From classical to optimized Schwarz

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Plan of presentation

Part 1:

- Motivation of DDM
- Partitioning algorithms
- Classical Schwarz algorithm
- Matrix/discrete level
- Convergence
- Two level approach

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Part 2:

- The Robin method
- Fourier analysis of Classical Schwarz
- Fourier analysis for optimized Schwarz
- Optimization over all Fourier modes
- Examples FDM
- High-Order methods (HOMs)
- Optimized Schwarz in a massively parallel GCM
- Conclusion

Part 1:

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DDM Motivation

- The global problem cannot fit into main memory, out of core computations: very slow swapping to disk
- (AND | OR) Concurrency can be exploited to solve the global problem: solving problem faster on parallel computers
- (AND | OR) The solution of the subproblems is “easier” than the global problem: direct methods on smaller subproblems cache friendly

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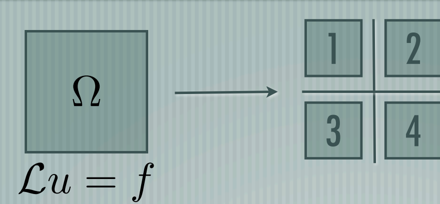
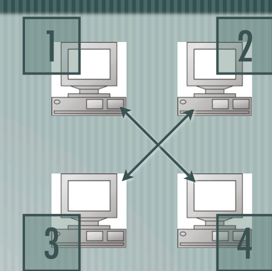
Domain Decomposition

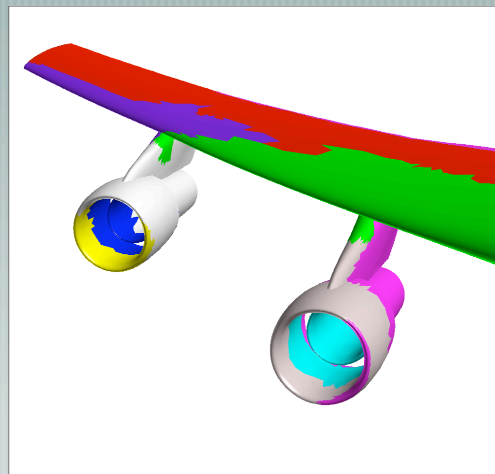




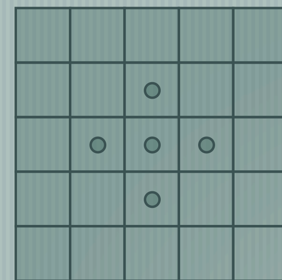
- Divide and conquer applied to PDEs:
- *Decompose* domain into many sub-domains
- *Solve* independently each smaller problem
- *Glue* the solutions together: convergence?

Mesh partitioning: *decompose* the domain

- Geometric Based Algorithms
  - Coordinate bisection
  - Inertia bisection
- Graph Theory Based Algorithms
  - Graph bisection
  - Greedy algorithm
  - Spectral bisection
  - K-L algorithm
- Other Partitioning Algorithms
  - Global optimization algorithms
  - Reducing the bandwidth of the matrix
  - Index based algorithms
- The State of the Art
  - Hybrid approach
  - Multilevel approach
  - Parallel partitioning algorithms



Example: spectral bisection



Laplacian

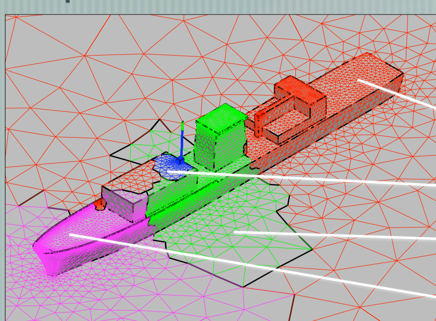
$$Lx = \begin{bmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & -1 & -1 & 4 & -1 & -1 & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} x = \lambda x$$

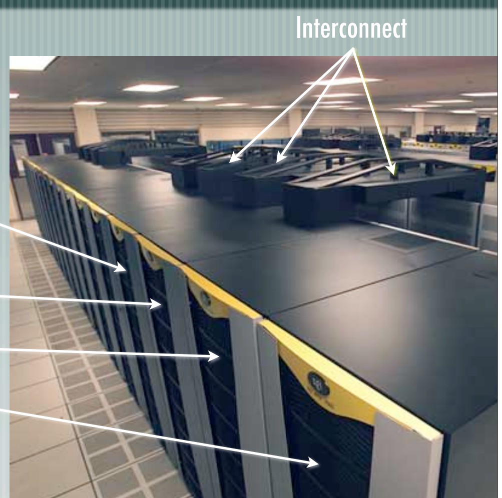
$$\sum_{i=1}^{n} (x_i) = 0 \qquad \sum_{i=1}^{n} (x_i)^2 = n$$

- Needs to be an eigenvector of Laplacian
- If composed of half +1 and half -1 it satisfies the two constraints
- Finding the Fielder vector: Lanczos algorithm
- Proceed recursively...

Practical DDM

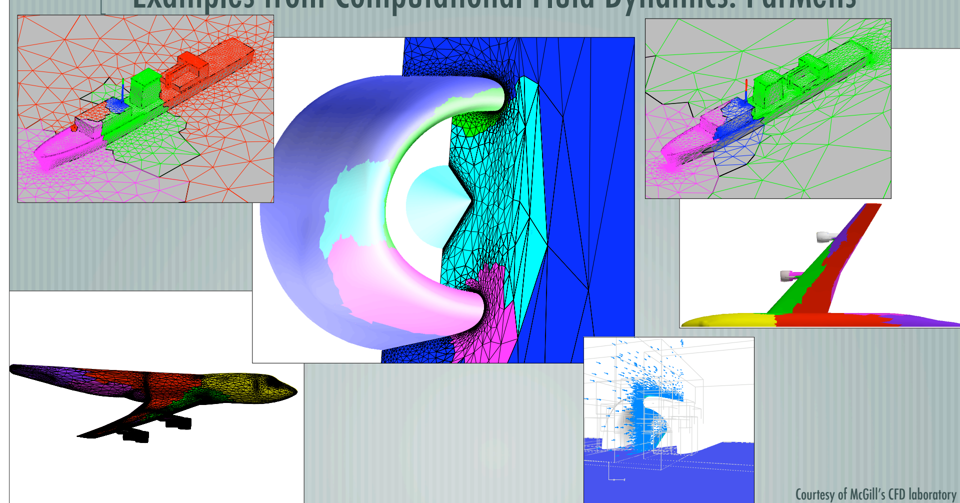
Each part of problem solved on a compute node:





Partitioned meshes

Examples from Computational Fluid Dynamics: ParMetis



Courtesy of McGill's CFD laboratory

Mesh partitioning

- Represents only the *technical* part of DDM
- Has deep ties with *parallel computing*: MIMD
- DDM denotes also the development of special algorithms to solve decomposed problems
- Algorithms: Schwarz, FETI, sub-structuring ...

Domain Decomposition

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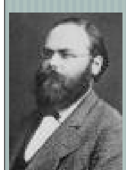
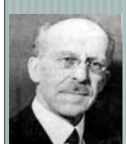
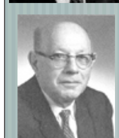
Basic DD methods



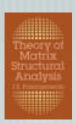
(Overlapping) Schwarz (1870): existence of elliptic problems on non trivial domains



(Non-overlapping) Schur / sub-structuring methods



Kron (53)



Przemieniecki (63)

2 classes of methods: overlapping and non-overlapping

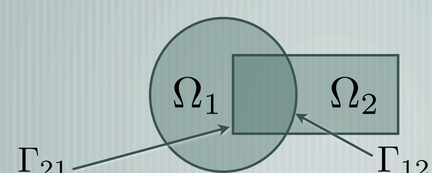
Classical Schwarz

Suppose we need to solve:

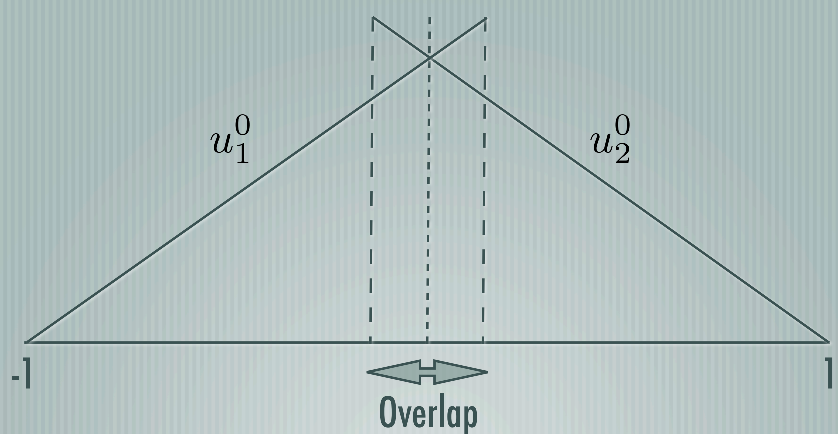
$$\mathcal{L}u = f \quad \text{in } \Omega, \qquad \mathcal{B}u = g \quad \text{on } \partial\Omega$$

Partition the original domain into 2 domains:

$$\begin{aligned} \mathcal{L}u_1^{n+1} &= f & \text{in } \Omega_1, \qquad \mathcal{L}u_2^{n+1} &= f & \text{in } \Omega_2, \\ \mathcal{B}(u_1^{n+1}) &= g & \text{on } \partial\Omega_1, \qquad \mathcal{B}(u_2^{n+1}) &= g & \text{on } \partial\Omega_2, \\ u_1^{n+1} &= u_2^n & \text{on } \Gamma_{12}, \qquad u_2^{n+1} &= u_1^n & \text{on } \Gamma_{21}. \end{aligned}$$



Schwarz with large overlap



$\Delta u = 0, \text{ on } [-1,1] \text{ with } u(-1) = u(1) = 0$

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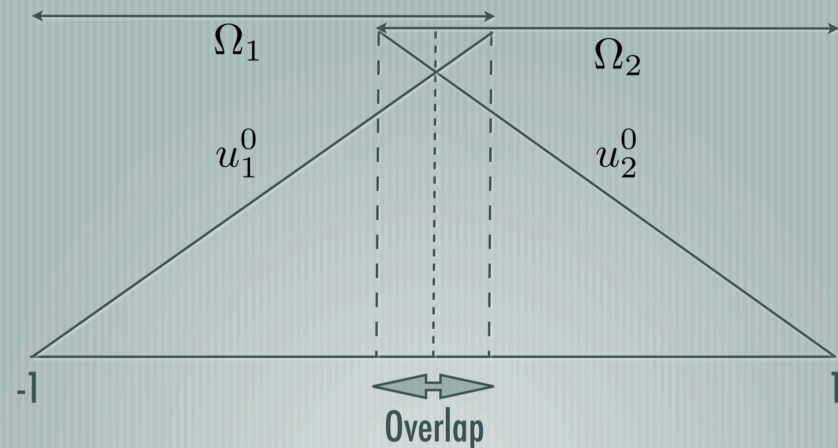

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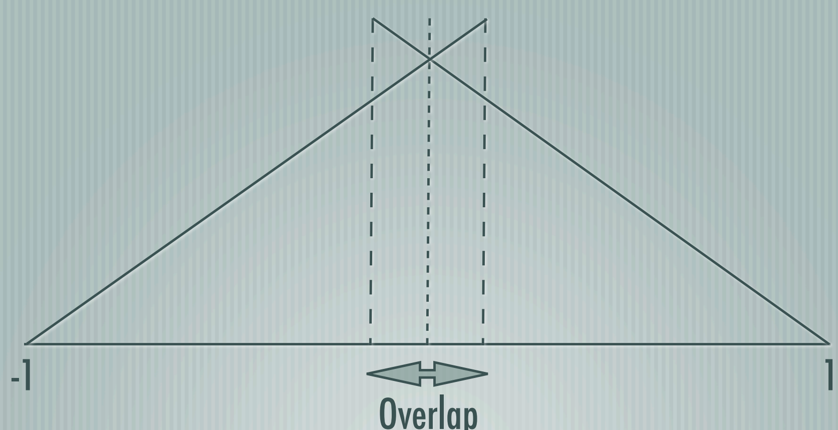

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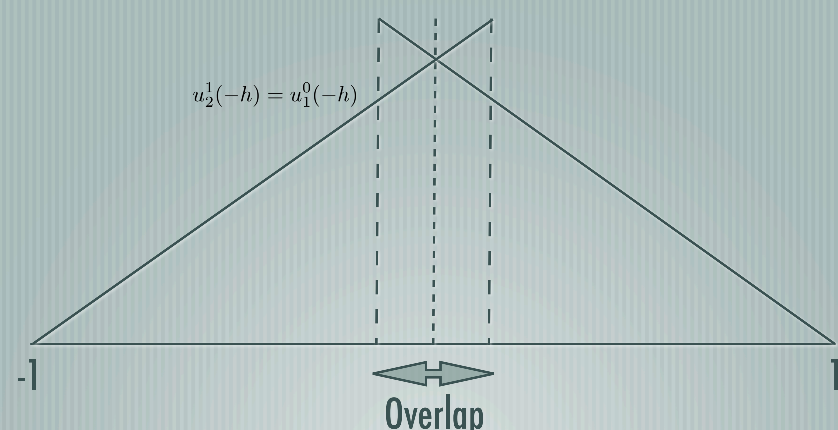

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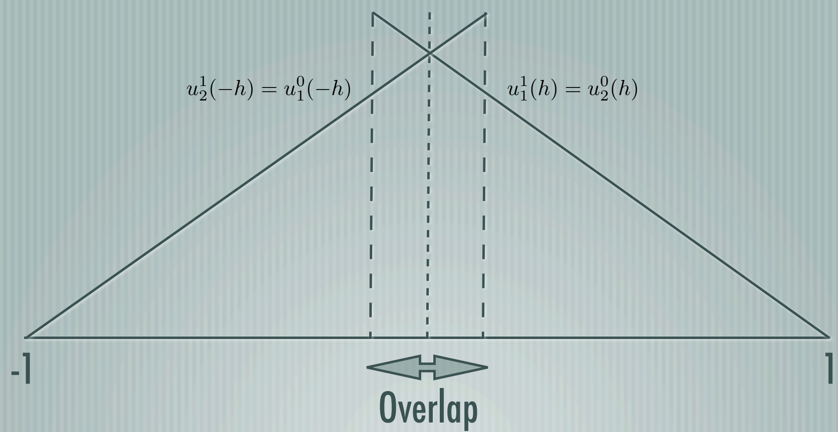

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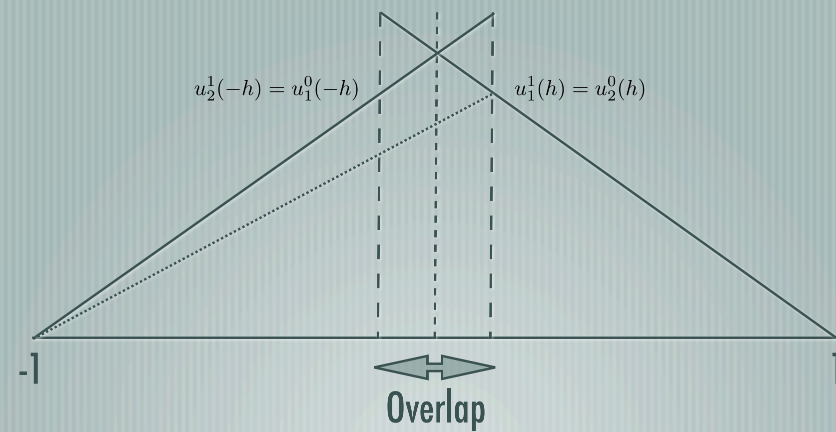

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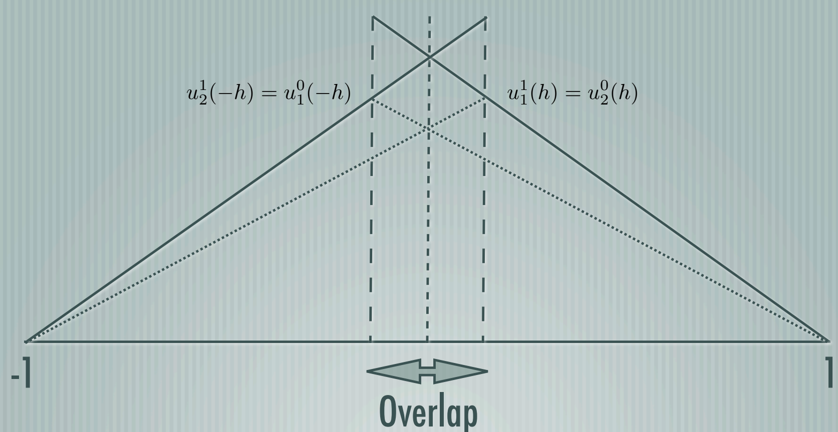

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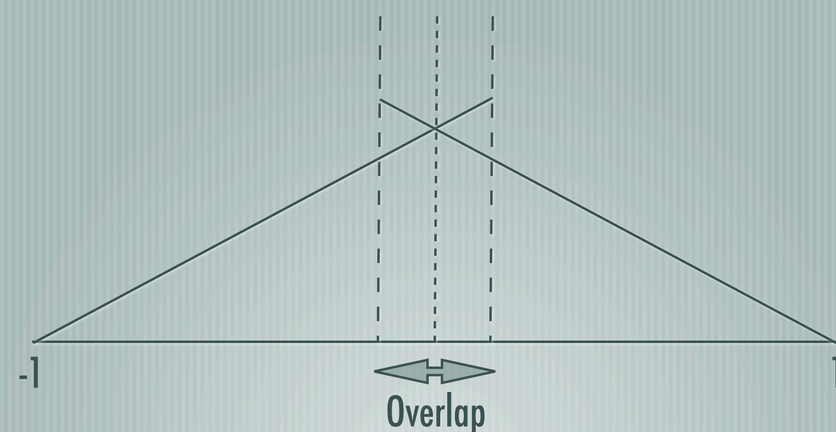

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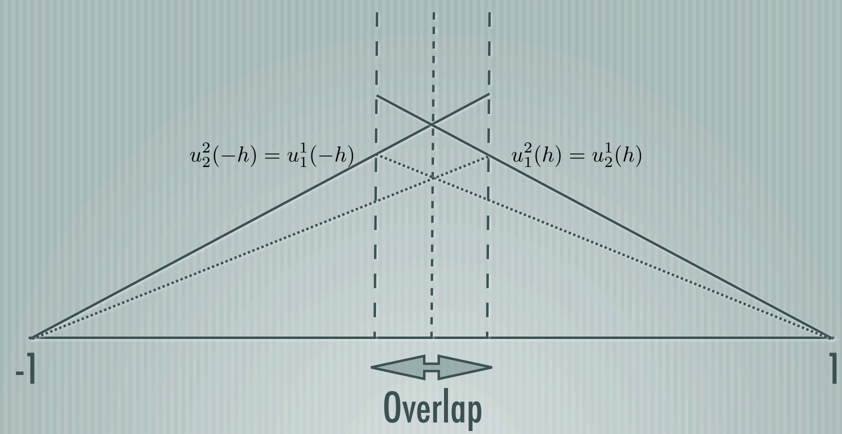

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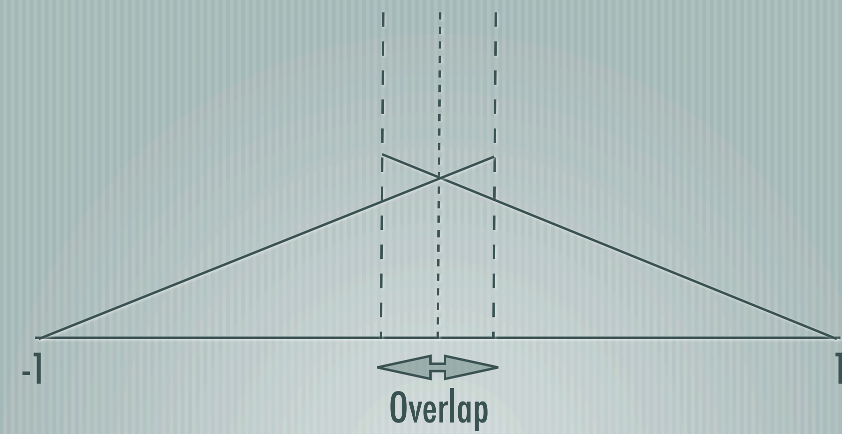

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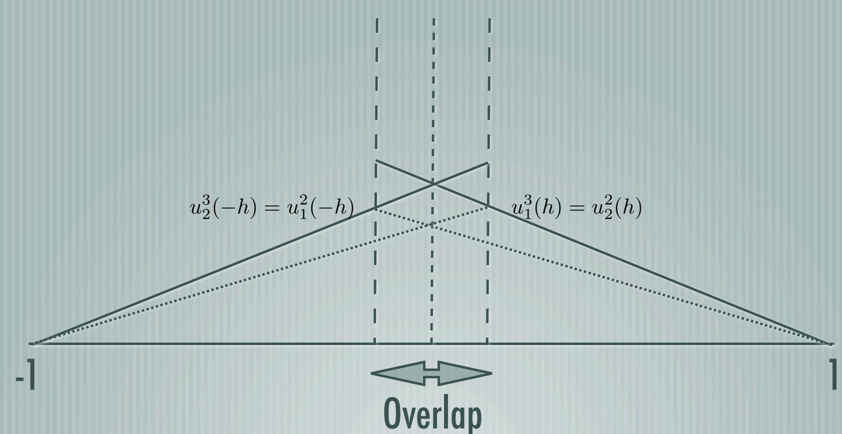

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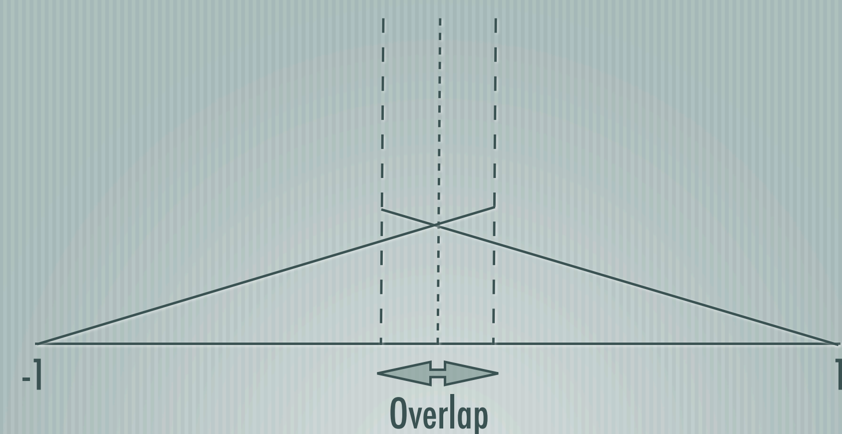

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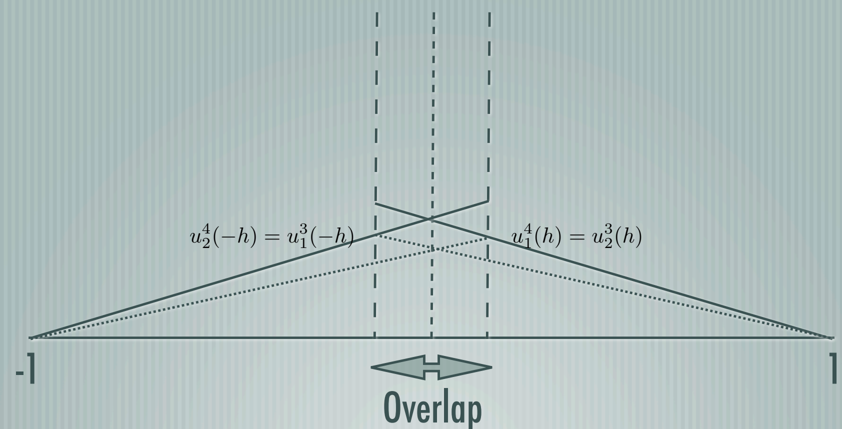

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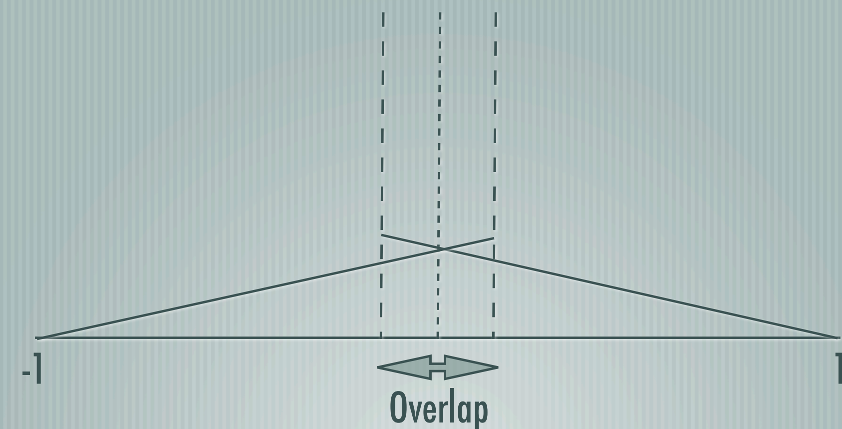

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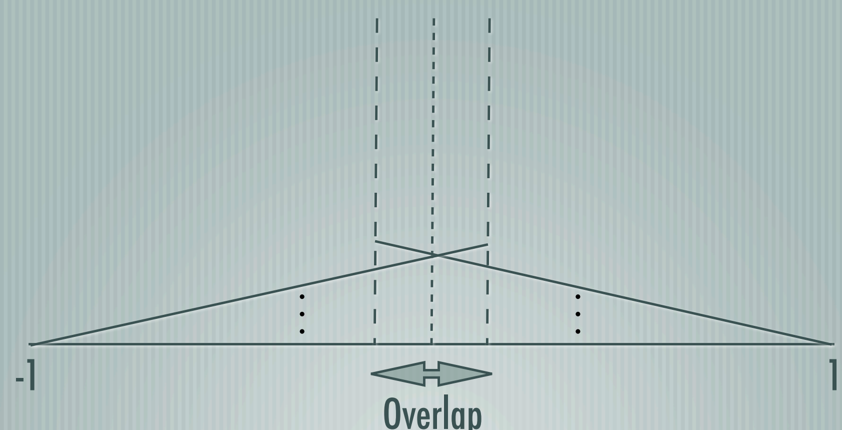

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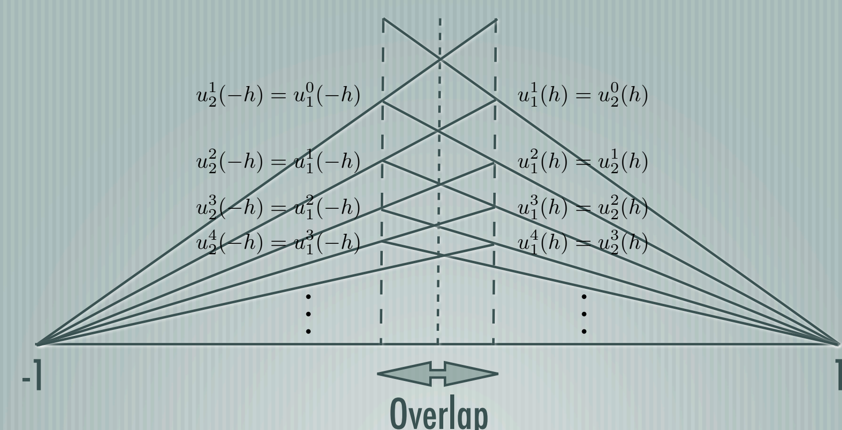

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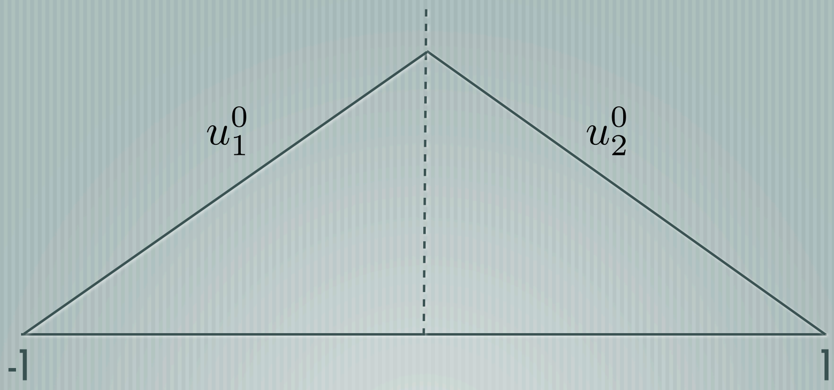
Schwarz no overlap



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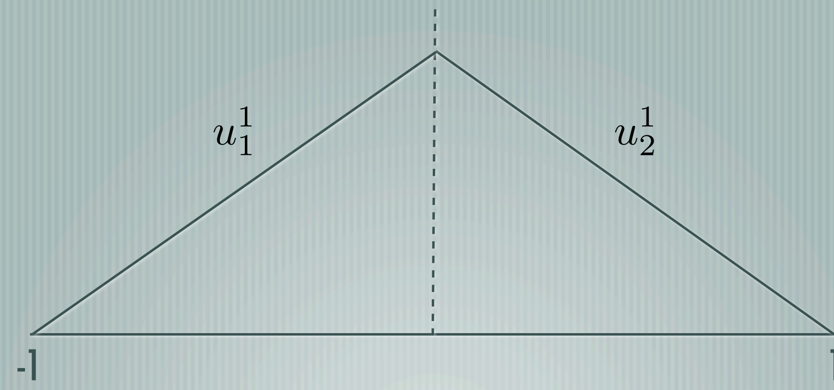

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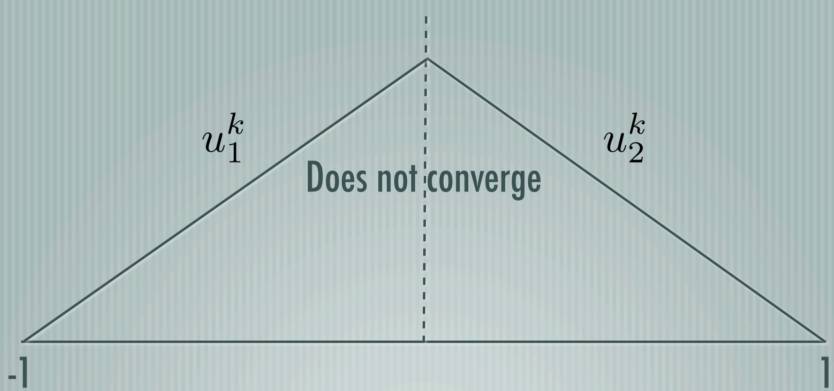

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Matrix formulation

Continuous problem:

$$\begin{aligned} \mathcal{L}u_1^{n+1} &= f && \text{in } \Omega_1, \\ \mathcal{B}(u_1^{n+1}) &= g && \text{on } \partial\Omega_1, \\ u_1^{n+1} &= u_2^n && \text{on } \Gamma_{12}, \end{aligned} \qquad \begin{aligned} \mathcal{L}u_2^{n+1} &= f && \text{in } \Omega_2, \\ \mathcal{B}(u_2^{n+1}) &= g && \text{on } \partial\Omega_2, \\ u_2^{n+1} &= u_1^n && \text{on } \Gamma_{21}. \end{aligned}$$

$$A_1 \underline{u}_1^{n+1} = \underline{f}_1 + B_{21}\underline{u}_2^n \qquad A_2 \underline{u}_2^{n+1} = \underline{f}_2 + B_{12}\underline{u}_1^n$$

Partition of unity: $e = \tilde{R}_1^T e_1 + \tilde{R}_2^T e_2 \rightarrow \underline{u}^{n+1} = \tilde{R}_1^T \underline{u}_1^{n+1} + \tilde{R}_2^T \underline{u}_2^{n+1}$

Restriction operator: $\underline{u}_k^n = R_k \underline{u}$

Matrix formulation

If consistent then: $A_1 R_1 - B_{21} R_2 = R_1 A, \quad A_2 R_2 - B_{12} R_1 = R_2 A.$

Leading to: $$\underline{u}^{n+1} = \underline{u}^n + \sum_{i=1}^{2} \tilde{R}_i^T A_i^{-1} R_i (f - A\underline{u}^n)$$

RAS: Restricted Additive Schwarz
- Nonsymmetric
- Default option in PETSC
- Cai and Sarkis (1997)
- Equivalent to continuous

AS: Additive Schwarz $\tilde{R}_i^T \to R_i^T$
- Symmetric
- No continuous equivalent (EG02)
- Use with Krylov accelerator
- Nepomnyaschikh (86)

Matrix formulation

On multiple domains:

$$\underline{u}^{n+1} = \underline{u}^n + \sum_{i=1}^{K} \tilde{R}_i^T A_i^{-1} R_i (f - A\underline{u}^n)$$

Preconditioning in Krylov methods:

$$M_{RAS}^{-1} = \sum_{i=1}^{K} \tilde{R}_i^T A_i^{-1} R_i \qquad M_{AS}^{-1} = \sum_{i=1}^{K} R_i^T A_i^{-1} R_i$$

- In practice the restriction and extension are not created
- Matrix Problem can be reformulated: lower operation counts

Convergence theory

- For symmetric positive definite matrices
- No results for Restricted Additive Schwarz
- Convergence rate *not optimal*
- Convergence rate *not scalable*
- Developed by Lions, Dryja, Widlund, BRamble, Pasciak, Wang, Xu, Zhang etc ...

Solved by using a coarse solver

Convergence additive Schwarz

Convergence of PCG: $||\underline{u}^{(k)} - \underline{u}^*|| \leq 2\gamma^k ||\underline{u}^{(0)} - \underline{u}^*||$

where $$\gamma = \frac{\sqrt{\kappa(M^{-1}A)} - 1}{\sqrt{\kappa(M^{-1}A)} + 1}$$

Subdomain diameter: $H = \max_{1 \leq i \leq K} diam(\Omega_i)$

Mesh size: h

Overlap size: $\beta H, \ \beta \in (0, 1]$

Convergence additive Schwarz

$$\kappa(M_{AS}^{-1}A) \leq CH^{-2}(1+\beta^{-1})$$

- Diameter tends to zero as the number of subdomain increases
- The overlap size does not remove the diameter problem
- Estimate worsen when A has varying coefficients: $\nabla \cdot (a(\mathbf{x})\nabla u)$
- Diameter dependence prevents algorithmic scalability
- Schwarz didn't care about the scalability!

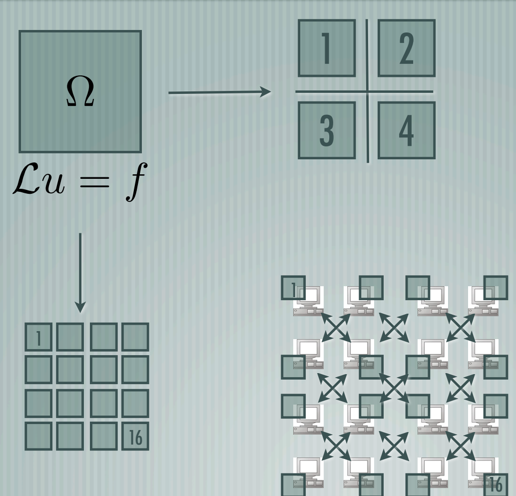
Scalable/Optimal DDM algorithm

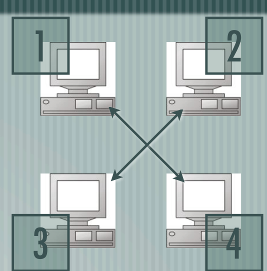
- A DDM is scalable if its rate of convergence does not deteriorate when the number of subdomains grows.
- A DDM for the solution of a linear system is optimal if its rate of convergence to the exact solution is independent of the size of the system.

The first definition involves H

The second involves h

Practical scalability





If scalable the solution is reached 4 times faster!

If additive Schwarz is used it takes the half time to solve!!

Two level methods

- Add a very coarse problem solved on the entire domain
- Removes completely the subdomain diameter problem
- Not easy to parallelize! (Duplication of coarse solves)

$$M_{AS,2}^{-1} = R_H^T A_H^{-1} R_H + \sum_{i=1}^{K} R_i^T A_i^{-1} R_i = \sum_{i=0}^{K} R_i^T A_i^{-1} R_i$$

Condition number: $\kappa(M_{AS,2}^{-1}A) \leq C(1+\beta^{-1})$

Varying coefficients: $C(\beta)(1+\log(H/h),\ C(\beta)(H/h)$

Two level methods

- Still not perfect since overlap must be kept constant!
- The perfect method would have zero overlap and a condition number independent of H and h
- Is it possible to construct such a Schwarz method ??
- If not how close can we get?

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- Fourier analysis for optimized Schwarz
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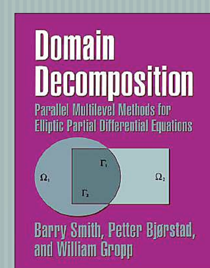

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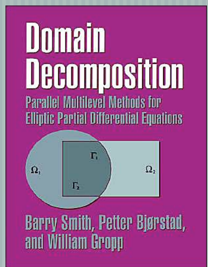

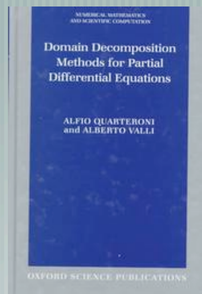

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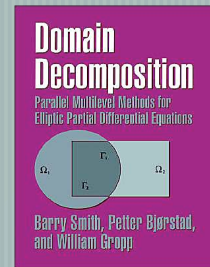

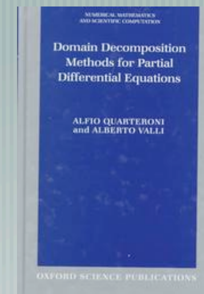

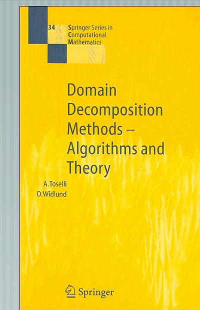

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