \frac{1}{2} \left[ (\mathcal{F}(u_h^+) + \mathcal{F}(u_h^-)) \cdot \hat{n} - \alpha(u_h^+ - u_h^-) \right]
\]
SSP Runge-Kutta with extended linear stability region

\[
\frac{dU_h}{dt} = L_h(U_h).
\]

\[ u^{(0)} = u^n \]

\[ u^{(i)} = \sum_{k=0}^{i-1} \alpha_{ik} u^{(k)} + \Delta t \beta_{ik} L(u^{(k)}), \quad i = 1, \ldots, m, \]

\[ u^{n+1} = u^{(m)}. \]

Higuera 2004, JSC
Compressible Euler:

\[ U \equiv (\rho, \rho u, \rho w, \Theta)^T = (\rho, U, W, \Theta)^T \]

\[ F(U) \equiv (F, G) \]

\[ F = (U, \frac{UU}{\rho} + p, \frac{WU}{\rho}, \frac{U\Theta}{\rho})^T \]

\[ G = (W, \frac{UW}{\rho}, \frac{WW}{\rho} + p, \frac{W\Theta}{\rho})^T \]

\[ S(U) = (0, 0, -g \rho, 0)^T \]

\[ p = p_0\left(\frac{R\Theta}{p_0}\right)^\gamma \]
Warm bubble (Robert 93)
To use or not to use HOMs?

- Low order: 2nd or 3rd
- Very dissipative
- Large dispersion error
- Oscillation control: grid point level
- Limiters - not - cache efficient
- CFL: \( dt = O(dx) \)
- Costly halos...

- Spectrally accurate
- No dissipation
- No dispersion error: with filtering
- Oscillation control: questionable
- Matrix-Matrix tensor operations: cache friendly
- Halos are minimal
Acknowledgments

The AMR work was funded under NSF grant CMG-0222282: An adaptive Mesh, Spectral Element Formulation of the Well-Posed Primitive Equations for Climate and Weather Modeling, NSF MRI Grant CNS-0421498, NSF MRI Grant CNS-0420873, NSF MRI Grant CNS-0420985

DOE Climate Change Prediction Program CCPP.

Email: amik@ucar.edu
Phone: 303-497-1287