From classical to optimized Schwarz

by
Amik St-Cyr
(PART-2)

The quest for cheaper and faster preconditioning

Part 2:

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- The Robin method
- Fourier analysis of Classical Schwarz
- Fourier analysis for optimized Schwarz
- Optimization over all Fourier modes
- Examples FDM
- High-Order methods (HOMs)
- Optimized Schwarz in a massively parallel GCM
- Conclusion

The Robin method

- Lions (1990)
- Used to accelerate convergence of Schwarz
- Free positive parameter: how to find its correct value?
- Convergence rate not demonstrated theoretically

$$\mathcal{L}u_j^{k+1} = u_j^{k+1} - \Delta u_j^{k+1} = f_j$$

$$p u_j^{k+1} + \frac{\partial u_j^{k+1}}{\partial \mathbf{n}_{jl}} = p u_l^k + \frac{\partial u_l^k}{\partial \mathbf{n}_{jl}} \text{ on } \partial\Omega_j \cap \partial\Omega_l \text{ for } l \in \mathcal{N}(\Omega_j)$$

$$u_j^{k+1} = u_0 \text{ on } \partial\Omega_j \cap \partial\Omega$$

Convergence of the Robin method

Write the error as: $e_j^{k+1} = u_j^{k+1} - u|_{\Omega_j}$

Homogeneous case:

$$e_j^{k+1} - \Delta e_j^{k+1} = 0$$
$$p e_j^{k+1} + \frac{\partial e_j^{k+1}}{\partial \mathbf{n}_{jl}} = p e_l^k + \frac{\partial e_l^k}{\partial \mathbf{n}_{jl}} \text{ on } \partial\Omega_j \cap \partial\Omega_l \text{ for } l \in \mathcal{N}(\Omega_j)$$
$$e_j^{k+1} = 0 \text{ on } \partial\Omega_j \cap \partial\Omega$$

Multiplication by error term + integration by parts:

$$0 = \int_{\Omega_j} e_j^{k+1} e_j^{k+1} + \int_{\Omega_j} \nabla e_j^{k+1} \cdot \nabla e_j^{k+1} - \sum_{l\in\mathcal{N}(j)} \int_{\Gamma_{jl}} e_j^{k+1} \frac{\partial e_j^{k+1}}{\partial n_{jl}}$$
$$= ||e_j^{k+1}||^2_{0,\Omega_j} + ||\nabla e_j^{k+1}||^2_{0,\Omega_i} - \sum_{l\in\mathcal{N}(j)} \int_{\Gamma_{jl}} e_j^{k+1} \frac{\partial e_j^{k+1}}{\partial n_{jl}}$$
$$= ||e_j^{k+1}||^2_{1,\Omega_j} - \sum_{l\in\mathcal{N}(j)} \int_{\Gamma_{jl}} e_j^{k+1} \frac{\partial e_j^{k+1}}{\partial n_{jl}}$$

Convergence of the Robin method

Using: $AB = \frac{1}{4p}[(A+pB)^2 - (A-pB)^2]$

and summing over all elements and the first M iterations:

$$\sum_{k=0}^{M} \sum_{i=0}^{K} ||e_i^{k+1}||^2_{1,\Omega_i} + \frac{1}{4p} \sum_{\{i,l\}\in\mathbf{e}_m} \int_{\Gamma_{il}} \{(\frac{\partial e_i^M}{\partial n_{il}} - p e_i^M)^2 + (\frac{\partial e_l^M}{\partial n_{li}} - p e_l^M)^2\}$$
$$= \frac{1}{4p} \sum_{\{i,l\}\in\mathbf{e}_m} \int_{\Gamma_{il}} \{(\frac{\partial e_l^0}{\partial n_{il}} + p e_l^0)^2 + (\frac{\partial e_i^0}{\partial n_{li}} + p e_i^0)^2\}$$

Implying:

$$\lim_{M\to\infty} \sum_{k=0}^{M} \sum_{i=0}^{K} ||e_i^{k+1}||^2_{1,\Omega_i} < C$$

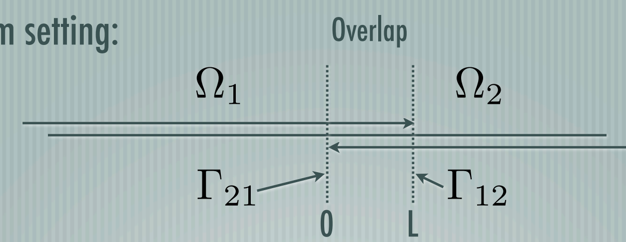
For any positive "p": what is the best choice?

Fourier analysis

- Study simple 2D problem
- Only 2 subdomains
- Fourier transform in the tangent direction to the separating interface between domains
- Solve the remaining ODE
- Obtain convergence rate of the algorithm

Fourier analysis

Problem setting:



$(\eta - \Delta)u(x,y) = 0, \text{ on } \Omega$

Boundary conditions: solution decays at infinity

Subdomains:

$$\Omega_1 = [-\infty, L] \times \mathbb{R} \text{ and } \Omega_2 = [0, \infty] \times \mathbb{R}$$

Fourier analysis

Two subproblems:

$$\begin{aligned}(\eta-\Delta)u_1^{n+1} &= 0 && \text{in } \Omega_1, & (\eta-\Delta)u_2^{n+1} &= 0 && \text{in } \Omega_2,\\ u_1^{n+1}(L,y) &= u_2^n(L,y) && \text{on } \Gamma_{12}, & u_2^{n+1}(0,y) &= u_1^n(0,y) && \text{on } \Gamma_{21}.\end{aligned}$$

Fourier transforming in the y direction:

$$\begin{aligned}(\eta+k^2-\partial_{xx})\hat{u}_1^{n+1} &= 0 && \text{in } \Omega_1, & (\eta+k^2-\partial_{xx})\hat{u}_2^{n+1} &= 0 && \text{in } \Omega_2,\\ \hat{u}_1^{n+1}(L,k) &= \hat{u}_2^n(L,k) && \text{on } \Gamma_{12}, & \hat{u}_2^{n+1}(0,k) &= \hat{u}_1^n(0,k) && \text{on } \Gamma_{21}.\end{aligned}$$

Solving in the x direction:

$$\hat{u}_1^n(x,k) = \hat{u}_2^{n-1}(L,k)e^{-\sqrt{k^2+\eta}(x-L)}, \qquad \hat{u}_2^n(x,k) = \hat{u}_1^{n-1}(0,k)e^{-\sqrt{k^2+\eta}x}$$

Convergence rate of classical Schwarz (Gander 2006 SINUM):

$$\rho_{cla} = \rho_{cla}(k,\eta,L) = e^{-\sqrt{k^2+\eta}L}$$

Remarks about convergence rate

$$\rho_{cla} = \rho_{cla}(k,\eta,L) = e^{-\sqrt{k^2+\eta}L}$$

- Converges for all frequencies
- Is a smoother: damps quickly high frequencies
- Convergence depends on eta and overlap size
- For no overlap the algorithm does not converge

Optimized approach

- Inspired by the Robin problem:

$$\begin{aligned}(\eta-\Delta)u_1^{n+1} &= 0 && \text{in } \Omega_1, & (\eta-\Delta)u_2^{n+1} &= 0 && \text{in } \Omega_2,\\ (\partial_x+S_1)u_1^{n+1} &= (\partial_x+S_1)u_2^n && \text{on } \Gamma_{12}, & (\partial_x+S_2)u_2^{n+1} &= (\partial_x+S_2)u_1^n && \text{on } \Gamma_{21}.\end{aligned}$$

We are looking for the best possible forms of in Fourier space

Proceeding as before leads to the solutions: ($\sigma_r(k) = \mathcal{F}(S_r)$)

$$\hat{u}_1^n(x,k) = \frac{\sigma_1(k)-\sqrt{k^2+\eta}}{\sigma_1(k)+\sqrt{k^2+\eta}}e^{-\sqrt{k^2+\eta}(x-L)}\hat{u}_2^{n-1}(L,k), \qquad \hat{u}_2^n(x,k) = \frac{\sigma_2(k)+\sqrt{k^2+\eta}}{\sigma_2(k)-\sqrt{k^2+\eta}}e^{-\sqrt{k^2+\eta}x}\hat{u}_1^{n-1}(0,k)$$

New convergence rate:

$$\rho_{opt} = \rho_{opt}(k,\eta,L) = \frac{\sigma_1(k)-\sqrt{k^2+\eta}}{\sigma_1(k)+\sqrt{k^2+\eta}}\frac{\sigma_2(k)+\sqrt{k^2+\eta}}{\sigma_2(k)-\sqrt{k^2+\eta}}e^{-2\sqrt{k^2+\eta}L}$$

Optimized approach

The choice

$$\sigma_1(k) = \sqrt{k^2+\eta},\ \sigma_2(k) = -\sqrt{k^2+\eta}$$

leads to the convergence of the algorithm in 2 iterations $\rho_{opt} = 0$

The operators are <u>not local</u> operators in physical space!

An approximation is sought such that all frequencies have an optimal decay rate:

$$\sigma_1^{app}(k) = p_1 + q_1k^2,\ \sigma_2^{app}(k) = -p_2 - q_2k^2$$

Various choices (one sided)

Taylor zeroth order: $\sigma_1^{app}(k) = \sqrt{\eta}$

Taylor second order: $\sigma_1^{app}(k) = \sqrt{\eta} + \frac{1}{2\sqrt{\eta}} k^2$

Zeroth order optimized: $k(L, \eta, p) = \frac{\sqrt{L(2p + L(p^2 - \eta))}}{L}$

$$\rho_{OO0}(k_{\min}, L, \eta, p^*) = \rho_{OO0}(k(p^*), L, \eta, p^*)$$

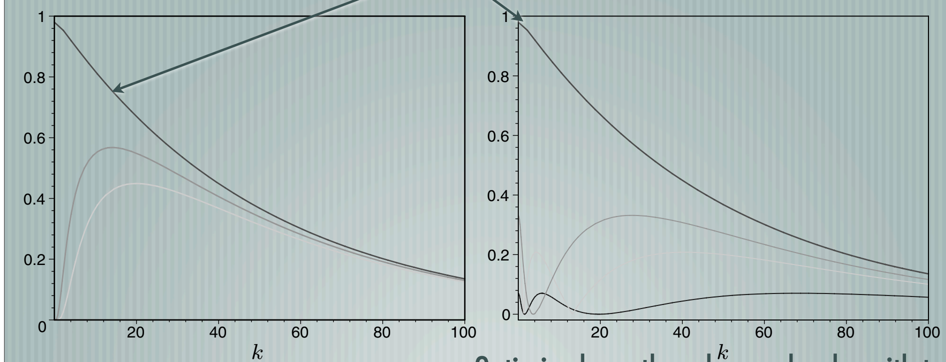
Zeroth order optimized (no overlap): $p^* = ((k_{\min}^2 + \eta)(k_{\max}^2 + \eta))^{\frac{1}{4}}$

Second order optimized: very long and complex formulas for p and q ...

Details see Gander (SINUM 2006)

Convergence rates

Classical Schwarz



Taylor zeroth order and second order

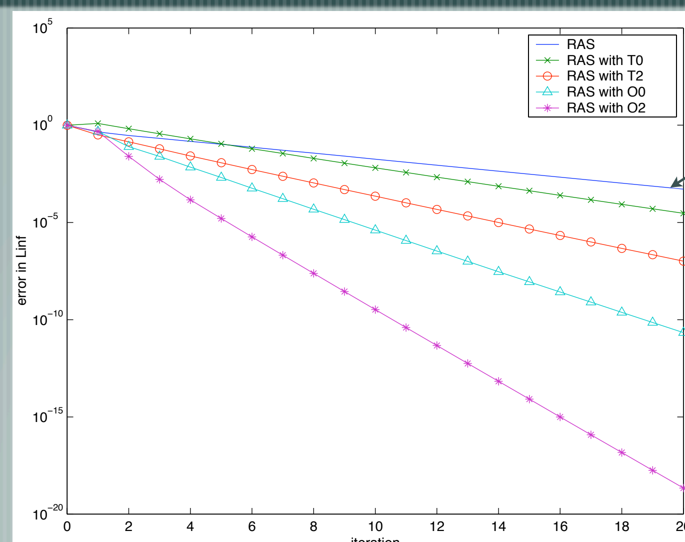
Optimized zeroth and second order with two-sided zeroth order

Examples for FDM

$$u - \Delta u = 0, \text{ on } [0,1] \times [0,1], \; u(0) = u(1) = 0$$

- 2 subdomains
- 2nd order Laplacian
- Mesh spacing $h = 1/30$
- Optimization done at the matrix level (SGT 2006 SISC)

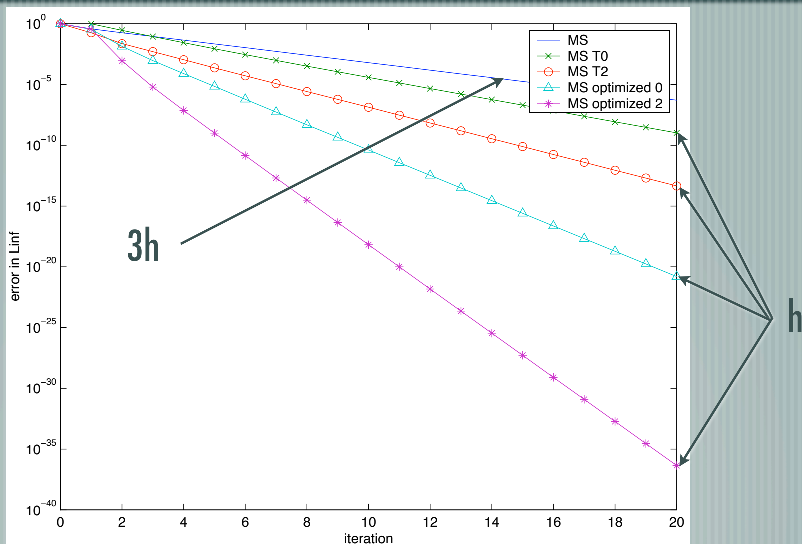
Examples for FDM: ORAS



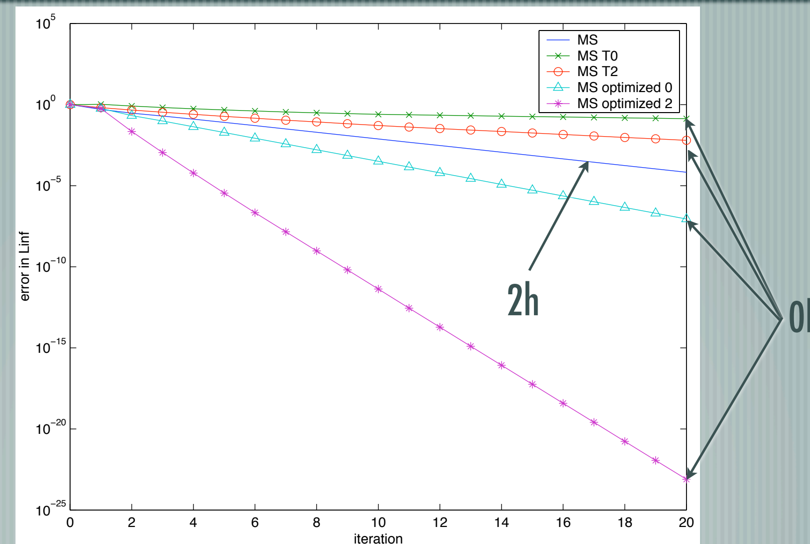
3h

h

Examples for FDM: OMS



Examples for FDM: OMS



HOMs: spectral elements

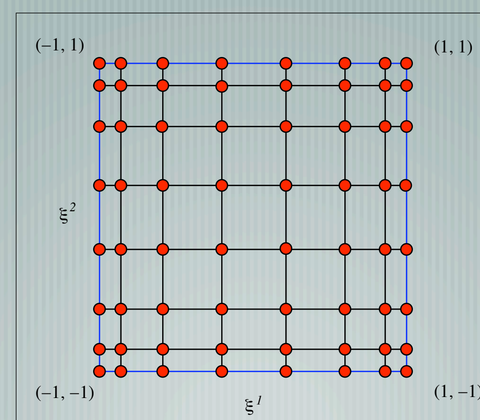
- Galerkin idea: identical to FEM
- High-order basis on each element
- Integration with Gauss-Legendre-Lobatto quadratures

$$\mathbf{v}_h^k(r_1, r_2) = \sum_{i=0}^{N} \sum_{j=0}^{N} \mathbf{v}_{ij}\, h_i(r_1)\, h_j(r_2)$$

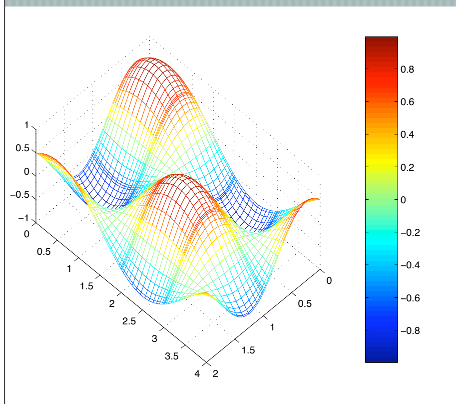
$$\langle f, g \rangle_{GL} = \sum_{k=1}^{K} \sum_{i=0}^{N} \sum_{j=0}^{N} f^k(\xi_i, \xi_j)\, g^k(\xi_i, \xi_j)\, \rho_i\, \rho_j$$

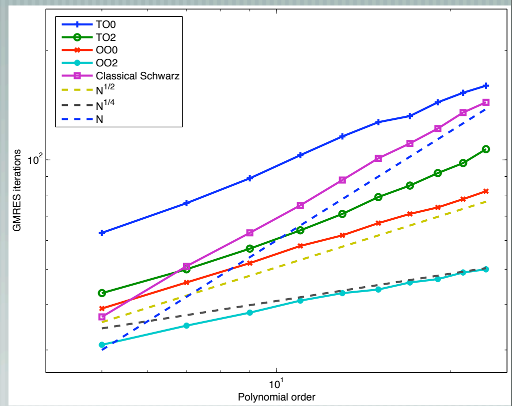
HOMs: spectral elements

Reference element:



HOMs: spectral elements





Asymptotic behavior

$\kappa(M^{-1}A)$	h	N
AS, no overlap	$O(h^{-1})$	$O(N^2)$
SS, no overlap	$O(h^{-1})$	$O(N^2)$
OO0, no overlap	$O(h^{-1/2})$	$O(N)$
OO2, no overlap	$O(h^{-1/4})$	$O(N^{1/2})$

- Number of subdomains dependance: $1/H^2$
- Removed by coarse solver.
- Optimal is the $\mathbb{Q}_1$ fem problem on GLL mesh (S.D. Kim 2006)

Primitive equations

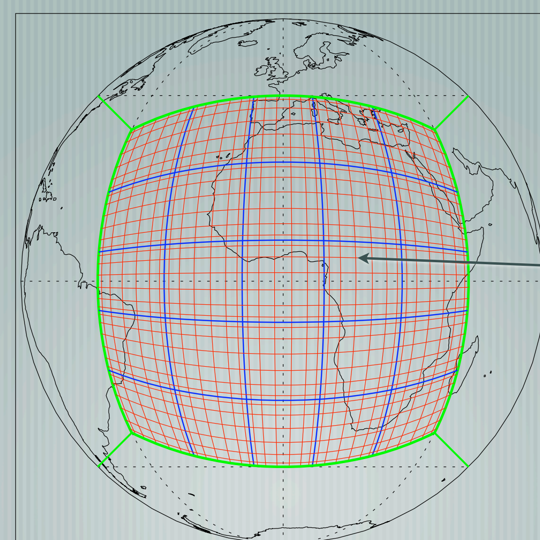
Momentum: $$\frac{d\mathbf{v}}{dt} + f\,\mathbf{k}\times\mathbf{v} + \nabla\Phi + R\,T\,\nabla\ln p = 0$$

Thermodynamic: $$\frac{d\,T}{d\,t} - \frac{\kappa\,T\,\omega}{p} = 0$$

Hydrostatic:

$$\frac{\partial}{\partial t}\left(\frac{\partial p}{\partial \eta}\right) + \nabla\cdot\left(\mathbf{v}\,\frac{\partial p}{\partial \eta}\right) + \frac{\partial}{\partial \eta}\left(\dot{\eta}\,\frac{\partial p}{\partial \eta}\right) = 0$$

Cubed sphere

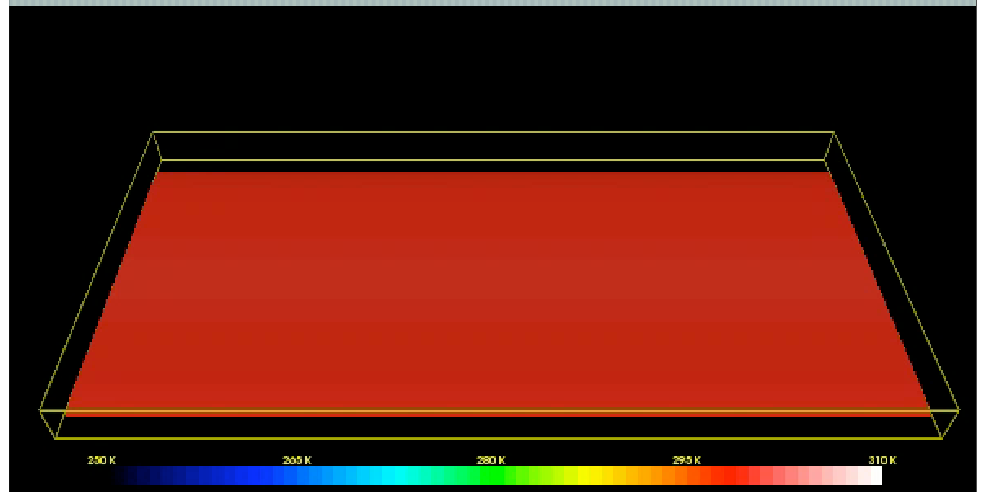


Spectral element

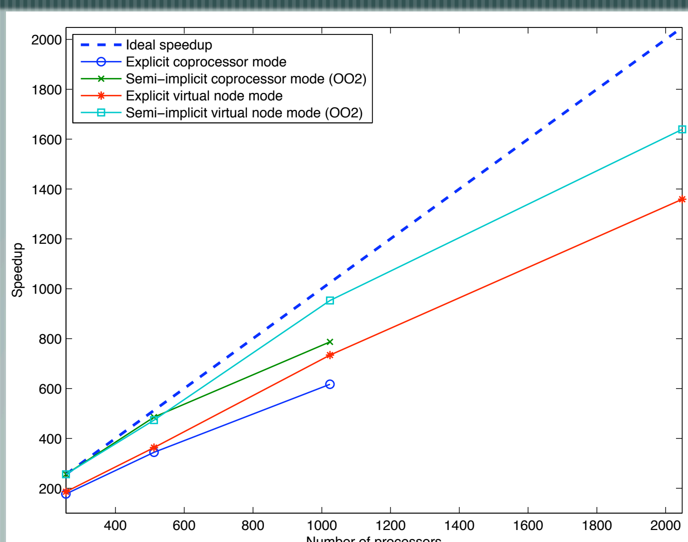
Time discretization

- Semi-implicit time discretization
- Leads to positive definite Helmholtz problem to solve at each time step
- Optimized Schwarz with tangential derivative used
- Results on Blue Gene/L machine
- Held-Suarez test case

Held-Suarez



Parallel performance BG/L



Conclusion

- A simple modification to classical Schwarz leads to a faster converging solver
- This is an easy intervention in a model
- With coarse solver, optimized Schwarz is nearly optimal: no need to keep constant overlap (none is required!)
- Good performance in a general circulation model
- Future work: semi-discrete optimizations, rate of convergence for SEM and optimal control (S.D. Kim)